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A new four-step iteration scheme for generalized α-nonexpansive multivalued mappings in Banach spaces

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Abstract

In this paper, we introduce a new four-step iterative process for approximating fixed points of generalized α-nonexpansive multivalued mappings in uniformly convex Banach spaces. The proposed iteration unifies and extends several classical schemes, including the Mann, Ishikawa and Noor processes, while preserving a simple computational structure. Strong and weak convergence theorems are established under mild conditions such as the Opial property and the demiclosedness of $I - P_T$. Unlike contraction-based approaches, the stability of the process is analyzed directly within the generalized α-nonexpansive framework, demonstrating robustness under small perturbations. A numerical experiment in $\mathbb{R}^2$ illustrates the superior performance of the proposed algorithm, achieving faster residual decay and improved asymptotic regularity compared with existing methods. Both theoretical and numerical findings confirm the efficiency and stability of the scheme, suggesting promising applications in nonlinear analysis and optimization problems.

Keywords: Four-step iteration; Multivalued mapping; Fixed point

1 Introduction

Fixed point theory has long been a central topic in mathematical analysis, with wide-ranging applications in optimization, differential equations and game theory, and its iterative approximation methods have been extensively studied in the literature [4]. One of the foundational results in this area is Banach's contraction principle [3], proposed in 1922, which guarantees the existence and uniqueness of fixed points for contractive mappings in complete metric spaces. This result inspired the development of iterative methods to approximate fixed points and continues to influence nonlinear analysis.

In 1953, Mann [12] introduced an iterative scheme for approximating fixed points of nonexpansive mappings, laying the groundwork for subsequent developments. This idea was later extended to a two-step process by Ishikawa [10]. The geometric foundations of fixed point theory were further strengthened by the classical results of Browder, Göhde, and Kirk [5], which ensured the existence of fixed points for nonexpansive mappings defined on closed, bounded, and convex subsets of uniformly convex Banach spaces. A systematic exposition of these results and their extensions can be found in Goebel and Reich [7], which remains a standard reference in the field.

To address broader classes of mappings, Suzuki [25] introduced condition (C), which led to the definition of generalized nonexpansive mappings. This concept was later expanded by Aoyama and Kohsaka [2], who defined α-nonexpansive mappings, and by Pant and Shukla [17], who proposed the more general class of generalized α-nonexpansive mappings. The fixed point theory of multivalued mappings evolved in parallel with its single-valued counterpart. Markin and Nadler [13] established the foundation by introducing the Hausdorff metric for set-valued mappings, enabling many classical fixed point results to be extended to the multivalued setting. Abkar and Eslamian [1] adapted Suzuki's nonexpansive framework to multivalued operators, while Iqbal et al. [9] introduced the notion of multivalued α-nonexpansive mappings, further broadening the class of admissible mappings. This line of work was expanded in [8], where several well-known mapping classes were unified under a generalized formulation. Recent developments continue this trend, including accelerated iterative processes [6, 19] and convergence results in broader geometric settings [16, 22].

Simultaneously, a wide variety of iterative methods have been developed to approximate fixed points of nonexpansive and generalized nonexpansive mappings. The classical Mann iteration [12] was followed by Ishikawa's two-step method [10] and Noor's three-step process [14], providing enhanced flexibility in algorithmic design. Further variants such as the SP-iteration [18], the S*-iteration [11], and the Normal-S process [21] have been shown to improve convergence under more general conditions. Recent progress in accelerated iteration schemes can be found in [6], where improved convergence rates are reported. Several of these methods have also been adapted to multivalued settings in later research.

Building upon these foundations, we revisit classical iterative frameworks to address their analytical limitations. Traditional one–step and two–step processes, although extensively studied, often exhibit slow convergence or require strong regularity assumptions to ensure stability. Three–step procedures provide a partial enhancement in the rate of convergence; however, the reduction of the approximation error may still be insufficient when applied to generalized α–nonexpansive multivalued mappings.

To overcome these difficulties, we define a new four–step iterative process that incorporates an additional intermediate projection stage through the operator P_T. This supplementary stage reduces the approximation error at each iteration while preserving the stability of the generated sequence. The proposed process unifies and extends the classical Mann, Ishikawa, and Noor constructions, retains a simple computational structure, and achieves an improved rate of convergence under suitable conditions. Both the convergence and stability properties of the process are analyzed, and its performance is illustrated by a numerical example.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and induced metric d. Let $K \subset E$ be a nonempty, closed and convex subset. A subset K is said to be *proximinal* if for every $x \in E$ there exists $y \in K$ such that

$$d(x, K) = \|x - y\|.$$

We denote by $\rho(K)$, $\rho_{cb}(K)$, $\rho_{cp}(K)$, and $\rho_{px}(K)$ the families of nonempty subsets, closed and bounded subsets, compact subsets, and proximinal bounded subsets of K, respectively. In particular, $CB(K) = \rho_{cb}(K)$.

The Hausdorff–Pompeiu metric H on $\rho_{cb}(K)$ induced by d is defined by

$$H(F,G) = \max\left\{\sup_{x\in F} d(x,G),\ \sup_{y\in G} d(y,F)\right\}, \qquad F,G \in \rho_{cb}(K).$$

A point $p \in K$ is called a fixed point of a multivalued mapping $T : K \to CB(K)$ if $p \in T(p)$; the set of fixed points is

$$F(T) = \{p \in K : p \in T(p)\}.$$

The mapping T is said to be *multivalued nonexpansive* if

$$H(Tz,Tw) \le \|z - w\|, \qquad \forall z,w \in K,$$

and *multivalued quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$H(Tz,Tq) \le \|z - q\|, \qquad \forall z,q \in F(T).$$

For each $w \in K$, the metric projection onto $T(w)$ is defined by

$$P_T(w) = \{v \in T(w) : \|w - v\| = d(w,T(w))\}.$$

A Banach space E is *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $x,y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \varepsilon$,

$$\left\|\frac{x+y}{2}\right\| \le 1 - \delta.$$

For a bounded sequence $\{z_n\} \subset E$, the *asymptotic radius* and *asymptotic center* relative to K are defined by

$$r(K,\{z_n\}) = \inf_{z\in K} \limsup_{n\to\infty} \|z_n - z\|,$$

$$A(K,\{z_n\}) = \{z \in K : \limsup_{n\to\infty} \|z_n - z\| = r(K,\{z_n\})\}.$$

If E is uniformly convex, then $A(K,\{z_n\})$ is a singleton.

Definition 2.1 [15] A Banach space E is said to satisfy the *Opial condition* if for every sequence $\{z_n\} \subset E$ such that $z_n \rightharpoonup z$ (weak convergence), and for every $w \in E$ with $w \neq z$, the following strict inequality holds:

$$\limsup_{n\to\infty} \|z_n - z\| < \limsup_{n\to\infty} \|z_n - w\|.$$

Lemma 2.2 [23] *Let E be a uniformly convex Banach space and let sequences $\{x_n\}, \{y_n\} \subseteq E$ and $\{k_n\} \subset [a,b] \subset (0,1)$ be such that*: $\limsup_{n\to\infty} \|x_n\| \le l$, $\limsup_{n\to\infty} \|y_n\| \le l$, $\limsup_{n\to\infty} \|(1-k_n)x_n + k_n y_n\| = l$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3 [9] *Let $T : K \longrightarrow \rho_{px}(K)$ be a multivalued mapping, and define metric projection*: $P_T(w) = \{v \in Tw : d(w, Tw) = \|w - v\|\}$. *Then, the following are equivalent*:

i) $w \in F(T)$;
ii) $P_T(w) = \{w\}$;
iii) $w \in F(P_T(w))$.
Furthermore, $F(T) = F(P_T)$.

Definition 2.4 A Multivalued mapping T is said to be demiclosed at a point $y \in K$ if for any sequence $\{x_n\}$ that converges weakly to x, and for any sequence $\{y_n\} \subseteq Tx_n$ converging strongly to y, it necessarily holds that $y \in Tx$.

Definition 2.5 [24] A multivalued mapping $T : K \to \rho(K)$ is said to satisfy condition (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$, $f(l) > 0$ for all $l > 0$, and $d(z, Tz) \geq f(d(z, F(T)))$ for all $z \in K$.

Lemma 2.6 [26] *Let $\{b_n\}$and $\{\varepsilon_n\}$ be sequences of nonnegative real numbers satisfying the recursive inequality $b_{n+1} \leq (1 - \phi_n) b_n + \varepsilon_n$, where $\phi_n \in (0, 1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$ and* $\lim_{n\to\infty} \frac{\varepsilon_n}{\phi_n} = 0$. *Then the sequence $\{b_n\}$ converges to zero, that is,* $\lim_{n\to\infty} b_n = 0$.

Definition 2.7 [9] A multivalued mapping $T : K \to CB(K)$ is called a generalized α-nonexpansive multivalued mapping if there exists a constant $\alpha \in [0, 1)$ such that for all $z, w \in K$, the following inequality holds:

$$\frac{1}{2} d(z, Tz) \leq \|z - w\| \Longrightarrow H(Tz, Tw) \leq \alpha d(z, Tw) + \alpha d(w, Tz) + (1 - 2\alpha) \|z - w\|. \quad (1)$$

Lemma 2.8 [20] *Let K be a nonempty, closed subset of a Banach space E, and let $T : K \to CB(K)$ be a mapping satisfying condition (1) with $F(T) \neq \emptyset$. If $Tz = \{z\}$ for every $z \in F(T)$, then $F(T)$ is both closed and convex.*

Proposition 2.9 [20] *Let $T : K \to CB(K)$ be a mapping that satisfies condition (1). Then, for any $w, t \in K$ the following statements hold*:

1. $H(Tw, Tz) \leq \|w - z\|$ for all $z \in Tw$.
2. Either $\frac{1}{2} d(w, Tw) \leq \|w - t\|$, or $\frac{1}{2} d(z, Tz) \leq \|z - t\|$ for all $z \in Tw$.
3. Either $H(Tw, Tt) \leq \alpha d(w, Tt) + \alpha d(t, Tw) + (1 - 2\alpha) \|w - t\|$, or $H(Tz, Ty) \leq \alpha d(z, Ty) + \alpha d(y, Tz) + (1 - 2\alpha) \|z - y\|$ for all $z \in Tx$.

Lemma 2.10 [9] *Let $T : K \to CB(K)$ be a mapping that satisfies condition (1). Then, for any $w, t \in K$ and $z \in Tw$ the following inequality holds*:

$$d(w, Tt) \leq \left(\frac{3 + \alpha}{1 - \alpha}\right) d(w, Tw) + \|w - t\|. \quad (2)$$

3 Convergence theorems in uniformly convex Banach spaces

Motivated by these considerations, we define a new iterative scheme by constructing the sequence $\{x_n\}$ starting from an arbitrary initial point $x_1 = x \in K$, according to the following process:

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, u_n + \alpha_n\, u_n', \\ y_n = (1-\beta_n)\, v_n + \beta_n\, v_n', \\ z_n = (1-\gamma_n)\, w_n + \gamma_n\, w_n', \\ t_n = (1-\delta_n)\, x_n + \delta_n\, x_n', \end{cases} \qquad (n \geq 0) \tag{3}$$

with $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$ and

$$\begin{cases} u_n \in P_T(y_n), & u_n' \in P_T(u_n), \\ v_n \in P_T(z_n), & v_n' \in P_T(v_n), \\ w_n \in P_T(t_n), & w_n' \in P_T(w_n), \\ x_n' \in P_T(x_n). \end{cases}$$

The iterative scheme (3) provides the mathematical framework of the process. To clarify the role of the projection operator and its use within the iteration, we recall the following standard selection rule.

For each $w \in K$, the projection $P_T(w)$ denotes a nearest point in $T(w)$ to w, that is,

$$P_T(w) = \{v \in T(w) : \|w - v\| = \operatorname{dist}(w, T(w))\}.$$

In uniformly convex Banach spaces this point is unique, and therefore the notation $P_T(w)$ is unambiguous.

To make the construction in (3) more transparent, we describe it in expanded form. Starting from an arbitrary $x_1 \in K$, for each $n \geq 1$ we set

$$\begin{cases} x_n' \in P_T(x_n) = \arg\min\limits_{y \in T(x_n)} \|x_n - y\|, \\ t_n = (1-\delta_n)\, x_n + \delta_n\, x_n', \\ w_n \in P_T(t_n) = \arg\min\limits_{y \in T(t_n)} \|t_n - y\|, \\ w_n' \in P_T(w_n) = \arg\min\limits_{y \in T(w_n)} \|w_n - y\|, \\ z_n = (1-\gamma_n)\, w_n + \gamma_n\, w_n', \\ v_n \in P_T(z_n) = \arg\min\limits_{y \in T(z_n)} \|z_n - y\|, \\ v_n' \in P_T(v_n) = \arg\min\limits_{y \in T(v_n)} \|v_n - y\|, \\ y_n = (1-\beta_n)\, v_n + \beta_n\, v_n', \\ u_n \in P_T(y_n) = \arg\min\limits_{y \in T(y_n)} \|y_n - y\|, \\ u_n' \in P_T(u_n) = \arg\min\limits_{y \in T(u_n)} \|u_n - y\|, \\ x_{n+1} = (1-\alpha_n)\, u_n + \alpha_n\, u_n' \end{cases} \qquad (n \geq 0)$$

This formulation shows that each iteration proceeds by alternating between a projection onto the image set of T and a convex combination step, repeating this pattern four times before updating x_{n+1}.

We note that several classical iterative schemes arise as special cases of (3). For instance, when the four–step correction pattern is reduced appropriately, the Mann, Ishikawa, and Noor iterations are recovered in their multivalued forms. The corresponding formulations are summarized as follows.

The multivalued versions of classical iterative schemes

Iteration	Multivalued Version
Mann	$x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n$
Ishikawa	$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n v_n \end{cases}$
Noor	$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n u_n, \\ y_n = (1-\beta_n)x_n + \beta_n w_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n v_n \end{cases}$

Here $u_n \in P_T(x_n)$, $v_n \in P_T(y_n)$, $w_n \in P_T(z_n)$, and $\alpha_n, \beta_n, \gamma_n \in (0,1)$. The proposed four–step process (3) not only includes these well–known schemes as particular cases but also achieves a faster rate of convergence in the setting of uniformly convex Banach spaces. In the subsequent sections, we establish the strong convergence of the sequence $\{x_n\}$ and verify the stability of the method under generalized α−nonexpansiveness.

Lemma 3.1 *Let E be a uniformly convex Banach space, and let K be a nonempty, closed, and convex subset of E. Suppose that $T : K \to \rho_{px}(K)$ is a multivalued mapping with $F(T) \neq \emptyset$, and assume that P_T is a generalized α-nonexpansive mapping. Let the sequence $\{x_n\}$ be generated by the iterative scheme (3). Then, for every point $q \in F(T)$, the limit $\lim_{n\to\infty} \|x_n - q\|$ exists, and in addition, $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$.*

Proof Let $q \in F(T)$. By invoking Lemma 2.3, we have $P_T(q) = \{q\}$ and $F(T) = F(P_T)$. Since P_T meets condition (3), it directly yields

$$\frac{1}{2} d(q, P_T(q)) = 0 = \|x_n - q\|.$$

Moreover, from the inequality

$$\begin{aligned} H(P_T(x_n), P_T(q)) &\leq \alpha d(x_n, P_T(q)) + \alpha d(q, P_T(x_n)) + (1-2\alpha)\|x_n - q\| \\ &\leq \alpha \|x_n - q\| + H(P_T(q), P_T(x_n)) + (1-2\alpha)\|x_n - q\| \\ &\leq \|x_n - q\|. \end{aligned}$$

Similarly, for every $q \in F(T)$, the following relations hold true:

$$\begin{aligned} H\left(P_T\left(y_n\right), P_T(q)\right) &\leq \left\|y_n - q\right\| \\ H(P_T(z_n), P_T(q)) &\leq \|z_n - q\| \\ H(P_T(t_n), P_T(q)) &\leq \|t_n - q\| \end{aligned}$$

$$H\left(P_T\left(v_n\right), P_T\left(q\right)\right) \leq \left\|v_n - q\right\|$$
$$H\left(P_T\left(w_n\right), P_T\left(q\right)\right) \leq \left\|w_n - q\right\|.$$

Based on the iterative structure in (3), we obtain

$$\begin{aligned}
\left\|t_n - q\right\| &\leq \left\|\left(1-\delta_n\right)x_n + \delta_n x_n' - q\right\| \\
&\leq \left(1-\delta_n\right)\left\|x_n - q\right\| + \delta_n\left\|x_n' - q\right\| \\
&\leq \left(1-\delta_n\right)\left\|x_n - q\right\| + \delta_n H\left(P_T\left(x_n\right), P_T\left(q\right)\right) \\
&\leq \left(1-\delta_n\right)\left\|x_n - q\right\| + \delta_n\left\|x_n - q\right\| \\
&= \left\|x_n - q\right\|,
\end{aligned} \tag{4}$$

$$\begin{aligned}
\left\|z_n - q\right\| &\leq \left\|\left(1-\gamma_n\right)w_n + \gamma_n w_n' - q\right\| \\
&\leq \left(1-\gamma_n\right)\left\|w_n - q\right\| + \gamma_n\left\|w_n' - q\right\| \\
&\leq \left(1-\gamma_n\right)d\left(w_n, P_T\left(q\right)\right) + \gamma_n d\left(w_n', P_T\left(q\right)\right) \\
&\leq \left(1-\gamma_n\right)H\left(P_T\left(t_n\right), P_T\left(q\right)\right) + \gamma_n H\left(P_T\left(w_n\right), P_T\left(q\right)\right) \\
&\leq \left(1-\gamma_n\right)\left\|t_n - q\right\| + \gamma_n\left\|w_n - q\right\| \\
&\leq \left(1-\gamma_n\right)\left\|t_n - q\right\| + \gamma_n H\left(P_T\left(t_n\right), P_T\left(q\right)\right) \\
&\leq \left(1-\gamma_n\right)\left\|t_n - q\right\| + \gamma_n\left\|t_n - q\right\| \\
&= \left\|t_n - q\right\| \leq \left\|x_n - q\right\|,
\end{aligned} \tag{5}$$

$$\begin{aligned}
\left\|y_n - q\right\| &\leq \left\|\left(1-\beta_n\right)v_n + \beta_n v_n' - q\right\| \\
&\leq \left(1-\beta_n\right)\left\|v_n - q\right\| + \beta_n\left\|v_n' - q\right\| \\
&\leq \left(1-\beta_n\right)d\left(v_n, P_T\left(q\right)\right) + \beta_n d\left(v_n', P_T\left(q\right)\right) \\
&\leq \left(1-\beta_n\right)H\left(P_T\left(z_n\right), P_T\left(q\right)\right) + \beta_n H\left(P_T\left(v_n\right), P_T\left(q\right)\right) \\
&\leq \left(1-\beta_n\right)\left\|z_n - q\right\| + \beta_n\left\|v_n - q\right\| \\
&\leq \left(1-\beta_n\right)\left\|z_n - q\right\| + \beta_n H\left(P_T\left(z_n\right), P_T\left(q\right)\right) \\
&\leq \left(1-\beta_n\right)\left\|z_n - q\right\| + \beta_n\left\|z_n - q\right\| \\
&= \left\|z_n - q\right\| \leq \left\|x_n - q\right\|.
\end{aligned} \tag{6}$$

Now, combining (4), (5) and (6), it follows that

$$\begin{aligned}
\left\|x_{n+1} - q\right\| &\leq \left\|\left(1-\alpha_n\right)u_n + \alpha_n u_n' - q\right\| \\
&\leq \left(1-\alpha_n\right)\left\|u_n - q\right\| + \alpha_n\left\|u_n' - q\right\| \\
&\leq \left(1-\alpha_n\right)d\left(u_n, P_T\left(q\right)\right) + \alpha_n d\left(u_n', P_T\left(q\right)\right) \\
&\leq \left(1-\alpha_n\right)H\left(P_T\left(y_n\right), P_T\left(q\right)\right) + \alpha_n H\left(P_T\left(u_n\right), P_T\left(q\right)\right) \\
&\leq \left(1-\alpha_n\right)\left\|y_n - q\right\| + \alpha_n\left\|u_n - q\right\| \\
&\leq \left(1-\alpha_n\right)\left\|y_n - q\right\| + \alpha_n H\left(P_T\left(y_n\right), P_T\left(q\right)\right)
\end{aligned} \tag{7}$$

$$\leq (1-\alpha_n)\left\|y_n - q\right\| + \alpha_n \left\|y_n - q\right\|$$
$$= \left\|y_n - q\right\| \leq \left\|x_n - q\right\|.$$

This implies that the sequence $\{\|x_n - q\|\}$ is non-increasing and bounded below. Hence, it converges, and we can write:

$$\lim_{n\to\infty} \|x_n - q\| = c \tag{8}$$

From inequalities (4) and (5), we infer:

$$\limsup_{n\to\infty} \|t_n - q\| \leq \limsup_{n\to\infty} \|x_n - q\| = c. \tag{9}$$

Using (5), (6), (7), we observe the inequality chain:

$$\|x_{n+1} - q\| \leq \left\|y_n - q\right\| \leq \|z_n - q\| \leq \|t_n - q\| \leq \|x_n - q\|.$$

From this chain of inequalities, we can express:

$$\|x_{n+1} - q\| \leq \|t_n - q\|. \tag{10}$$

By combining (10) with $\lim_{n\to\infty} \|x_n - p\| = c$, we get:

$$c \leq \lim_{n\to\infty} \inf \|t_n - q\|. \tag{11}$$

Hence, from (9) and (11), it follows that $\lim_{n\to\infty} \|t_n - q\| = c$. Next, observe the following:

$$\left\|x_n' - q\right\| \leq d\left(x_n', P_T(q)\right) \leq H\left(P_T(x_n), P_T(q)\right) \leq \|x_n - q\|.$$

Taking the upper limit, we obtain:

$$\limsup_{n\to\infty} \left\|x_n' - q\right\| \leq c. \tag{12}$$

Also, it holds that:

$$c = \lim_{n\to\infty} \|t_n - q\| = \lim_{n\to\infty} \left\|(1-\delta_n)(x_n - q) + \delta_n\left(x_n' - q\right)\right\|. \tag{13}$$

Combining (9), (12), (13) and using Lemma 2.2, we arrive at:

$$\lim_{n\to\infty} \left\|x_n - x_n'\right\| = 0.$$

Finally, since $d(x, P_T(x)) = \inf_{z\in P_T(x)} d(x,z)$, it follows that:

$$d(x_n, P_T(x_n)) \leq \left\|x_n - x_n'\right\| \to 0 \quad as \quad n \to \infty. \qquad \square$$

Theorem 3.2 *Let E be a uniformly convex Banach space satisfying the Opial property, and let $K \subset E$ be a nonempty, closed, and convex subset. Assume that $T : K \to \rho_{px}(K)$ is a multivalued mapping with $F(T) \neq \emptyset$, and denote by P_T its metric projection. Suppose* *that P_T is generalized α-nonexpansive for some $\alpha \in [0,1)$. Let $\{x_n\}$ be the sequence generated by the four-step iteration process* (3) *starting from $x_1 \in K$, where the control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$ satisfy*

$$\alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \beta_n, \gamma_n, \delta_n \to 0.$$

If the operator $I - P_T$ is demiclosed at the origin, then the sequence $\{x_n\}$ converges strongly to a fixed point $q \in F(T)$.

Proof Let $F := F(T) = F(P_T)$ by Lemma 2.3, and fix any $q \in F$. Since $P_T(q) = \{q\}$, the generalized α–nonexpansive property of P_T yields the Fejér-type chain of inequalities:

$$\|t_n - q\| \le \|x_n - q\|, \qquad \|z_n - q\| \le \|t_n - q\|, \qquad \|y_n - q\| \le \|z_n - q\|,$$
$$\|x_{n+1} - q\| \le \|y_n - q\|.$$

Hence $\{\|x_n - q\|\}$ is nonincreasing and bounded below, so the sequence $\{x_n\}$ is bounded in E.

From the recursive structure of (2.3) and the convexity of K, together with $\delta_n, \gamma_n, \beta_n, \alpha_n \to 0$, Lemma 2.2 implies that

$$\|x_n - P_T(x_n)\| \to 0, \qquad \text{equivalently,} \qquad d(x_n, P_T(x_n)) \to 0.$$

Since $\{x_n\}$ is bounded in the uniformly convex Banach space E, it has a weakly convergent subsequence. Let $x_{n_j} \rightharpoonup z$. From the above argument we have $x_{n_j} - P_T(x_{n_j}) \to 0$; by the demiclosedness of $I - P_T$ at the origin, it follows that $z = P_T(z)$, and hence $z \in F$.

Fejér monotonicity of $\{x_n\}$ with respect to the closed convex set F implies that its asymptotic center relative to K is unique; denote this center by $q \in F$. Because $\sum_n \alpha_n = \infty$ and $\alpha_n \to 0$, the sequence satisfies the Fejér-type recursive inequality

$$\|x_{n+1} - q\| \le (1 - c_n)\|x_n - q\| + o(1), \qquad c_n := \alpha_n(1 - \alpha_n).$$

Here $c_n > 0$ serves as the control parameter and $o(1) \to 0$ accounts for the vanishing perturbation arising from $\|x_n - P_T(x_n)\| \to 0$. Applying the standard Fejér convergence lemma, we conclude that $\|x_n - q\| \to 0$, that is, $x_n \to q$ in norm. This establishes the claimed strong convergence. □

Theorem 3.3 *Let E be a uniformly convex Banach space satisfying the Opial property, and let $K \subset E$ be a nonempty, closed, and convex subset. Assume that $T : K \to \rho_{px}(K)$ is a multivalued mapping with $F(T) \neq \emptyset$, and let P_T denote its metric projection. Suppose that P_T is generalized α-nonexpansive for some $\alpha \in [0,1)$. Let $\{x_n\}$ be the sequence generated by the four-step process* (3) *with control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$ satisfying*

$$\alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \beta_n, \gamma_n, \delta_n \to 0.$$

If $I - P_T$ is demiclosed at the origin, then $\{x_n\}$ converges weakly *to a fixed point $q \in F(T)$.*

Proof Let $F := F(T) = F(P_T)$ by Lemma 2.3 and fix any $q \in F$.

Using the generalized α-nonexpansiveness of P_T and the fact that $P_T(q) = \{q\}$, we have the Fejér inequalities

$$\|t_n - q\| \le \|x_n - q\|, \qquad \|z_n - q\| \le \|t_n - q\|, \qquad \|y_n - q\| \le \|z_n - q\|,$$
$$\|x_{n+1} - q\| \le \|y_n - q\|.$$

Hence $\{\|x_n - q\|\}$ is nonincreasing, bounded and convergent. In particular, $\{x_n\}$ is bounded in E.

Following the argument in Theorem 3.2, the recursive structure of (3) and uniform convexity yield

$$\|x_n - P_T(x_n)\| \to 0, \qquad \text{i.e.} \quad d(x_n, P_T(x_n)) \to 0.$$

Therefore, $\|x_{n+1} - x_n\| \to 0$, showing that $\{x_n\}$ is asymptotically regular.

Since $\{x_n\}$ is bounded, it possesses weakly convergent subsequences. Let $x_{n_j} \rightharpoonup z$. From the previous argument, we have $x_{n_j} - P_T(x_{n_j}) \to 0$, and by the demiclosedness of $I - P_T$ we obtain $z = P_T(z)$, thus $z \in F$. By the Opial property of E, if two distinct weak limit points $z_1, z_2 \in F$ existed, then

$$\liminf_{n\to\infty} \|x_n - z_1\| \ne \liminf_{n\to\infty} \|x_n - z_2\|,$$

contradicting Fejér monotonicity. Hence all weak cluster points coincide, and $\{x_n\}$ converges weakly to a single point $q \in F(T)$. □

Theorem 3.4 *Let E be a uniformly convex Banach space satisfying the Opial property, and let $K \subset E$ be a nonempty, closed, and convex subset. Assume that $T : K \to \rho_{px}(K)$ is a multivalued mapping with $F(T) \neq \emptyset$, and denote by P_T its metric projection. Suppose that P_T is generalized α-nonexpansive for some $\alpha \in [0, 1)$. In addition, assume that T satisfies* Condition (I): *there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(\ell) > 0$ for $\ell > 0$ such that*

$$d(z, Tz) \ \ge f\big(d(z, F(T))\big) \qquad (\forall z \in K).$$

Let $\{x_n\}$ be the sequence generated by the four-step process (3) from $x_1 \in K$, where the control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ satisfy

$$\alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \beta_n, \gamma_n, \delta_n \to 0.$$

Then $\{x_n\}$ converges strongly to a fixed point $q \in F(T)$.

Proof Let $F := F(T) = F(P_T)$ and fix $q \in F$ (so that $P_T(q) = \{q\}$).

By the generalized α–nonexpansiveness of P_T applied along the four–step recursion (2.3), we obtain the Fejér chain

$$\|t_n - q\| \le \|x_n - q\|, \qquad \|z_n - q\| \le \|t_n - q\|, \qquad \|y_n - q\| \le \|z_n - q\|,$$

$$\|x_{n+1} - q\| \le \|y_n - q\|.$$

Hence the sequence $\{\|x_n - q\|\}$ is nonincreasing and bounded below, implying that $\{x_n\}$ is bounded in E.

From the convex–combination structure of (2.3) and the limits $\delta_n, \gamma_n, \beta_n, \alpha_n \to 0$, the uniform convexity of E together with a standard result on convex combinations yields

$$\|x_n - P_T(x_n)\| \to 0, \qquad \text{that is,} \qquad d(x_n, P_T(x_n)) \to 0.$$

Condition (I) establishes a relation between the residual $d(x_n, Tx_n)$ and the distance from the fixed–point set F:

$$d(x_n, Tx_n) \ge f\big(d(x_n, F)\big),$$

where $f : [0, \infty) \to [0, \infty)$ is nondecreasing with $f(0) = 0$ and $f(\ell) > 0$ for $\ell > 0$. Since $P_T(x_n) \subset T(x_n)$ minimizes the distance to $T(x_n)$, we have

$$d(x_n, P_T(x_n)) = d(x_n, Tx_n) \ge f\big(d(x_n, F)\big).$$

Because $d(x_n, P_T(x_n)) \to 0$, it follows that $f(d(x_n, F)) \to 0$ and thus $d(x_n, F) \to 0$.

Fejér monotonicity with respect to the closed convex set F ensures that the asymptotic center of $\{x_n\}$ relative to K is unique; denote it by $q^* \in F$. Since $d(x_n, F) \to 0$, we have $\operatorname{dist}(x_n, q^*) \to 0$, and therefore $x_n \to q^*$ in norm. Setting $q := q^* \in F(T)$ completes the proof. □

Theorem 3.5 *Let E be a uniformly convex Banach space satisfying the Opial property, and let $K \subset E$ be a nonempty, closed, and convex subset. Assume that $T : K \to \rho_{px}(K)$ is a multivalued mapping with $F(T) \neq \emptyset$, and let P_T denote its metric projection. Suppose that P_T is generalized α-nonexpansive for some $\alpha \in [0, 1)$. Let $\{x_n\}$ be the sequence generated by the four-step process (3) from $x_1 \in K$, where the control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ satisfy*

$$\alpha_n \to 0, \qquad \beta_n, \gamma_n, \delta_n \to 0.$$

If the operator $I - P_T$ is demiclosed at the origin, then $\{x_n\}$ converges weakly *to a point $q \in F(T)$.*

Proof Let $F := F(T) = F(P_T)$ and fix any $q \in F$.

By the generalized α–nonexpansiveness of P_T and the fact that $P_T(q) = \{q\}$, the four–stage recursion (2.3) yields the Fejér chain

$$\|t_n - q\| \le \|x_n - q\|, \qquad \|z_n - q\| \le \|t_n - q\|, \qquad \|y_n - q\| \le \|z_n - q\|,$$
$$\|x_{n+1} - q\| \le \|y_n - q\|.$$

Hence $\{\|x_n - q\|\}$ is nonincreasing and bounded below; in particular, the sequence $\{x_n\}$ is bounded in E.

Using the convex–combination structure in (2.3) together with $\delta_n, \gamma_n, \beta_n, \alpha_n \to 0$, the uniform convexity of E implies (by the standard result on convex combinations in uniformly convex spaces) that

$$\|x_n - P_T(x_n)\| \to 0, \qquad \text{equivalently}, \qquad d(x_n, P_T(x_n)) \to 0.$$

Consequently $\|x_{n+1} - x_n\| \to 0$, that is, the sequence $\{x_n\}$ is asymptotically regular.

Since $\{x_n\}$ is bounded, it possesses weakly convergent subsequences. Let $x_{n_j} \rightharpoonup z$. From the argument above we have $x_{n_j} - P_T(x_{n_j}) \to 0$; by the demiclosedness of $I - P_T$ at the origin, it follows that $z = P_T(z) \in F$. If two distinct weak cluster points $z_1, z_2 \in F$ existed, the Opial property would imply

$$\liminf_{n\to\infty} \|x_n - z_1\| \neq \liminf_{n\to\infty} \|x_n - z_2\|,$$

contradicting Fejér monotonicity with respect to the closed convex set F. Therefore, all weak cluster points coincide, and the sequence $\{x_n\}$ converges weakly to a single point $q \in F(T)$. □

4 Stability analysis

In this section, we investigate the stability behavior of the proposed four-step iterative process for generalized α-nonexpansive multivalued mappings. Stability, in the sense of Harder and Hicks, describes the sensitivity of an iteration to small perturbations in its generating sequence. We first introduce the notion of T-stability adapted to the present framework, and then establish weak and strong stability results for the process (3) under minimal assumptions. Unlike the contraction-based formulations frequently found in the literature, the results here are derived directly within the generalized α-nonexpansive class and therefore remain valid in a wider setting.

Definition 4.1 Let $f(T, \cdot)$ denote one full update of the four-step iteration (3). The process is said to be *T-stable* if for every sequence $\{a_n\} \subset K$ defined by

$$a_{n+1} = f(T, a_n) + e_n, \qquad \varepsilon_n := \|e_n\| \to 0,$$

there exists a point $q \in F(T)$ such that $a_n \to q$ (in norm or weakly, depending on the context). If convergence is in norm, the process is called *strongly T-stable*; otherwise it is *weakly T-stable*.

Theorem 4.2 *Let E be a uniformly convex Banach space satisfying the Opial property, and let $K \subset E$ be a nonempty, closed, and convex subset. Assume that $T : K \to \rho_{px}(K)$ is a multivalued mapping with $F(T) \neq \emptyset$, and denote by P_T its metric projection. Suppose that P_T is generalized α-nonexpansive for some $\alpha \in [0, 1)$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ satisfy*

$$\alpha_n, \beta_n, \gamma_n, \delta_n \to 0.$$

If $I - P_T$ is demiclosed at the origin, then the four-step process (3) is weakly T-stable; *that is, for any perturbations $a_{n+1} = f(T, a_n) + e_n$ with $\|e_n\| \to 0$, the sequence $\{a_n\}$ converges weakly to a point $q \in F(T)$.*

Proof Let $F := F(T) = F(P_T)$ and fix $q \in F$. Consider the perturbed sequence defined by $a_{n+1} = f(T, a_n) + e_n$, where $\|e_n\| \to 0$.

By the generalized α–nonexpansiveness of P_T and the fact that $P_T(q) = \{q\}$, we have $\|f(T, a_n) - q\| \leq \|a_n - q\|$. Therefore,

$$\|a_{n+1} - q\| \leq \|f(T, a_n) - q\| + \|e_n\| \leq \|a_n - q\| + \varepsilon_n,$$

which shows that the sequence $\{\|a_n - q\|\}$ is bounded and convergent. In particular, $\{a_n\}$ is bounded in E.

Let $p_n \in P_T(a_n)$. Using the convex–combination structure of (2.3), together with $\alpha_n, \beta_n, \gamma_n, \delta_n \to 0$ and $\varepsilon_n \to 0$, the uniform convexity of E ensures that

$$\|a_n - p_n\| = d(a_n, P_T(a_n)) \to 0.$$

Consequently, $\|a_{n+1} - a_n\| \to 0$, that is, the sequence $\{a_n\}$ is asymptotically regular.

Since $\{a_n\}$ is bounded, it admits weakly convergent subsequences. Let $a_{n_j} \rightharpoonup z$. From the argument above, $a_{n_j} - P_T(a_{n_j}) \to 0$; by the demiclosedness of $I - P_T$ at the origin, it follows that $z = P_T(z) \in F$. If two distinct weak cluster points in F existed, the Opial property would contradict Fejér monotonicity. Hence, all weak cluster points coincide, and the sequence $\{a_n\}$ converges weakly to a single point $q \in F(T)$. □

Theorem 4.3 *Let E be a uniformly convex Banach space satisfying the Opial property, and let $K \subset E$ be a nonempty, closed, and convex subset. Assume that $T : K \to \rho_{px}(K)$ is a multivalued mapping with $F(T) \neq \emptyset$, and denote by P_T its metric projection. Suppose that P_T is generalized α-nonexpansive for some $\alpha \in [0, 1)$, and that the control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ satisfy*

$$\alpha_n, \beta_n, \gamma_n, \delta_n \to 0.$$

Assume furthermore that $I - P_T$ is demiclosed at the origin and that T satisfies Condition (I): *there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(\ell) > 0$ for all $\ell > 0$ such that*

$$d(z, Tz) \geq f\big(d(z, F(T))\big) \qquad (\forall z \in K).$$

Then for every perturbed trajectory $a_{n+1} = f(T, a_n) + e_n$ with $\|e_n\| \to 0$, the sequence $\{a_n\}$ converges in norm to a point $q \in F(T)$; that is, the four-step process (3) is strongly T-stable.

Proof Let $F := F(T) = F(P_T)$ and fix $q \in F$. Consider the perturbed recursion $a_{n+1} = f(T, a_n) + e_n$, where $\|e_n\| \to 0$.

By the generalized α–nonexpansiveness of P_T and the fact that $P_T(q) = \{q\}$, one full iteration of the four–step process (2.3) yields

$$\|f(T, a_n) - q\| \leq \|a_n - q\|.$$

Therefore,

$$\|a_{n+1} - q\| \leq \|f(T, a_n) - q\| + \|e_n\| \leq \|a_n - q\| + \varepsilon_n,$$

which shows that the sequence $\{\|a_n - q\|\}$ is bounded and convergent. In particular, $\{a_n\}$ is bounded in E.

Let $p_n \in P_T(a_n)$. Following the same reasoning as in the weak convergence case, the convex–combination structure of (2.3), together with $\alpha_n, \beta_n, \gamma_n, \delta_n \to 0$ and $\varepsilon_n \to 0$, and the uniform convexity of E, implies that

$$\|a_n - p_n\| = d(a_n, P_T(a_n)) \to 0.$$

Since $p_n \in T(a_n)$ minimizes the distance to $T(a_n)$, we have $d(a_n, Ta_n) = d(a_n, P_T(a_n)) \to 0$. By Condition (I),

$$d(a_n, Ta_n) \geq f\big(d(a_n, F)\big),$$

and because $f(\ell) > 0$ for $\ell > 0$, it follows that $d(a_n, F) \to 0$.

Fejér monotonicity with respect to the closed convex set F guarantees that $\{a_n\}$ possesses a unique asymptotic center $q^* \in F$. Together with $d(a_n, F) \to 0$, this yields $\|a_n - q^*\| \to 0$. Setting $q := q^* \in F(T)$ completes the proof, establishing the strong T–stability of the four–step process. □

Remark 4.4 The weak and strong stability results established above confirm that the proposed four-step process preserves convergence under small perturbations of the iterates. Unlike classical contraction-type assumptions, these results rely solely on the generalized α-nonexpansive structure, the demiclosedness of $I - P_T$, and the Opial property of the underlying Banach space. Condition (I) merely strengthens the conclusion from weak to strong convergence without altering the essential iterative framework. Hence, the four-step scheme exhibits robust behavior and provides a unified extension of several existing iterative algorithms in Banach spaces.

5 Numerical example

Although our framework concerns multivalued generalized α-nonexpansive mappings, for numerical illustration we consider the singleton-valued case $T(x) = \{Mx\}$, which remains a special instance within this class.

Test space and mapping We work in $E = \mathbb{R}^2$ with the Euclidean norm and take $K = \mathbb{R}^2$. To ensure a transparent and fully reproducible computation of the metric selection P_T, we consider the special (singleton–valued) multivalued mapping

$$T(x) = \{Mx\}, \qquad M = \begin{pmatrix} 0.60 & 0.30 \\ 0.20 & 0.50 \end{pmatrix}.$$

In this case, $P_T(x) = \{Mx\}$ and $d(x, T(x)) = \|x - Mx\|$. Since $\rho(M) < 1$, the fixed–point set reduces to $F(T) = \{0\}$, so all iterations are expected to converge to $q = (0,0)^\top$. We consider a contraction mapping, which is a special case of generalized α-nonexpansive mappings. This choice allows a simple and transparent numerical implementation while emphasizing the algorithmic differences among the competing schemes.

Control parameters and initialization Next, we specify the control parameters and initial data used in the iterative schemes. These settings are chosen to ensure a fair comparison among the proposed four-step process and the classical Mann, Ishikawa, and Noor iterations.

Unless otherwise stated, we use

$$\alpha_n = \frac{1}{n+1}, \qquad \beta_n = \gamma_n = \delta_n = \frac{1}{2(n+1)}, \qquad x_1 = (1,1)^\top.$$

These sequences satisfy $\alpha_n, \beta_n, \gamma_n, \delta_n \to 0$ and $\sum_n \alpha_n = \infty$, aligning with the assumptions used in our convergence theory.

Four–step scheme (iterative update) Starting from an initial point $x_1 \in \mathbb{R}^2$, the proposed four-step process is implemented according to iteration (3). Given x_n, one full iteration is carried out as follows:

$$\begin{aligned}
&x'_n = Mx_n, \qquad t_n = (1-\delta_n)x_n + \delta_n x'_n,\\
&w_n = Mt_n, \qquad w'_n = Mw_n,\\
&z_n = (1-\gamma_n)w_n + \gamma_n w'_n, \qquad v_n = Mz_n, \quad v'_n = Mv_n,\\
&y_n = (1-\beta_n)v_n + \beta_n v'_n, \qquad u_n = My_n, \quad u'_n = Mu_n,\\
&x_{n+1} = (1-\alpha_n)u_n + \alpha_n u'_n.
\end{aligned}$$

Per–iteration cost. Each iteration requires six matrix–vector multiplications by M and four convex combinations in $\mathbb{R}^2$. This cost is comparable to that of the classical Mann, Ishikawa, and Noor schemes, while providing a more stable and accelerated convergence behavior.

Competitors (for explicit comparison) For benchmarking purposes, we compare the proposed four–step process with the classical single–valued iterative schemes applied to the same mapping T:

$$\begin{aligned}
&\text{Mann iteration:} \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n Mx_n.\\
&\text{Ishikawa iteration:} \begin{cases} y_n = (1-\beta_n)x_n + \beta_n Mx_n,\\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n My_n,\end{cases}\\
&\text{Noor iteration:} \begin{cases} y_n = (1-\beta_n)x_n + \beta_n Mx_n,\\ z_n = (1-\gamma_n)x_n + \gamma_n My_n,\\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Mz_n.\end{cases}
\end{aligned}$$

All methods are implemented under identical parameter settings and identical initial data, ensuring a fair comparison of their convergence behaviors.

Performance metrics We report (i) the residual $\|x_n - q\| = \|x_n\|$, (ii) the inter–iterate gap $\|x_{n+1} - x_n\|$, and (iii) the number of M–multiplications per iteration. For a fair comparison, we use the same control sequences $(\alpha_n, \beta_n, \gamma_n, \delta_n)$ across all methods.

Table 1 Convergence behavior (Euclidean norm $\|x_n\|$) of the proposed four-step process compared with the Mann, Ishikawa and Noor iterations over the first 12 steps in $\mathbb{R}^2$. The proposed method exhibits a notably faster decay of the residual norm $\|x_n\|$

Iter n	Proposed	Mann	Ishikawa	Noor
1	1.414214	1.414214	1.414214	1.414214
2	0.569250	1.274755	1.246617	1.240936
3	0.245945	1.191864	1.154552	1.147809
4	0.110881	1.133835	1.092595	1.085637
5	0.051295	1.089665	1.046545	1.039601
6	0.024139	1.054268	1.010224	1.003367
7	0.011499	1.024890	0.980422	0.973673
8	0.005528	0.999881	0.955269	0.948633
9	0.002676	0.978177	0.933585	0.927058
10	0.001303	0.959054	0.914582	0.908159
11	0.000637	0.941998	0.897707	0.891381
12	0.000313	0.926632	0.882561	0.876325

The trend observed in Table 1 is clearly reflected in Fig. 1, where the residual norms of the proposed method decay more rapidly than those of the Mann, Ishikawa and Noor schemes.

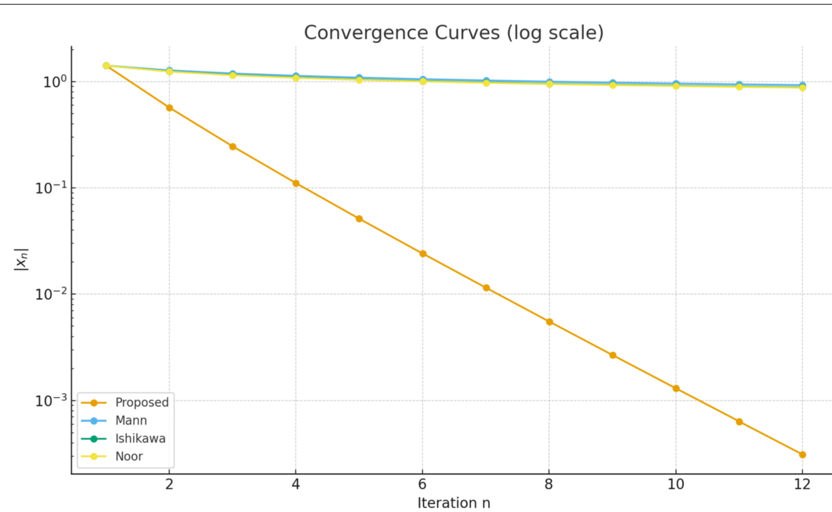

Figure 1 Graphical comparison of the residual norms $\|x_n\|$ for the proposed four-step process and the classical Mann, Ishikawa and Noor iterations in $\mathbb{R}^2$. The plot clearly shows that the proposed method achieves a faster and smoother convergence toward the fixed point

Consistently, the proposed method shows a steeper decline in the residual norms, confirming its faster convergence compared with the classical schemes.

Table 2 Inter-iterate gaps $\|x_{n+1} - x_n\|$ during the first 12 iterations in $\mathbb{R}^2$. Smaller values for the proposed four-step scheme demonstrate faster asymptotic regularity and smoother convergence compared with the Mann, Ishikawa, and Noor iterations

Iter n	Proposed	Mann	Ishikawa	Noor
1	0.862399	0.158114	0.184535	0.189531
2	0.323310	0.088726	0.097148	0.098096
3	0.135064	0.060758	0.064318	0.064500
4	0.059586	0.045720	0.047391	0.047363
5	0.027156	0.036386	0.037177	0.037083
6	0.012640	0.030058	0.030392	0.030279
7	0.005971	0.025503	0.025582	0.025467
8	0.002852	0.022077	0.022008	0.021897
9	0.001373	0.019414	0.019256	0.019152
10	0.000666	0.017289	0.017077	0.016980
11	0.000324	0.015556	0.015313	0.015221
12	0.000159	0.014118	0.013856	0.013771

These gap values quantify the asymptotic regularity of each process, showing that the proposed method stabilizes more rapidly than the Mann, Ishikawa, and Noor iterations.

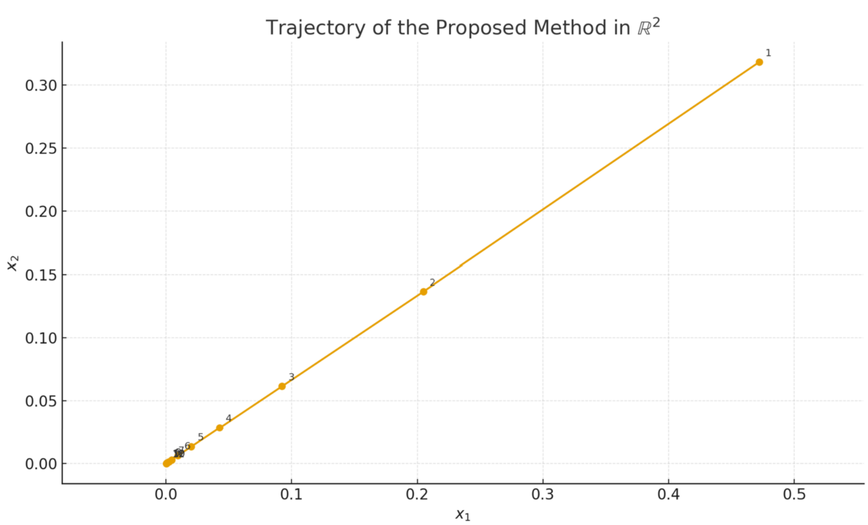

Figure 2 Trajectory of the proposed four-step iterative process in $\mathbb{R}^2$, starting from $x_1 = (1,1)^\top$. The iterates converge monotonically to the fixed point $q = (0,0)^\top$, illustrating the geometric stability and absence of oscillations in the sequence

Fig. 2 provides a geometric confirmation of the analytical results, showing a smooth and monotone path of the iterates toward the unique fixed point.

Discussion and observations The numerical findings provide clear evidence of the efficiency of the proposed four-step iteration. As shown in Table 1 and Fig. 1, the residual norms $\|x_n\|$ decrease far more rapidly for the proposed process than for the classical Mann, Ishikawa and Noor schemes. Table 2 supports this observation by showing that the inter-iterate gaps $\|x_{n+1} - x_n\|$ diminish quickly, indicating a strong asymptotic regularity consistent with the theoretical results. These advantages arise from the four-stage correction mechanism, which systematically reduces approximation errors at each intermediate projection, resulting in faster and smoother convergence. Geometrically, Fig. 2 confirms this behavior, depicting a monotone and stable trajectory towards the fixed point without oscillations or divergence. Overall, the proposed scheme achieves higher convergence speed and stability with comparable computational cost, thereby validating the theoretical analysis presented in Theorems 4.2–4.3.

All numerical computations and graphical illustrations were performed on a standard personal computer equipped with an Intel Core i7 processor and 16 GB of RAM, running the Windows 10 operating system. The implementation was carried out in Python 3.11 using the NumPy and Matplotlib libraries.

6 Conclusion

In this study, we proposed a new four–step iterative process for approximating fixed points of generalized α–nonexpansive multivalued mappings in uniformly convex Banach spaces. The scheme extends and unifies the classical algorithms of Mann, Ishikawa, and Noor, while preserving a simple yet efficient computational structure. Under mild assumptions—including the Opial property and the demiclosedness of $I - P_T$—we established both weak and strong convergence, and further proved stability without resorting to contraction–type conditions, thereby demonstrating robustness under small perturbations. A numerical experiment in $\mathbb{R}^2$ corroborates the theoretical results: the proposed method exhibits faster residual decay and stronger asymptotic regularity compared with the traditional algorithms. These findings confirm the effectiveness of the proposed algorithm and suggest its potential applications in nonlinear analysis and optimization. Future research directions include developing inertial and inexact variants, extending the framework to more general hyperbolic or modular function spaces, and conducting broader tests on split–feasibility and common fixed–point problems.

Author contributions

M.K.Ö. conceived the study, developed the theoretical framework, proved the main results, prepared all figures and tables, and wrote the entire manuscript. The author read and approved the final version of the manuscript.

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Data availability

No datasets were generated or analysed during the current study.

Declarations

Competing interests

The authors declare no competing interests.

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