Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 2, 213–223

Strong and A-statistical comparisons for double sequences and multipliers

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Abstract. In this work, we obtain strong and A-statistical comparisons for double sequences. Also, we study multipliers for bounded A-statistically convergent and bounded A-statistically null double sequences. Finally, we prove a Steinhaus type result.

Mathematics Subject Classification (2010): 40A05, 40A35, 40B05, 42A45.

Keywords: A-statistical convergence of double sequences, multipliers.

1. Introduction

Strong and A-statistical comparisons for sequences have been studied in [3]. Demirci, Khan and Orhan [4] have studied multipliers for bounded A-statistically convergent and bounded A-statistically null sequences. Also, Connor, Demirci and Orhan [1] have studied multipliers and factorizations for bounded statistically convergent sequences. Yardımcı [16] has extended the results in [1] using the concept of ideal convergence. Dündar and Altay [6] have obtained analogous results in [16] for bounded ideal convergent double sequences.

In this paper we show that the double sequence $\chi_{\mathbb{N}^2}$, which is the characteristic function of $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, is a multiplier from $W(T, p, q) \cap l_2^\infty$, the space of all bounded strongly T-summable double sequences with index $p, q > 0$, into the bounded summability domain $c_A^2(b)$, when T and A two nonnegative RH-regular summability matrices. Also A-statistical comparisons for both bounded as well as arbitrary double sequences have been characterized.

We first recall the concept of A-statistical convergence for double sequences.

A double sequence $x = (x_{m,n})$ is said to be convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$, the set of all natural numbers, such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. L is called the Pringsheim limit of x and denoted by $P - \lim x = L$ (see [14]). We shall such an x more briefly as "$P-$convergent". By a bounded double sequence we mean there exists a positive number K such that

$|x_{m,n}| < K$ for all $(m,n) \in \mathbb{N}^2$, two-dimensional set of all positive integers. For bounded double sequences, we use the notation

$$||x||_{2,\infty} = \sup_{m,n} |x_{m,n}| < \infty.$$

Note that in contrast to the case for single sequences, a convergent double sequence is not necessarily bounded. Let $A = (a_{j,k,m,n})$ be a four-dimensional summability method. For a given double sequence $x = (x_{m,n})$, the $A-$transform of x, denoted by $Ax := ((Ax)_{j,k})$, is given by

$$(Ax)_{j,k} = \sum_{m,n=1,1}^{\infty,\infty} a_{j,k,m,n} x_{m,n}$$

provided the double series converges in the Pringsheim's sense for $(m,n) \in \mathbb{N}^2$.

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence in to a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations is known as Silverman-Toeplitz conditions ([8]). In 1926 Robison [15] presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is $P-$convergent is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, $RH-$regularity ([7], [15]).

Recall that a four dimensional matrix $A = (a_{j,k,m,n})$ is said to be $RH-$regular if it maps every bounded $P-$convergent sequence into a $P-$convergent sequence with the same $P-$limit. The Robison- Hamilton conditions state that a four dimensional matrix $A = (a_{j,k,m,n})$ is $RH-$regular if and only if

(i) $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each $(m,n) \in \mathbb{N}^2$,

(ii) $P - \lim_{j,k} \sum_{m,n=1,1}^{\infty,\infty} a_{j,k,m,n} = 1$,

(iii) $P - \lim_{j,k} \sum_{m=1}^{\infty} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,

(iv) $P - \lim_{j,k} \sum_{n=1}^{\infty} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{m,n=1,1}^{\infty,\infty} |a_{j,k,m,n}|$ is $P-$convergent for every $(j,k) \in \mathbb{N}^2$,

(vi) There exits finite positive integers A and B such that $\sum_{m,n>B} |a_{j,k,m,n}| < A$ holds for every $(j,k) \in \mathbb{N}^2$.

Now let $A = (a_{j,k,m,n})$ be a nonnegative $RH-$regular summability matrix, and let $K \subset \mathbb{N}^2$. Then $A-$density of K is given by

$$\delta_A^2(K) := P - \lim_{j,k} \sum_{(m,n) \in K} a_{j,k,m,n}$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $x = (x_{m,n})$ is said to be $A-$statistically convergent to L if, for every

$\varepsilon > 0$,

$$\delta_A^2(\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}) = 0.$$

In this case, we write $st^2_{(A)} - \lim x = L$. Clearly, a $P-$convergent double sequence is $A-$statistically convergent to the same value but its converse it is not always true. Also, note that an $A-$statistically convergent double sequence need not be bounded. For example, consider the double sequence $x = (x_{m,n})$ given by

$$x_{m,n} = \begin{cases} mn, & \text{if } m \text{ and } n \text{ are squares,} \\ 1, & \text{otherwise.} \end{cases}$$

We should note that if we take $A = C(1,1)$,which is double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in ([12], [13]).

By st_A^2, $st_A^{2,0}$, $st_A^2(b)$, $st_A^{2,0}(b)$, c^2, $c^2(b)$, l_2^∞ we denote the set of all A-statistically convergent double sequences, the set of all A-statistically null double sequences, the set of all bounded A-statistically convergent double sequences, the set of all bounded A-statistically null double sequences, the set of all convergent double sequences, the set of all bounded convergent double sequences and the set of all bounded double sequences, respectively. From now on the summability field of matrix A will be denoted by c_A^2, i.e.,

$$c_A^2 = \left\{ x : P - \lim_{j,k} (Ax)_{j,k} \text{ exists} \right\},$$

and $c_A^2(b) := c_A^2 \cap l_2^\infty$.

Let p, q positive real numbers and let $A = (a_{j,k,m,n})$ be a nonnegative RH-regular infinite matrix. Write

$$W(A,p,q) := \left\{ x = (x_{m,n}) : P - \lim_{j,k} \sum_{m,n} a_{j,k,m,n} |x_{m,n} - L|^{pq} = 0 \text{ for some } L \right\};$$

we say that x is strongly A-summable with $p, q > 0$.

Definition 1.1. *Let E and F be two double sequence spaces. A multiplier from E into F is a sequence $u = (u_{m,n})$ such that*

$$ux = (u_{m,n} x_{m,n}) \in F$$

whenever $x = (x_{m,n}) \in E$. The linear space of all such multipliers will be denoted by $m(E,F)$. Bounded multipliers will be denoted by $M(E,F)$. Hence

$$M(E,F) = l_2^\infty \cap m(E,F).$$

If $E = F$, then we write $m(E)$ instead of $m(E,E)$. Hence the inclusion $X \subset Y$ may be interpreted as saying that the sequence $\chi_{\mathbb{N}^2}$ is a multiplier from X to Y.

2. Strong and A-statistical comparisons for double sequences

In this section, we demonstrate equivalent forms of $\chi_{\mathbb{N}^2} \in m\left(W\left(T,p,q\right) \cap l_2^\infty, c_A^2\left(b\right)\right)$ that compares bounded strong summability field of the nonnegative RH-regular summability matrices A and T. Also we will show that these characterize the A-statistical comparisons for both bounded as well as arbitrary double sequences.

Theorem 2.1. *Let $A = (a_{j,k,m,n})$ and $T = (t_{j,k,m,n})$ be nonnegative RH-regular summability matrices. Then the followings are equivalent:*

(i) $\chi_{\mathbb{N}^2} \in m\left(W\left(T,p,q\right) \cap l_2^\infty, c_A^2\left(b\right)\right)$,
(ii) $W\left(T,p,q\right) \cap l_2^\infty \subseteq c_A^2\left(b\right)$,
(iii) $A \in \left(W\left(T,p,q\right) \cap l_2^\infty, c^2\right)$,
(iv) For any subset $K \subseteq \mathbb{N}^2$, $\delta_T^2\left(K\right) = 0$ implies that $\delta_A^2\left(K\right) = 0$,
(v) $A \in \left(W\left(T,p,q\right) \cap l_2^\infty, c^2\right)$ and A preserves the strong limits of T.

Proof. It is obvious that the first three parts are equivalent. To show that (iii) implies (iv), suppose that (iii) holds. Assume the contrary and let K be a subset of nonnegative integers with $\delta_T^2\left(K\right) = 0$ but

$$\limsup_{j,k} \sum_{(m,n)\in K} a_{j,k,m,n} > 0. \tag{2.1}$$

So, K must be an infinitive set since A is RH-regular and $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each $(m,n) \in \mathbb{N}^2$. (Since $\delta_T^2\left(K\right) = 0$, and T is RH-regular, it must be that $\mathbb{N}\times\mathbb{N} - K$ must also be infinitive). Now take a sequence x which is the indicator of the set K . Note that for any $p,q > 0$, we have

$$\begin{aligned} P - \lim_{j,k} \sum_{m,n} \left|t_{j,k,m,n}\right| \left|x_{m,n} - 0\right|^{pq} &= P - \lim_{j,k} \sum_{m,n} t_{j,k,m,n} x_{m,n} \\ &= P - \lim_{j,k} \sum_{(m,n)\in K} t_{j,k,m,n} \\ &= \delta_T^2\left(K\right) = 0. \end{aligned}$$

Hence, $x \in W\left(T,p,q\right) \cap l_2^\infty$. By $A \in \left(W\left(T,p,q\right) \cap l_2^\infty, c^2\right)$, it must be that $\left(Ax\right)_{j,k}$ is convergent. Combining this with (2.1) we obtain that the density $\delta_A^2\left(K\right)$ exists and so $P - \lim_{j,k} \left(Ax\right)_{j,k} = \delta_A^2\left(K\right) > 0$. Consider the matrix D that keeps all the columns of A whose positions correspond with the set K and fills the rest of the columns with zero matrices. Because of $P - \lim_{j,k} \left(Dx\right)_{j,k} = P - \lim_{j,k} \left(Ax\right)_{j,k} > 0$, a straight forward extension of an argument of Maddox provides a contradiction. Suppose now (iv) holds, and let $x \in W\left(T,p,q\right) \cap l_2^\infty$, so that

$$P - \lim_{j,k} \sum_{m,n} t_{j,k,m,n} \left|x_{m,n} - L\right|^{pq} = 0,$$

for some number L. So x is T-statistically convergent. Then for any $\varepsilon > 0$, define the set $K = \left\{(m,n) : \left|x_{m,n} - L\right| > \varepsilon\right\}$. And we have $\delta_T^2\left(K\right) = 0$. Then by assumption, it must be that $\delta_A^2\left(K\right) = 0$. Since x is bounded, let $\left|x_{m,n}\right| \leq C$ for all m,n. So, for any

$p, q > 0$, we have

$$\begin{aligned}\sum_{m,n} a_{j,k,m,n} \left|x_{m,n} - L\right|^{pq} &= \sum_{(m,n)\in K} a_{j,k,m,n} \left|x_{m,n} - L\right|^{pq} + \\ &\quad \sum_{(m,n)\in K^c} a_{j,k,m,n} \left|x_{m,n} - L\right|^{pq} \\ &\leq (2C)^{pq} \sum_{(m,n)\in K} a_{j,k,m,n} + \varepsilon^{pq} \sum_{(m,n)\in K^c} a_{j,k,m,n} \\ &\leq (2C)^{pq} \sum_{(m,n)\in K} a_{j,k,m,n} + \varepsilon^{pq} \sum_{(m,n)\in K^c} a_{j,k,m,n}.\end{aligned}$$

Letting $j, k \to \infty$, we obtain

$$P - \lim_{j,k} \sum_{(m,n)} a_{j,k,m,n} \left|x_{m,n} - L\right|^{pq} = 0.$$

So that, $(Ax)_{j,k} \to L$ and A preserves the strong limit of T , which gives (v). Observe that (v) trivially implies(iii) . □

The following proposition collects the last result's various equivalent forms. For this purpose we introduce the notation

$$W^L(T, p, q) := \left\{ x : P - \lim_{j,k} \sum_{m,n} t_{j,k,m,n} \left|x_{m,n} - L\right|^{pq} = 0 \right\}.$$

Proposition 2.2. *Let $A = (a_{j,k,m,n})$ and $T = (t_{j,k,m,n})$ be nonnegative RH-regular summability matrices. The following statements are equivalent:*

(i) $st_T^2(b) \subseteq st_A^2(b)$,

(ii) $W(T, p, q) \cap l_2^\infty \subseteq W(A, s, t) \cap l_2^\infty$ *for some* $p, q, s, t > 0$,

(iii) $A \in \left(W(T, p, q) \cap l_2^\infty, c^2\right)$ *and A preserves the strong limits of T. That is,* $W^L(T, p, q) \cap l_2^\infty \subseteq W^L(A, s, t) \cap l_2^\infty$ *for every* L,

(iv) *For any subset* $K \subseteq \mathbb{N}^2$, $\delta_T^2(K) = 0$ *implies that* $\delta_A^2(K) = 0$,

(v) $st_T^{2,0}(b) \subseteq st_A^{2,0}(b)$,

(vi) $st_T^2(b) \subseteq st_A^2(b)$ *and A preserves the T-statistical limits,*

(vii) $W^L(T, p, q) \cap l_2^\infty \subseteq W^L(A, s, t) \cap l_2^\infty$ *for some* $p, q, s, t > 0$ *and some real number* L,

$(viii)$ $W(T, p, q) \cap l_2^\infty \subseteq c_A^2(b)$ *for some* $p, q > 0$,

(ix) $st_T^2 \subseteq st_A^2$ *and A preserves the T-statistical limits,*

(x) $st_T^2 \subseteq st_A^2$.

Proof. At fist we give the following notation:

$$st_T^L(b) := \{x \in l_2^\infty : x \text{ is } T - \text{statistically convergent to } L\}.$$

Note that

$$st_T^L(b) = W^L(T, p, q) \cap l_2^\infty$$

for any $p, q > 0$. Because of this, taking union over all L gives that (i) and (ii) are equivalent. By theorem, we know that (iii) and (iv) are equivalent. Taking union over

L shows that (iii) implies (ii). To show that (ii) implies (iii), clearly (ii) implies that $W(T,p,q)\cap l_2^\infty \subseteq c_A^2(b)$. Hence, $A\in\left(W(T,p,q)\cap l_2^\infty, c^2\right)$. Therefore, by theorem, (iv) holds. Therefore, (iii) holds. Also theorem implies that (iv) and $(viii)$ are equivalent. While (vii) holds some L, if $x\in W^M(T,p,q)\cap l_2^\infty$ then define a new squence $y_{m,n}=x_{m,n}-M+L$. Since $y\in W^L(T,p,q)\cap l_2^\infty$, we have $y\in W^L(A,s,t)\cap l_2^\infty$. This implies that

$$\sum_{m,n} a_{j,k,m,n}\left|x_{m,n}-M\right|^{st}=\sum_{m,n} a_{j,k,m,n}\left|y_{m,n}-L\right|^{pq}\to 0.$$

So that, $y\in W^M(A,s,t)\cap l_2^\infty$. That is,

$$W^M(T,p,q)\cap l_2^\infty\subseteq W^L(A,s,t)\cap l_2^\infty$$

for every M. If supremum over all M takes then (ii) holds. Now (ii) implies (iii) and clearly (iii) implies (vii). Hence (i) and (iii) together imply (vi). Trivially (vi) implies (i) .Also, (vi) implies (v). Conversely (v) implies (vii) with $L=0$. Hence, (i) through $(viii)$ are all equivalent. So far all arguments were for bounded sequences. Now (ix) implies (x), and (x) implies (i). To show that (i) implies (ix), let $x\in st_T^2$ with T-statistical limit L. For $\varepsilon>0$, define $h_{m,n}=0$ if $|x_{m,n}-L|<\varepsilon$ and $h_{m,n}=1$ otherwise. Hence, any such $h\in st_T^{2,0}(b)\subseteq st_A^{2,0}(b)$ by (v). This implies that $x\in st_A^2$ with L being the A-statistical limit, the proof is complete. □

3. Multipliers

In this section, we introduce multipliers on above some different spaces. Firstly, we give some notations.

Definition 3.1. ([5]) *Let $A=(a_{j,k,m,n})$ be a non-negative RH-regular summability matrix and let $(\alpha_{m,n})$ be a positive non-increasing double sequence. A double sequence $x=(x_{m,n})$ is A-statistically convergent to a number L with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon>0$,*

$$P-\lim_{j,k\to\infty}\frac{1}{\alpha_{j,k}}\sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n}=0,$$

where

$$K(\varepsilon):=\left\{(m,n)\in\mathbb{N}^2:|x_{m,n}-L|\geq\varepsilon\right\}.$$

In this case, we write

$$x_{m,n}-L=st_A^2-o(\alpha_{m,n})\quad as\quad m,n\to\infty.$$

Definition 3.2. ([5]) *Let $A=(a_{j,k,m,n})$ and $(\alpha_{m,n})$ be the same as in Definition* 3.1*. Then, a double sequence $x=(x_{m,n})$ is A-statistically bounded with the rate of $O(\alpha_{m,n})$ if for every $\varepsilon>0$,*

$$\sup_{j,k}\frac{1}{\alpha_{j,k}}\sum_{(m,n)\in L(\varepsilon)} a_{j,k,m,n}<\infty,$$

where

$$L(\varepsilon):=\left\{(m,n)\in\mathbb{N}^2:|x_{m,n}|\geq\varepsilon\ \right\}.$$

In this case, we write

$$x_{m,n} = st_A^2 - O(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

Now, we define the subspaces of A-statistically convergent double sequences as follows:

$$\begin{aligned}
st_{A,a}^2 &: = \left\{x : x_{m,n} - L = st_A^2 - o\left(\alpha_{m,n}\right), \text{ as } m,n \to \infty, \text{ for some } L\right\}, \\
st_{A,O(a)}^2 &: = \left\{x : x_{m,n} - L = st_A^2 - O\left(\alpha_{m,n}\right), \text{ as } m,n \to \infty, \text{ for some } L\right\}, \\
st_{A,a}^{2,0} &: = \left\{x : x_{m,n} = st_A^2 - o\left(\alpha_{m,n}\right), \text{ as } m,n \to \infty\right\}, \\
st_{A,O(a)}^{2,0} &: = \left\{x : x_{m,n} = st_A^2 - O\left(\alpha_{m,n}\right), \text{ as } m,n \to \infty\right\}, \\
st_{A,a}^2\left(b\right) &: = st_{A,a}^2 \cap l_2^\infty, \\
st_{A,O(a)}^2\left(b\right) &: = st_{A,O(a)}^2 \cap l_2^\infty, \\
st_{A,a}^{2,0}\left(b\right) &: = st_{A,a}^{2,0} \cap l_2^\infty, \\
st_{A,O(a)}^{2,0}\left(b\right) &: = st_{A,O(a)}^{2,0} \cap l_2^\infty.
\end{aligned}$$

For each $Z \subset \mathbb{N}^2$, we let c_Z^2 denote the set of double sequences which convergence along Z and $c_Z^2\ (b)$ bounded members of c_Z^2. Note that c_Z^2 is the convergence domain of a nonnegative $RH-$regular summability method. It is also easy to verify that $m\left(c_Z^2\right) = c_Z^2$; $M\left(c_Z^2\right) = c_Z^2\left(b\right)$, and $st_A^2\left(b\right) = \cup\left\{c_Z^2\left(b\right) : \delta_A^2\left(Z\right) = 1\right\}$.

Theorem 3.3. $m\left(st_{A,a}^2\left(b\right)\right) = st_{A,a}^2\left(b\right)$, *and* $m\left(st_{A,O(a)}^2\left(b\right)\right) = st_{A,O(a)}^2\left(b\right)$.

Proof. Let $u \in m\left(st_{A,a}^2\left(b\right)\right)$. Then $ux \in st_{A,a}^2\left(b\right)$ for all $x \in st_{A,a}^2\left(b\right)$. Especially, $x = \chi_{\mathbb{N}^2} \in st_{A,a}^2\left(b\right)$, hence $u \in st_{A,a}^2\left(b\right)$, which shows $m\left(st_{A,a}^2\left(b\right)\right) \subset st_{A,a}^2\left(b\right)$. Conversely, suppose that $u \in st_{A,a}^2\left(b\right)$ and take $x \in st_{A,a}^2\left(b\right)$. Then, by the discussion preceding Section 2 we get $ux \in st_{A,a}^2\left(b\right)$, by this $u \in m\left(st_{A,a}^2\left(b\right)\right)$, i.e., $st_{A,a}^2\left(b\right) \subset m\left(st_{A,a}^2\left(b\right)\right)$. The same argument works for the second part of the theorem. □

One may now expect that $m\left(st_{A,a}^{2,0}\left(b\right)\right) = st_{A,a}^{2,0}\left(b\right)$. However , as the next example shows, it is not the case.

Example 3.4. Take $\alpha = \chi_{\mathbb{N}^2}$ and $A = C\left(1,1\right)$. Then $st_{A,a}^{2,0}\left(b\right) = st^{2,0}\left(b\right)$, the set of all bounded statistically null double sequences. Now define a double bounded sequence $u = \left(u_{m,n}\right)$ by

$$u_{m,n} = \begin{cases} 1 & , \quad m,n \text{ are odds,} \\ -1 & , \quad m,n \text{ are evens,} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then $ux \in st^{2,0}\left(b\right)$ for every $x \in st^{2,0}\left(b\right)$. Hence $u \in m\left(st^{2,0}\left(b\right)\right)$, but $u \notin st^{2,0}\left(b\right)$. So, the next result characterizes the multipliers from $st_{A,a}^{2,0}\left(b\right)$ into itself.

Theorem 3.5. $m\left(st_{A,a}^{2,0}\left(b\right)\right) = l_2^\infty$.

Proof. If $u \in m\left(st_{A,a}^{2,0}(b)\right)$, then $ux \in st_{A,a}^{2,0}(b) \subset l_2^\infty$ for all $x \in st_{A,a}^{2,0}(b)$.To show that this implies that $u \in l_2^\infty$, first observe that $c_0^2 \subseteqq st_{A,a}^{2,0}(b)$; and from this case $u \in m\left(st_{A,a}^{2,0}(b)\right)$ if and only if the matrix $Tu = (t_{j,k,m,n}) = \left(u_{j,k}\delta_{(m,n)}^{(j,k)}\right)$ maps $st_{A,a}^{2,0}(b)$ into itself, where $\delta_{(m,n)}^{(j,k)}$ is the Kronecker delta. Hence, it also maps c_0^2 into l_2^∞, which implies that $\sup\limits_{j,k}\sum\limits_{m,n}|t_{j,k,m,n}| = \sup\limits_{j,k}\sum\limits_{m,n}\left|u_{j,k}\delta_{(m,n)}^{(j,k)}\right| = \sup\limits_{j,k}|u_{j,k}| < \infty$. Conversely, suppose $u \in l_2^\infty$ and let $z \in st_{A,a}^{2,0}(b)$, then

$$\{(m,n) : |u_{m,n}z_{m,n}| \geq \varepsilon\} \subseteqq \left\{(m,n) : |z_{m,n}| \geq \frac{\varepsilon}{1+\|u\|_{2,\infty}}\right\}.$$

Thus, since $z_{m,n} = st_A^2 - o(a_{m,n})$, we obtain $u_{m,n}x_{m,n} = st_A^2 - o(a_{m,n})$. Also it is clear that uz is bounded, and hence $l_2^\infty \subseteqq m\left(st_{A,a}^{2,0}(b)\right)$, and the proof is complete. □

Theorem 3.6. $m\left(st_A^2(b)\right) = \cup\left\{M\left(c_Z^2\right) : \delta_A^2(Z) = 1\right\}$.

Proof. $m\left(st_A^2(b)\right) = st_A^2(b) = \cup\left\{c_Z^2(b) : \delta_A^2(Z) = 1\right\} = \cup\left\{M\left(c_Z^2\right) : \delta_A^2(Z) = 1\right\}$.

Before proving the following theorem, we observe that, in general,

$$c_0^2 \subseteq m\left(st_A^2(b), c^2\right) \subseteq c^2.$$

The first inclusion follows from noting $ux \in c_0^2 \subseteq st_A^2(b)$ for any $u \in c_0^2$ and $x \in l_2^\infty$.The second inclusion follows from $\chi_{\mathbb{N}^2} \in st_A^2(b)$. Note that if $st_A^2(b) = c^2$, then $m\left(st_A^2(b), c^2\right) = c^2$. The next theorem shows that this the only situation for which $m\left(st_A^2(b), c^2\right) = c^2$. □

Theorem 3.7. $m\left(st_A^2(b), c^2\right) = c_0^2$ *and* $m\left(c^2, st_A^2(b)\right) = st_A^2(b)$.

Proof. First we show that $m\left(st_A^2(b), c^2\right) = c_0^2$. All we need to establish is that if $u \in c^2$ and $\lim u = l \neq 0$, then $u \notin m\left(st_A^2(b), c^2\right)$. Let $z \in st_A^2(b)$, $z \notin c^2$, and, without loss of generality, suppose z is $A-$statistically convergent to 1. Then there is an $\varepsilon > 0$ such that $K = \{(m,n) : |z_{m,n} - 1| \geq \varepsilon\}$ is an infinite set. Note that $\delta_A^2(K) = 0$.

Define x by $x_{m,n} = \chi_{K^c}(m,n)$ and observe that x is convergent in $A-$density to 1, hence $x \in st_A^2(b)$. Also note xu converges to $l \neq 0$ along K^c and to 0 along K, hence $xu \notin c^2$ and thus $u \notin m\left(st_A^2(b), c^2\right)$.

Now we show that $m\left(c^2, st_A^2(b)\right) = st_A^2(b)$. As $\chi_{\mathbb{N}^2} \in c^2$, $m\left(c^2, st_A^2(b)\right) \subseteq st_A^2(b)$. The reserve inclusion follows from noting that if $u \in st_A^2(b)$ and $x \in c^2 \subseteq st_A^2(b)$, then ux is $A-$statistically convergent. □

Theorem 3.8. (i) $m\left(c_0^2, st_A^{2,0}(b)\right) = l_2^\infty$,

(ii) $m\left(st_A^{2,0}(b), c_0^2\right) = \left\{u \in l_2^\infty : u\chi_E \in c_0^2 \text{ for all } E \text{ such that } \delta_A^2(E) = 0\right\}$.

Proof. The proof of (i) follows from noting

$$l_2^\infty = m\left(c_0^2, c_0^2\right) \subseteq m\left(c_0^2, st_A^{2,0}(b)\right) \subseteq l_2^\infty.$$

Next we prove (ii). First note that if $\delta_A^2(E) = 0$, then $\chi_E \in st_A^{2,0}(b)$ and thus, if $u \in m\left(st_A^{2,0}(b), c_0^2\right)$, $u\chi_E \in c_0^2$, or u goes 0 along E.

Hence,
$m\left(st_A^{2,0}(b), c_0^2\right) \subseteq \left\{u \in l_2^\infty : u\chi_E \in c_0^2 \text{ for all } E \text{ such that } \delta_A^2(E) = 0\right\}.$

Now suppose that u is a bounded sequence such that u tends to 0 along every $A-$null set and suppose x is bounded and convergent to 0 in $A-$density. Then there is an $K \subseteq \mathbb{N}^2$ such that, $x\chi_{K^c} \in c_0^2$, $\delta_A^2(K) = 0$. As $ux = ux\chi_{K^c} + ux\chi_K$ and both terms of the right hand side are null double sequences , $ux \in c_0^2$.

Now suppose $x \in st_A^{2,0}(b)$. Then there is a sequence $\left(x^{j,k}\right)$, each $x^{j,k}$ convergent in $A-$density to 0, such that $x^{j,k}$ converges to x in l_2^∞. Now $ux^{j,k} \to ux$ in l_2^∞, and as $ux^{j,k} \in c_0^2$ for all j,k and c_0^2 is closed, $ux \in c_0^2$. Thus

$$\left\{u \in l_2^\infty : u\chi_E \in c_0^2 \text{ for all } E \text{ such that } \delta_A^2(E) = 0\right\} \subseteq m\left(st_A^{2,0}(b), c_0^2\right)$$

and hence the theorem.

Note that $m\left(st_A^{2,0}(b), c_0^2\right)$ can be a variety of spaces. In particular $m\left(c_0^2, c_0^2\right) = l_2^\infty$ and, if $c_{0,Z}^2$ denotes the sequences that converge to 0 along Z, then

$$m\left(c_{0,Z}^2(b), c_0^2\right) = c_{0,Z}^2(b). \qquad \square$$

4. A Steinhaus-type result

The well known Theorem of Steinhaus knows that if T is a regular matrix then $\chi_{\mathbb{N}}$ is not a multipler from l^∞ into $c_T := \{x : Tx \in c\}$. It may be true if regularity condition on A is replaced by coregularity. Maddox [10] proved that $\chi_{\mathbb{N}}$ is not a multipler from l^∞ into $f_T := \{x : Tx \in f\}$ either, where f denotes the space of all almost convergent sequences [9]. It is known that almost convergence and statistical convergence are not compatible summability methods [11]. So there seems some hope that $\chi_{\mathbb{N}}$ might be a multiplier from l^∞ into $(st_A)_T := \{x : Tx \in st_A\}$. However, it has been shown in [1] that it is not the case. Of course $\chi_{\mathbb{N}}$ is not a multipler from l^∞ into the space $(st_{A,a})_T := \{x : Tx \in st_{A,a}\}$ either. Furthermore Demirci, Khan and Orhan gave an alternate proof of it. What we offer in this study is to prove the theorem which is characterized $\chi_{\mathbb{N}^2}$ is not a multiplier from l_2^∞ into $\left(st_{A,a}^2\right)_T$.

Definition 4.1. *Let $A = (a_{j,k,m,n})$ be a non-negative RH-regular summability matrix. The characteristic χ defined by*

$$\chi(A) = \lim_{j,k} \sum_{m,n} a_{j,k,m,n} - \sum_{m,n} \lim_{j,k} a_{j,k,m,n}.$$

If $\chi(A) = 0$ then we say A is co-null, if $\chi(A) \neq 0$ then we say A is co-regular.

$$\begin{aligned} K_0^2 &= \{A : \chi(A) = 0\}, \\ K^2 &= \{A : \chi(A) \neq 0\}. \end{aligned}$$

Now, we give the following lemma before the proof of theorem;

Lemma 4.2. ([2]) $A \in \left(l_2^{\infty}, c^2(b)\right)$ *if and only if the condition* $\sum_{j,k} |a_{j,k,m,n}| \leq C < \infty$ *holds and*

(i) $\lim_{j,k} a_{j,k,m,n} = \alpha_{m,n}$ *for each* $(m,n) \in \mathbb{N}^2$,

(ii) $\lim_{j,k} \sum_{n=1}^{k} |a_{j,k,m,n}|$ *exists for each* $m \in \mathbb{N}$ *and*

(iii) $\lim_{j,k} \sum_{m=1}^{j} |a_{j,k,m,n}|$ *exists for each* $n \in \mathbb{N}$,

(iv) $\sum_{j,k} |a_{j,k,m,n}|$ *converges,*

(v) $\lim_{j,k} \sum_{m} \sum_{n} |a_{j,k,m,n} - \alpha_{m,n}| = 0$.

Theorem 4.3. *Let A and B be conservative matrices and suppose that $A \in (l_2^{\infty}, c_B^2(b))$. Then*

(i) $BA \in K_0^2$,

(ii) *If* $B \in K^2$ *then* $A \in K_0^2$.

Proof. (i) Because of $A \in \left(l_2^{\infty}, c_B^2(b)\right)$ we have $B(Ax) \in c^2(b)$ for all $x \in l_2^{\infty}$. Now A and B conservative implies $B(Ax) = (BA)x$ for all $x \in l_2^{\infty}$, therefore $(BA)x \in c^2(b)$ for all $x \in l_2^{\infty}$, so that $BA \in \left(l_2^{\infty}, c^2(b)\right) \subset K_0^2$ from Lemma 4.2.

(ii) By (i) and the fact that χ is a scalar homomorphism we have $\chi(B)\chi(A) = 0$, whence the result. $\square$

Theorem 4.4. *Let A be a nonnegative RH-regular summability method. If T is a co-regular summabilty matrix, then $\chi_{\mathbb{N}^2}$ is not a multiplier from l_2^{∞} into $\left(st_{A,a}^2\right)_T := \left\{x : Tx \in st_{A,a}^2\right\}$.*

Proof. Suppose $\chi_{\mathbb{N}^2} \in m\left(l_2^{\infty}, \left(st_{A,a}^2\right)_T\right)$, then $l_2^{\infty} \subset \left(st_{A,a}^2\right)_T$. Hence $Tx \in l_2^{\infty}$ and $Tx \in st_{A,a}^2 \subset st_A^2$ for all $x \in l_2^{\infty}$. Then we have $Tx \in c_A^2$. So $T : l_2^{\infty} \to c_A^2$. Since A is RH-regular, it follows from Theorem 4.3 that T is co-null double matrix which is a contradiction. $\square$

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