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On Anti Fuzzy Structures in BCC-Algebras

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Abstract

In this paper, we define the notions of anti intuitionistic fuzzy BCC-subalgebras and anti intuitionistic fuzzy ideals of the BCC-algebras with respect to arbitrary t-conorms and t-norms, and obtain some related results.

Keywords: t-norm, t-conorm, anti intuitionistic fuzzy subalgebra, anti intuitionistic fuzzy ideal, BCC-algebra.

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1 Introduction and preliminaries

The notion of fuzzy sets was introduced by Zadeh [24]. Since then, this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology and so on. In 1991, Xi [23] applied this concept to BCK-algebras and Dudek et al. [8-11] studied fuzzy structures in BCC-algebras. A BCK-algebra is an important class of logical algebras introduced by Iseki [14]. Iseki [14] posed the interesting problem of whether the class of BCK-algebras is a variety. In connection with this problem, Komori [18] introduced the notion of BCC-algebras and Dudek [6,7] modified the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori [18].

In the present paper, using the idea of Kutukcu and Yildiz [19], we introduce the notion of anti intuitionistic fuzzy BCC-subalgebras of the BCC-algebras with the help of arbitrary t-conorms and t-norms as a generalization of anti fuzzy subalgebras. We also introduce the notion of anti intuitionistic fuzzy ideals as a generalization of anti fuzzy ideals and prove that an intuitionistic fuzzy subset of a BCC-algebra is an intuitionistic fuzzy ideal if and only if the complement of this intuitionistic fuzzy subset is an anti intuitionistic fuzzy ideal. We prove that if an intuitionistic fuzzy subset is an anti intuitionistic fuzzy ideal then so is the fuzzifications of its upper and lower level cuts.

Let us recall [18] that a BCC-algebra is a nonempty set X with a constant 0 and a binary operation $*$ which satisfies the following conditions, for all $x, y, z \in X$: (i) $((x*y)*(z*y))*(x*z) = 0$; (ii) $x*x = 0$; (iii) $0*x = 0$; (iv) $x*0 = x$; (v) $x*y = 0$ and $y*x = 0$ imply $x = y$. A nonempty subset G of a BCC-algebra X is called a BCC-subalgebra of X if $x*y \in G$ for all $x, y \in G$ (see also [15,23]).

By a triangular conorm (shortly t-conorm) S [22], we mean a binary operation on the unit interval $[0,1]$ which satisfies the following conditions, for all $x, y, z \in [0,1]$: (i) $S(x,0) =$

x; (ii) $S(x,y) \leq S(x,z)$ if $y \leq z$; (iii) $S(x,y) = S(y,x)$; (iv) $S(x,S(y,z)) = S(S(x,y),z)$. Some important examples of t-conorms are $S_L(x,y) = \min\{x+y,1\}$, $S_P(x,y) = x+y-xy$ and $S_M(x,y) = \max\{x,y\}$.

By a triangular norm (shortly t-norm) T [22], we mean a binary operation on the unit interval $[0,1]$ which satisfies the following conditions, for all $x,y,z \in [0,1]$: (i) $T(x,1) = x$; (ii) $T(x,y) \leq T(x,z)$ if $y \leq z$; (iii) $T(x,y) = T(y,x)$; (iv) $T(x,T(y,z)) = T(T(x,y),z)$. Some important examples of t-norms are $T_L(x,y) = \max\{x+y-1,0\}$, $T_P(x,y) = xy$ and $T_M(x,y) = \min\{x,y\}$.

A t-conorm S and a t-norm T are called associated [20], i.e. $S(x,y) = 1-T(1-x,1-y)$ for all $x,y \in [0,1]$. For example t-conorm S_M and t-norm T_M are associated [12,17,19,20]. Also it is well known [12,17] that if S is a t-conorm and T is a t-norm, then $\max\{x,y\} \leq S(x,y)$ and $\min\{x,y\} \geq T(x,y)$ for all $x,y \in [0,1]$, respectively.

Note that, the concepts of t-conorms and t-norms are known as the axiomatic skeletons that we use for characterizing fuzzy unions and intersections, respectively. These concepts were originally introduced by Menger [21] and several properties and examples for these concepts were proposed by many authors (see [1,5,12,16,17,19-22]).

A fuzzy set A in an arbitrary non-empty set X is a function $\mu_A : X \to [0,1]$. The complement of μ_A, denoted by μ_A^c, is the fuzzy set in X given by $\mu_A^c(x) = 1 - \mu_A(x)$ for all $x \in X$.

For any fuzzy set μ_A in X and any $\alpha \in [0,1]$, Dudek et al. [11] defined two sets

$$U(\mu_A;\alpha) = \{x \in X : \mu_A(x) \geq \alpha\} \text{ and } V(\mu_A;\alpha) = \{x \in X : \mu_A(x) \leq \alpha\}$$

which are called an upper and lower α-level cut of μ_A, respectively, and can be used to the characterization of μ_A.

Definition 1.1 ([8]). *A fuzzy set A in a BCC-algebra X is called a fuzzy BCC-subalgebra of X if*

$$\mu_A(x*y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

for all $x,y \in X$.

Definition 1.2 ([8]). *A fuzzy set A in a BCC-algebra X is called a fuzzy BCC-subalgebra of X with respect to a t-norm T (or simply, a T-fuzzy BCC-subalgebra of X) if*

$$\mu_A(x*y) \geq T(\mu_A(x), \mu_A(y))$$

for all $x,y \in X$. Every BCC-algebra is a fuzzy BCC-algebra and so a T-fuzzy BCC-subalgebra but the converse is not true (see [6,8-10]).

Definition 1.3 ([9]). *A fuzzy set A in a BCC-algebra X is called a fuzzy ideal of X if*

$$\mu_A(0) \geq \mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\}$$

for all $x,y \in X$.

Definition 1.4 ([9,13]). *A fuzzy set A in a BCK-algebra X is called an anti fuzzy subalgebra of X if*

$$\mu_A(x*y) \leq \max\{\mu_A(x), \mu_A(y)\}$$

for all $x,y \in X$.

As a generalization of the notion of fuzzy sets in X, Atanassov [2] introduced the concept of intuitionistic fuzzy sets defined on X as objects having the form $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely

$\lambda_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

In [3], for every two intuitionistic fuzzy sets A and B in X, we have

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,

(ii) $\Box A = \{(x, \mu_A(x), \mu_A^c(x)) : x \in X\}$,

(iii) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) : x \in X\}$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$ as Dudek et al. [11].

2 (S,T)-anti intuitionistic fuzzy BCC-subalgebras

DEFINITION 2.1. *A fuzzy set A in a BCC-algebra X is said to be an anti fuzzy BCC-subalgebra of X if*

$$\mu_A(x * y) \leq \max\{\mu_A(x), \mu_A(y)\}$$

for all $x, y \in X$.

DEFINITION 2.2. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in a BCC-algebra X is said to be an anti intuitionistic fuzzy BCC-subalgebra of X if*

(i) $\mu_A(x * y) \leq \max\{\mu_A(x), \mu_A(y)\}$,

(ii) $\lambda_A(x * y) \geq \min\{\lambda_A(x), \lambda_A(y)\}$

for all $x, y \in X$.

DEFINITION 2.3. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in a BCC-algebra X is said to be an anti intuitionistic fuzzy BCC-subalgebra of X with respect to a t-conorm S and a t-norm T (or simply, an (S,T)-anti intuitionistic fuzzy BCC-subalgebra of X) if*

(i) $\mu_A(x * y) \leq S(\mu_A(x), \mu_A(y))$,

(ii) $\lambda_A(x * y) \geq T(\lambda_A(x), \lambda_A(y))$

for all $x, y \in X$.

REMARK 2.1. *Every anti intuitionistic fuzzy BCC-subalgebra of a BCC-algebra is an (S,T)-anti intuitionistic fuzzy BCC-subalgebra of X, but it is clear that the converse is not true. If $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, then every anti intuitionistic fuzzy BCC-subalgebra of a BCC-algebra X is an anti fuzzy BCC-subalgebra of X. Also, if $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, $S = S_M$ and $T = T_M$, then every (S,T)-anti intuitionistic fuzzy BCC-subalgebra of a BCC-algebra X is an anti fuzzy BCC-subalgebra of X.*

Example. Let $X = \{0, 1, 2, 3\}$ be a BCC-algebra with the Cayley table as follows

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Define an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X by

$$\mu_A(x) = \begin{cases} 0, & x = 0 \\ 1/2, & x = 1 \text{ or } 2 \\ 1, & x = 3 \end{cases} \quad \text{and } \lambda_A(x) = \begin{cases} 1, & x = 0 \\ 1/3, & x = 1 \text{ or } 2 \\ 0, & x = 3 \end{cases}$$

It is easy to check that $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, $\mu_A(x * y) \leq S_M(\mu_A(x), \mu_A(y))$ *and* $\lambda_A(x * y) \geq T_L(\lambda_A(x), \lambda_A(y))$ *for all* $x, y \in X$. *Hence* $A = (\mu_A, \lambda_A)$ *is an* (S_M, T_L)*-anti intuitionistic fuzzy BCC-subalgebra of* X. *Also note that t-conorm* S_M *and t-norm* T_L *are not associated.*

Example. Let $X = \{0, a, b, c, d\}$ be a proper BCC-algebra with the Cayley table as follows

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

Define an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ *in* X *by*

$$\mu_A(x) = \begin{cases} t_0, & x \in \{0, a, b\} \\ t_1, & \text{otherwise} \end{cases} \quad \text{and } \lambda_A(x) = \begin{cases} t_2, & x \in \{0, a, b\} \\ t_3, & \text{otherwise.} \end{cases}$$

where $0 \leq t_0, t_1, t_2, t_3 \leq 1$ such that $t_0 < t_1$, $t_3 < t_2$ and $t_0 + t_1 + t_2 + t_3 = 1$. *It is easy to check that* $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, $\mu_A(x * y) \leq S_L(\mu_A(x), \mu_A(y))$ *and* $\lambda_A(x * y) \geq T_P(\lambda_A(x), \lambda_A(y))$ *for all* $x, y \in X$. *Hence* $A = (\mu_A, \lambda_A)$ *is an* (S_L, T_P)*-anti intuitionistic fuzzy BCC-subalgebra of* X. *Also note that t-conorm* S_L *and t-norm* T_P *are not associated.*

Remark 2.2. *Note that, the above examples hold even with the t-conorm* S_M *and t-norm* T_M, *and hence* $A = (\mu_A, \lambda_A)$ *is an* (S_M, T_M)*-anti intuitionistic fuzzy BCC-subalgebra of* X *in such examples. Hence every anti intuitionistic fuzzy BCC-subalgebra of* X *is an* (S, T)*-anti intuitionistic fuzzy BCC-subalgebra, but the converse is not true.*

Lemma 2.1. *If* $A = (\mu_A, \lambda_A)$ *is an* (S, T)*-anti intuitionistic fuzzy BCC- subalgebra of a BCC-algebra* X, *then so is* $\Box A = (\mu_A, \mu_A^c)$ *such that t-conorm* S *and t-norm* T *are associated.*

Proof. Since $A = (\mu_A, \lambda_A)$ is an (S, T)-anti intuitionistic fuzzy BCC-subalgebra of X, we have

$$\mu_A(x * y) \leq S(\mu_A(x), \mu_A(y))$$

for all $x, y \in X$ and so

$$1 - \mu_A^c(x * y) \leq S(1 - \mu_A^c(x), 1 - \mu_A^c(y))$$

which implies

$$1 - S(1 - \mu_A^c(x), 1 - \mu_A^c(y)) \leq \mu_A^c(x * y).$$

Since S and T are associated, we have

$$T(\mu_A^c(x), \mu_A^c(y)) \leq \mu_A^c(x * y).$$

This completes the proof.

Lemma 2.2. *If* $A = (\mu_A, \lambda_A)$ *is an* (S, T)*-anti intuitionistic fuzzy BCC- subalgebra of a BCC-algebra* X, *then so is* $\Diamond A = (\lambda_A^c, \lambda_A)$ *such that t-conorm* S *and t-norm* T *are associated.*

Proof. The proof is similar to the proof of Lemma 2.1.

Combining the above two lemmas, it is easy to see that the following theorem is valid.

THEOREM 2.1. *$A = (\mu_A, \lambda_A)$ is an (S, T)-anti intuitionistic fuzzy BCC- subalgebra of a BCC-algebra X if and only if $\Box A$ and $\Diamond A$ are (S, T)-anti intuitionistic fuzzy BCC-subalgebra of X such that t-conorm S and t-norm T are associated.*

COROLLARY 2.1. *$A = (\mu_A, \lambda_A)$ is an (S, T)-anti intuitionistic fuzzy BCC- subalgebra of a BCC-algebra X if and only if μ_A and λ_A^c are anti fuzzy BCC-subalgebra of X such that t-conorm S and t-norm T are associated.*

If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set in a BCC-algebra X and f is a self mapping of X, we define mappings

$$\mu_A[f] : X \to [0, 1] \text{ by } \mu_A[f](x) = \mu_A(f(x))$$

and

$$\lambda_A[f] : X \to [0, 1] \text{ by } \lambda_A[f](x) = \lambda_A(f(x))$$

for all $x \in X$, respectively.

PROPOSITION 2.1. *If $A = (\mu_A, \lambda_A)$ is an (S, T)-anti intuitionistic fuzzy BCC- subalgebra of a BCC-algebra X and f is an endomorphism of X, then $(\mu_A[f], \lambda_A[f])$ is an (S, T)-anti intuitionistic fuzzy BCC-subalgebra of X.*

Proof. For any given $x, y \in X$, we have

$$\begin{aligned} \mu_A[f](x * y) &= \mu_A(f(x * y)) = \mu_A(f(x) * f(y)) \leq S(\mu_A(f(x)), \mu_A(f(y))) \\ &= S(\mu_A[f](x), \mu_A[f](y)), \end{aligned}$$

$$\begin{aligned} \lambda_A[f](x * y) &= \lambda_A(f(x * y)) = \lambda_A(f(x) * f(y)) \geq T(\lambda_A(f(x)), \lambda_A(f(y))) \\ &= T(\lambda_A[f](x), \lambda_A[f](y)). \end{aligned}$$

This completes the proof.

If f is a self mapping of a BCC-algebra X and $B = (\mu_B, \lambda_B)$ is an intuitionistic fuzzy set in $f(X)$, then the intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X defined by $\mu_A = \mu_B \circ f$ and $\lambda_A = \lambda_B \circ f$ (i.e., $\mu_A(x) = \mu_B(f(x))$ and $\lambda_A(x) = \lambda_B(f(x))$ for all $x \in X$) is called the *preimage* of B under f.

THEOREM 2.2. *An onto homomorphic preimage of an (S, T)-anti intuitionistic fuzzy BCC-subalgebra is an (S, T)-anti intuitionistic fuzzy BCC-subalgebra.*

Proof. Let $f : X \to Y$ be an onto homomorphism of BCC-algebras, $B = (\mu_B, \lambda_B)$ be an (S, T)-anti intuitionistic fuzzy BCC-subalgebra of Y, and $A = (\mu_A, \lambda_A)$ be preimage of B under f. Then, we have

$$\begin{aligned} \mu_A(x * y) &= \mu_B(f(x * y)) = \mu_B(f(x) * f(y)) \leq S(\mu_B(f(x)), \mu_B(f(y))) \\ &= S(\mu_A(x), \mu_A(y)), \end{aligned}$$

$$\begin{aligned} \lambda_A(x * y) &= \lambda_B(f(x * y)) = \lambda_B(f(x) * f(y)) \geq T(\lambda_B(f(x)), \lambda_B(f(y))) \\ &= T(\lambda_A(x), \lambda_A(y)) \end{aligned}$$

for all $x, y \in X$. Hence, $A = (\mu_A, \lambda_A)$ is an (S, T)-anti intuitionistic fuzzy BCC-subalgebra of X.

If f is a self mapping of a BCC-algebra X and $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set in X, then the intuitionistic fuzzy set $A^f = (\mu_A^f, \lambda_A^f)$ in $f(X)$ defined by

$$\mu_A^f(y) = \inf_{x \in f^{-1}(y)} \mu_A(x) \text{ and } \lambda_A^f(y) = \sup_{x \in f^{-1}(y)} \lambda_A(x)$$

for all $y \in f(x)$, is called *image* of $A = (\mu_A, \lambda_A)$ under f.

An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X is said to be have *(inf-sup) property* if there exists a $t_0 \in T$ such that $\mu_A(t_0) = \inf_{t \in T} \mu_A(t)$ and $\lambda_A(t_0) = \sup_{t \in T} \lambda_A(t)$ for every subset $T \subseteq X$.

PROPOSITION 2.2. *An onto homomorphic image of an anti intuitionistic fuzzy BCC-subalgebra with (inf-sup) property is an anti intuitionistic fuzzy BCC-subalgebra.*

Proof. Let $f : X \to Y$ be an onto homomorphism of BCC-algebras and $A = (\mu_A, \lambda_A)$ be an anti intuitionistic fuzzy BCC-subalgebra of X with (inf-sup) property. For given $x', y' \in Y$, let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ such that $\mu_A(x_0) = \inf_{t \in f^{-1}(x')} \mu_A(t)$, $\mu_A(y_0) = \inf_{t \in f^{-1}(y')} \mu_A(t)$, $\lambda_A(x_0) = \sup_{t \in f^{-1}(x')} \lambda_A(t)$ and $\lambda_A(y_0) = \sup_{t \in f^{-1}(y')} \lambda_A(t)$, respectively. Then

$$\begin{aligned}
\mu_A^f(x' * y') &= \inf_{z \in f^{-1}(x' * y')} \mu_A(z) \leq \max\{\mu_A(x_0), \mu_A(y_0)\} \\
&= \max\left\{\inf_{t \in f^{-1}(x')} \mu_A(t), \inf_{t \in f^{-1}(y')} \mu_A(t)\right\} \\
&= \max\left\{\mu_A^f(x'), \mu_A^f(y')\right\},
\end{aligned}$$

$$\begin{aligned}
\lambda_A^f(x' * y') &= \sup_{z \in f^{-1}(x' * y')} \lambda_A(z) \geq \min\{\lambda_A(x_0), \lambda_A(y_0)\} \\
&= \min\left\{\sup_{t \in f^{-1}(x')} \lambda_A(t), \sup_{t \in f^{-1}(y')} \lambda_A(t)\right\} \\
&= \min\left\{\lambda_A^f(x'), \lambda_A^f(y')\right\}.
\end{aligned}$$

Hence, $A^f = (\mu_A^f, \lambda_A^f)$ is an anti intuitionistic fuzzy BCC-subalgebra of Y.

REMARK 2.3. *It is well known [12,17] that* $\max\{x, y\} \leq S(x, y)$ *and* $\min\{x, y\} \geq T(x, y)$ *for all* $x, y \in [0, 1]$. *Therefore, it is easy to see that the above proposition is also true in the case of* (S, T)*-anti intuitionistic fuzzy BCC-subalgebras.*

LEMMA 2.3 ([12]). *Let S and T be a t-conorm and a t-norm, respectively. Then*

$$\begin{aligned}
S(S(x, y), S(z, t)) &= S(S(x, z), S(y, t)), \\
T(T(x, y), T(z, t)) &= T(T(x, z), T(y, t))
\end{aligned}$$

for all $x, y, z, t \in [0, 1]$.

THEOREM 2.3. *Let S be a t-conorm, T be a t-norm and $X = X_1 \times X_2$ be the direct product BCC-algebra of BCC-algebras X_1 and X_2. If $A_1 = (\mu_{A_1}, \lambda_{A_1})$ (resp. $A_2 = (\mu_{A_2}, \lambda_{A_2})$) is an (S, T)-anti intuitionistic fuzzy BCC-subalgebra of X_1 (resp. X_2), then $A = (\mu_A, \lambda_A)$ is an (S, T)-anti intuitionistic fuzzy BCC-subalgebra of X defined by $\mu_A = \mu_{A_1} \times \mu_{A_2}$ and $\lambda_A = \lambda_{A_1} \times \lambda_{A_2}$ such that*

$$\begin{aligned}
\mu_A(x_1, x_2) &= (\mu_{A_1} \times \mu_{A_2})(x_1, x_2) = S(\mu_{A_1}(x_1), \mu_{A_2}(x_2)), \\
\lambda_A(x_1, x_2) &= (\lambda_{A_1} \times \lambda_{A_2})(x_1, x_2) = T(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2))
\end{aligned}$$

for all $(x_1, x_2) \in X$.

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of X. Since X is a BCC-algebra, we have

$$\begin{aligned}
\mu_A(x * y) &= \mu_A((x_1, x_2) * (y_1, y_2)) = \mu_A(x_1 * y_1, x_2 * y_2) \\
&= (\mu_{A_1} \times \mu_{A_2})(x_1 * y_1, x_2 * y_2) \\
&= S(\mu_{A_1}(x_1 * y_1), \mu_{A_2}(x_2 * y_2)) \\
&\leq S(S(\mu_{A_1}(x_1), \mu_{A_1}(y_1)), S(\mu_{A_2}(x_2), \mu_{A_2}(y_2))) \\
&= S(S(\mu_{A_1}(x_1), \mu_{A_2}(x_2)), S(\mu_{A_1}(y_1), \mu_{A_2}(y_2))) \\
&= S((\mu_{A_1} \times \mu_{A_2})(x_1, x_2), (\mu_{A_1} \times \mu_{A_2})(y_1, y_2)) \\
&= S(\mu_A(x), \mu_A(y)),
\end{aligned}$$

$$\begin{aligned}
\lambda_A(x * y) &= \lambda_A((x_1, x_2) * (y_1, y_2)) = \lambda_A(x_1 * y_1, x_2 * y_2) \\
&= (\lambda_{A_1} \times \lambda_{A_2})(x_1 * y_1, x_2 * y_2) \\
&= T(\lambda_{A_1}(x_1 * y_1), \lambda_{A_2}(x_2 * y_2)) \\
&\geq T(T(\lambda_{A_1}(x_1), \lambda_{A_1}(y_1)), T(\lambda_{A_2}(x_2), \lambda_{A_2}(y_2))) \\
&= T(T(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2)), T(\lambda_{A_1}(y_1), \lambda_{A_2}(y_2))) \\
&= T((\lambda_{A_1} \times \lambda_{A_2})(x_1, x_2), (\lambda_{A_1} \times \lambda_{A_2})(y_1, y_2)) \\
&= T(\lambda_A(x), \lambda_A(y)).
\end{aligned}$$

This completes the proof.

3 (S,T)-anti intuitionistic fuzzy ideals

In this section, we shall define the notion (S, T)-anti intuitionistic fuzzy ideal of a BCC-algebra with the help of arbitrary t-conorms and t-norms. We investigate some relations between (S, T)-anti intuitionistic fuzzy ideals and (S, T)-anti intuitionistic fuzzy BCC-subalgebras and prove some results on them.

DEFINITION 3.1. *A fuzzy set A in a BCC-algebra X is said to be an anti fuzzy ideal of X if*

(i) $\mu_A(0) \leq \mu_A(x)$,

(ii) $\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\}$

for all $x, y \in X$.

DEFINITION 3.2. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in a BCC-algebra X is said to be an anti intuitionistic fuzzy ideal of X if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\}$,

(iii) $\lambda_A(x) \geq \min\{\lambda_A(x * y), \lambda_A(y)\}$

for all $x, y \in X$.

DEFINITION 3.3. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in a BCC-algebra X is said to be an anti intuitionistic fuzzy ideal of X with respect to a t-conorm S and a t-norm T (or simply, an (S, T)-anti intuitionistic fuzzy ideal of X) if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq S(\mu_A(x * y), \mu_A(y))$,

(iii) $\lambda_A(x) \geq T(\lambda_A(x * y), \lambda_A(y))$

for all $x, y \in X$.

Remark 3.1. *Every anti intuitionistic fuzzy ideal of a BCC-algebra is an (S,T)-anti intuitionistic fuzzy ideal of X, but it is clear that the converse is not true. If $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, then every anti intuitionistic fuzzy ideal of a BCC-algebra X is an anti fuzzy ideal of X. Also, if $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, $S = S_M$ and $T = T_M$, then every (S,T)-anti intuitionistic fuzzy ideal of a BCC-algebra X is an anti fuzzy ideal of X.*

Example. In Example 1, it is easy to show that $A = (\mu_A, \lambda_A)$ is also an (S,T)-anti intuitionistic fuzzy ideal of X.

Lemma 3.1. *Let $A = (\mu_A, \lambda_A)$ be an (S,T)-anti intuitionistic fuzzy ideal of a BCC-algebra X. If $\leq$ is a partial ordering on X then $\mu_A(x) \leq \mu_A(y)$ and $\lambda_A(y) \leq \lambda_A(x)$ for all $x, y \in X$.*

Proof. Let X be a BCC-algebra. It is known [13] that $\leq$ is a partial ordering on X defined by $x \leq y$ if and only if $x * y = 0$ for all $x, y \in X$. Let A be a (S,T)-anti intuitionistic fuzzy ideal of X. Then

$$\mu_A(x) \leq S(\mu_A(x * y), \mu_A(y)) = S(\mu_A(0), \mu_A(y)) = \mu_A(y)$$

and

$$\lambda_A(x) \geq T(\lambda_A(x * y), \lambda_A(y) = T(\lambda_A(0), \lambda_A(y)) = \lambda_A(y).$$

These complete the proof.

Theorem 3.1. *Let $A = (\mu_A, \lambda_A)$ be an (S,T)-anti intuitionistic fuzzy ideal of a BCC-algebra X. If $x * y \leq x$ holds in X, A is an (S,T)-anti intuitionistic fuzzy BCC-subalgebra of X.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an (S,T)-anti intuitionistic fuzzy ideal *of* X. Since $x * y \leq x$ for all $x, y \in X$, it follows from Lemma 4 that $\mu_A(x * y) \leq \mu_A(x)$ and $\lambda_A(x) \leq \lambda_A(x * y)$. Then

$$\mu_A(x * y) \leq \mu_A(x) \leq S(\mu_A(x * y), \mu_A(y)) \leq S(\mu_A(x), \mu_A(y))$$

and

$$\lambda_A(x * y) \geq \lambda_A(x) \geq T(\lambda_A(x * y), \lambda_A(y)) \geq T(\lambda_A(x), \lambda_A(y))$$

and so A is an (S,T)-anti intuitionistic fuzzy BCC-subalgebra of X.

Remark 3.2. *The converse of the above theorem does not hold in general. In fact, suppose that X be the BCC-algebra in Example 1. It is clear that $x * y \leq x$ for all $x, y \in X$. Define an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X by*

$$\mu_A(x) = \begin{cases} 0, & x = 0 \\ 1/2, & x = 1 \\ 1, & x = 2 \text{ or } 3 \end{cases} \quad \text{and } \lambda_A(x) = \begin{cases} 1, & x = 0 \\ 1/3, & x = 1 \\ 0, & x = 2 \text{ or } 3 \end{cases}$$

By routine calculations, we know that $A = (\mu_A, \lambda_A)$ is an (S,T)-anti intuitionistic fuzzy BCC-subalgebra of X but not an (S,T)-anti intuitionistic fuzzy ideal of X because $\mu_A(2) = 1 > S(\mu_A(2 * 1), \mu_A(1))$ and $\lambda_A(2) = 0 < T(\lambda_A(2 * 1), \lambda_A(1))$.

PROPOSITION 3.1. *Let $A = (\mu_A, \lambda_A)$ be an (S,T)-anti intuitionistic fuzzy ideal of a BCC-algebra X. If $x * y \leq z$ holds in X, then $\mu_A(x) \leq S(\mu_A(y), \mu_A(z))$ and $\lambda_A(x) \geq T(\lambda_A(y), \lambda_A(z))$ for all $x, y, z \in X$.*

Proof. Since $x * y \leq z$ holds for all $x, y, z \in X$, we have

$$\begin{aligned}\mu_A(x*y) &\leq S(\mu_A((x*y)*z), \mu_A(z)) \\ &\leq S(\mu_A(z*z), \mu_A(z)) \\ &= S(\mu_A(0), \mu_A(z)) \\ &= \mu_A(z)\end{aligned}$$

it follows that

$$\mu_A(x) \leq S(\mu_A(x*y), \mu_A(y)) \leq S(\mu_A(z), \mu_A(y))$$

and

$$\begin{aligned}\lambda_A(x*y) &\geq T(\lambda_A((x*y)*z), \lambda_A(z)) \\ &\geq T(\lambda_A(z*z), \lambda_A(z)) \\ &= T(\lambda_A(0), \lambda_A(z)) \\ &= \lambda_A(z)\end{aligned}$$

it follows that

$$\lambda_A(x) \geq T(\lambda_A(x*y), \lambda_A(y)) \geq T(\lambda_A(z), \lambda_A(y)).$$

They complete the proof.

PROPOSITION 3.2. *An intuitionistic fuzzy subset A of a BCC-algebra X is an (S,T)-intuitionistic fuzzy ideal of X if and only if its complement A^c is an (S,T)-anti intuitionistic fuzzy ideal of X such that t-conorm S and t-norm T are associated.*

Proof. Let A be an (S,T)-intuitionistic fuzzy ideal of X such that S and T are associated. Then

$$\mu_A^c(0) = 1 - \mu_A(0) \leq 1 - \mu_A(x) = \mu_A^c(x)$$

and

$$\lambda_A^c(0) = 1 - \lambda_A(0) \geq 1 - \lambda_A(x) = \lambda_A^c(x).$$

We also have

$$\begin{aligned}\mu_A^c(x) &= 1 - \mu_A(x) \leq 1 - T(\mu_A(x*y), \mu_A(y)) \\ &= 1 - T(1 - \mu_A^c(x*y), 1 - \mu_A^c(y)) \\ &= S(\mu_A^c(x*y), \mu_A^c(y))\end{aligned}$$

and

$$\begin{aligned}\lambda_A^c(x) &= 1 - \lambda_A(x) \geq 1 - S(\lambda_A(x*y), \lambda_A(y)) \\ &= 1 - S(1 - \lambda_A^c(x*y), 1 - \lambda_A^c(y)) \\ &= T(\lambda_A^c(x*y), \lambda_A^c(y)).\end{aligned}$$

for all $x, y \in X$. Thus A^c is an (S,T)-anti intuitionistic fuzzy ideal of X. The converse also can be proved similarly.

Theorem 3.2. *Let A be an anti intuitionistic fuzzy ideal of a BCC-algebra X. Then the set*

$$X_A := \{x \in X : \mu_A(x) = \mu_A(0), \lambda_A(x) = \lambda_A(0)\}$$

is an ideal of X.

Proof. Since A is an anti intuitionistic fuzzy ideal of X, we have $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$. Now, suppose that $x, y \in X$ such that $x * y \in X_A$ and $y \in X_A$. Then $\mu_A(x * y) = \mu_A(0) = \mu_A(y)$ and $\lambda_A(x * y) = \lambda_A(0) = \lambda_A(y)$, so we have

$$\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\} = \max\{\mu_A(0), \mu_A(0)\} = \mu_A(0)$$

and

$$\lambda_A(x) \geq \min\{\lambda_A(x * y), \lambda_A(y)\} = \min\{\lambda_A(0), \lambda_A(0)\} = \lambda_A(0)$$

respectively. Thus, we have $\mu_A(x) = \mu_A(0)$ and $\lambda_A(x) = \lambda_A(0)$, and therefore $x \in X_A$. Also, it is easy to see that $0 \in X_A$. This completes the proof.

Theorem 3.3. *Let A be an intuitionistic fuzzy subset of a BCC-algebra X. Then A is an (S, T)-anti intuitionistic fuzzy ideal of X if and only if for each $\alpha_0, \alpha_1 \in [0, 1]$ such that $\alpha_0 \leq \lambda_A(0)$ and $\alpha_1 \geq \mu_A(0)$, the upper α_0-level cut $U(\lambda_A; \alpha_0)$ and the lower α_1-level cut $V(\mu_A; \alpha_1)$ are ideals of X.*

Proof. Let A be an (S, T)-anti intuitionistic fuzzy ideal of X and let $\alpha_0 \in [0, 1]$ such that $\alpha_0 \leq \lambda_A(0)$. Clearly $0 \in U(\lambda_A; \alpha_0)$. Let $x, y \in X$ such that $x * y \in U(\lambda_A; \alpha_0)$ and $y \in U(\lambda_A; \alpha_0)$. Then

$$\lambda_A(x) \geq S(\lambda_A(x * y), \lambda_A(y)) \geq \alpha_0$$

and $x \in U(\lambda_A; \alpha_0)$. Hence $U(\lambda_A; \alpha_0)$ is an ideal of X. Similarly, $V(\mu_A; \alpha_1)$ is also an ideal of X.

Conversely, we first show that $\lambda_A(0) \geq \lambda_A(x)$ for all $x \in X$. If not, then there exists a $x_0 \in X$ such that $\lambda_A(0) < \lambda_A(x_0)$. Taking $\alpha_0 = \frac{1}{2}(\lambda_A(x_0) + \lambda_A(0))$ then $0 \leq \lambda_A(0) < \alpha_0 < \lambda_A(x_0) \leq 1$. It follows that $x_0 \in U(\lambda_A; \alpha_0)$, so $U(\lambda_A; \alpha_0) \neq \varnothing$. Since $U(\lambda_A; \alpha_0)$ is an ideal of X we have $0 \in U(\lambda_A; \alpha_0)$ or $\lambda_A(0) \geq \alpha_0$ which is a contradiction. Hence $\lambda_A(0) \geq \lambda_A(x)$ for all $x \in X$. Next, we prove that $\lambda_A(x) \geq T(\lambda_A(x * y), \lambda_A(y))$ for all $x, y \in X$. If not, then there exist $x_0, y_0 \in X$ such that $\lambda_A(x_0) \geq T(\lambda_A(x_0 * y_0), \lambda_A(y_0))$. Taking $\alpha_0 = \frac{1}{2}(\lambda_A(x_0) + T(\lambda_A(x_0 * y_0), \lambda_A(y_0)))$ then $\alpha_0 < \lambda_A(x_0)$ and $0 \leq T(\lambda_A(x_0 * y_0), \lambda_A(y_0)) < \alpha_0 \leq 1$. Thus, we have $\alpha_0 > \lambda_A(x_0 * y_0)$ and $\alpha_0 > \lambda_A(y_0)$ which imply that $x_0 * y_0 \in U(\lambda_A; \alpha_0)$ and $y_0 \in U(\lambda_A; \alpha_0)$. As $U(\lambda_A; \alpha_0)$ is an ideal of X, it follows that $x_0 \in U(\lambda_A; \alpha_0)$ or $\lambda_A(x_0) \geq \alpha_0$, which is a contradiction. Similarly, $\mu_A(x) \geq \lambda_A(0)$ and $\mu_A(x) \leq S(\mu_A(x * y), \mu_A(y))$ for all $x, y \in X$. This completes the proof.

Theorem 3.4. *Let A be an (S, T)-anti intuitionistic fuzzy ideal of a BCC-algebra X. Two upper level cuts $U(\lambda_A; \alpha_0)$ and $U(\lambda_A; \alpha_1)$ with $\alpha_0 > \alpha_1$ (resp. two lower level cuts $V(\mu_A; \beta_0)$ and $V(\mu_A; \beta_1)$ with $\beta_1 > \beta_0$) are equal if and only if there exists no $x \in X$ such that $\alpha_0 \geq \lambda_A(x) > \alpha_1$ (resp. $\beta_1 \geq \mu_A(x) > \beta_0$).*

Proof. From the definition of upper level cut, it follows that $U(\lambda_A; \alpha) = \lambda_A^{-1}([\alpha, \lambda_A(0)])$ for $\alpha \in [0, 1]$. Let $\alpha_0, \alpha_1 \in [0, 1]$ such that $\alpha_0 > \alpha_1$. Then

$$\begin{aligned} U(\lambda_A; \alpha_0) &= U(\lambda_A; \alpha_1) \Longleftrightarrow \lambda_A^{-1}([\alpha_0, \lambda_A(0)]) = \lambda_A^{-1}([\alpha_1, \lambda_A(0)]) \\ &\Longleftrightarrow \lambda_A^{-1}((\alpha_1, \alpha_1]) = \varnothing \\ &\Longleftrightarrow \text{there is no } x \in X \text{ such that } \alpha_0 \geq \lambda_A(x) > \alpha_1. \end{aligned}$$

This completes the proof.

References

[1] M.T. Abu Osman, On some product of fuzzy subgroups, Fuzzy Sets and Systems 24(1987), 79-86.

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Systems 20(1986), 87-96.

[3] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets Systems 61(1994), 137-142.

[4] R. Biswas, Fuzzy subgroups and anti fuzzy subgroups, Fuzzy Sets and Systems 35(1990), 121-124.

[5] D. Dubois, H. Prade, New results about properties and semantics of fuzzy set-theoretic operators, In: P.P. Wang, S.K. Chang editors. Fuzzy sets: theory and applications to policy analysis and information systems, New York, Plenum Press, 1980.

[6] W.A. Dudek, The number of subalgebras of finite BCC-algebras, Bull. Inst. Math. Acad. Sinica 20(1992), 129-136.

[7] W.A. Dudek, On proper BCC-algebras, Bull. Inst. Math. Acad. Sinica 20(1992), 137-150.

[8] W.A. Dudek, K.H. Kim, Y.B. Jun, Fuzzy BCC-subalgebras of BCC-algebras with respect to a t-norm, Sci. Math. 3(2000), 99-106.

[9] W.A. Dudek, Y.B. Jun, Fuzzy BCC-ideals in BCC-algebras, Math. Montisnigri 10(1999), 21-30.

[10] W.A. Dudek, Y.B. Jun, S.M. Hong, On fuzzy topological BCC-algebras, Discuss Math. Gen. Algebra Appl. 20(1)(2000), 77-86.

[11] W.A. Dudek, B. Davvaz, Y.B. Jun, On intuitionistic fuzzy sub-hyperquasigroups of hyperquasigroups, Inform. Sci. 170(2005), 251-262.

[12] O. Hadzic, E. Pap, Fixed point theory in probabilistic metric spaces, Dordrecht, Kluwer Acad. Publishers, 2001.

[13] S.M. Hong, Y.B. Jun, Anti fuzzy ideals in BCK-algebras, Kyungpook Math. J. 38(1998), 145-150.

[14] K. Iseki, BCK-algebras with condition (S), Math. Japon. 24(1)(1979), 107-119.

[15] Y.B. Jun, Q. Zhang, Fuzzy subalgebras of BCK-algebras with respect to a t-norm, Far East J. Math. Sci. 2(3)(2000), 489-495.

[16] E.P. Klement, Operations on fuzzy sets: an axiomatic approach, Inform. Sci. 27(1984), 221-232.

[17] E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Dordrecht, Kluwer Acad. Publishers, 2000.

[18] Y. Komori, The class of BCC-algebras is not a veriety, Math. Japon. 29(1984), 391-394.

[19] S. Kutukcu, C. Yildiz, Intuitionistic fuzzy BCC-subalgebras of BCC-algebras, J. Concr. Appl. Math., in press.

[20] R. Lowen, Fuzzy set theory, Dordrecht, Kluwer Acad. Publishers, 1996.

[21] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. 28(1942), 535-537.

[22] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10(1960), 314-334.

[23] O.G. Xi, Fuzzy BCK-algebras, Math. Japon. 36(5)(1991), 935-942.

[24] L.A. Zadeh, Fuzzy sets, Inform. Control 8(1965), 338-353.

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A KANTOROVICH ANALYSIS OF NEWTON METHODS ON LIE GROUPS

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Abstract

A local as well as a semilocal Kantorovich-type convergence analysis is provided for Newton methods (Newton's method and Modified Newton's method) to solve equations on Lie groups. Motivated by optimization considerations and by using more precise majorizing sequences than before [6], [9], [10], we show that under the same or weaker hypotheses: a larger convergence domain; finer error estimates on the distances involved can be obtained; an at least as precise information on the location of the solution is given semilocal case, and a larger radius of convergence (in the local case).

We also note that our results are obtained under the same computational cost as in [6], [9], [10]. Finally the results are extended to the Hölder case not examined before.

AMS (MOS) subject classification codes: 65J15, 65G99, 65B05, 65L50, 47H17, 49M15.

Key Words: Newton's method, Modified Newton's method, Lie groups, Abelian groups, majorizing sequence, local/semilocal convergence, Kantorovich hypothesis, radius of convergence, Lipschitz conditions.

1 Introduction

In this study we are concerned with the problem of approximating a locally unique zero x^* of a map f defined on a Lie grup (to be precised in section 1). Numerical algorithms on manifolds are very important in computational mathematics [2], [5], [6], [9], [10], because they appear in connection to eigenvalue problems, minimization problems, optimization problems. A convergence analysis of Newton's method on Riemannian manifolds under various condition similar to the corresponding ones on Banach spaces [1],[4], [7], [8] has been given in [9], [10] and the references there.

Here we are motivated in particular by the elegant work in [9], and optimization considerations with advantages over earlier works [6], [9], [10] as already mentioned in the abstract of the paper.

2 Preliminaries

A Lie group (G,·) is a Hausdorff topological group with countable bases which also has the structure of a smooth manifold such that the group product and the inversion are smooth operations in the differentiable structure given on the manifold. The dimension of a Lie group is that of the underlying manifold, and we shall always assume that it is finite. The symbol e designates the identity element of G. Let g be the Lie algebra of the Lie group G which is the tangent space T_eG of G at e, equipped with Lie bracket $[\cdot,\cdot] : g \times g \to g$. In the sequel we will make use of the left translation of the Lie group G. We define for each $y \in G$

$$\begin{aligned} L_y &: G \to G \\ z &\to y \cdot z, \end{aligned}$$

the left multiplication in the group. The differential of L_y at e denoted by $(dL_y)_e$ determines an isomorphism of $g = T_eG$ with the tangent space T_yG via the relation

$$(dL_y)_e(g) = T_yG$$

or, equivalently,

$$g = (dL_y)_e^{-1}(T_yG) = dL_{y-1})_y(T_yG).$$

The exponential map is a map

$$\begin{aligned} \exp &: g \to G \\ u &\to \exp(u), \end{aligned}$$

which is certainly the most important construct associated to G and g. Given $u \in g$, the left invariant vector field $X_u : y \to (dL_y)_e(u)$ determines an one-parameter subgroup of G $\sigma_u : \mathbb{R} \to G$ such that $\sigma_u(0) = e$ and

$$\sigma_u'(t) = X_u(\sigma_u(t)) = (dL_{\sigma_u(t)})_e(u).$$

The exponential map is then defined by the relation

$$\exp(u) = \sigma_u(1).$$

Note that the exponential map is not surjective in general. However, the exponential map is a diffeomorphism on an open neighborhood $\mathcal{N}(\prime)$ of $0 \in g$. Let

$$N(e) = \exp(\mathcal{N}(0)).$$

Then for each $y \in N(e)$, there exists $\upsilon \in \mathcal{N}(0)$ such that $y = \exp(\upsilon)$. Furthermore, if

$$\exp(u) = \exp(\upsilon) \in N(e)$$

for some $u, \upsilon \in \mathcal{N}(0)$, then $u = \upsilon$. If G is Abelian, exp is also a homomorphism from g to G, i.e.,

$$\exp(u + \upsilon) = \exp(u) \cdot \exp(\upsilon) \tag{1}$$

for all $u, v \in g = T_eG$. In the non-abelian case, exp is not a homomorphism and (1) must be replaced by

$$\exp(\omega) = \exp(u) \cdot \exp(v),$$

where ω is given by the Baker-Campbell-Hausdorff (BCH) formula

$$\omega = u + v + \frac{1}{2}[u, v] + \frac{1}{12}([u[u, v]] + [v[v, u]]) + ...,$$

for all u, v in an open neighborhood of $0 \in g$. To analyse convergence, we need a Riemannian metric on the Lie group G.

Following [5] take an inner product $\langle , \rangle_e$ on g and define

$$\langle u, v\rangle_x = \langle (dL_{x-1})_x(u), (dL_{x-1})_x(v)\rangle_e, \quad \text{for each } x \in G \quad \text{and} \quad u, v \in T_xG.$$

This construction actually produces a Riemannian metric on the Lie group G, see for example [5]. Let $\|\cdot\|_x$ *be* associated norm, where the subscript x is sometimes omitted if there is no confusion. For any two distinct elements x, $y \in G$, let $c : [0, 1] \to G$ be a piecewise smooth curve connecting X and y. Then the arc-length of c is defined by $l(c) := \int_0^1 \|c'(t)\| \, dt$, and the distance from x to y by $d(x, y) := inf_c l(c)$, where the infimum is taken over all piecewise smooth curves $c : [0, 1 \to G$ connecting x and y. Thus, we assume throughout the whole paper that G is connected and hence (G, d) is a complete metric space. Since we only deal with finite dimensional Lie algebras, every linear mapping $\varphi : g \to g$ is bounded and we define its norm by

$$\|\varphi\| = \sup_{u \neq 0} \frac{\|\varphi(u)\|}{\|u\|} = \sup_{\|u\|=1} \|\varphi(u)\| < \infty.$$

For $r > 0$ we introduced the corresponding ball of radius r around $y \in G$ defined by one parameter subgroups of G as

$$C_r(y) = \{z \in G : z = y \cdot \exp(u), \|u\| \leq r\}.$$

We give the following definition on convergence:

Definition 1 *Let $\{x_n\}_{n\geq 0}$ be a sequence of G and $x \in G$. Then $\{x_n\}_{n\geq 0}$ is said to be*

(i) *convergent to x if for any $\varepsilon > 0$ there exists a natural number K such that $x^{-1} \cdot x_n \in N(e)$ and $\left\|\exp^{-1}(x^{-1} \cdot x_n)\right\| \leq \varepsilon$ for all $n \geq K$;*

(ii) *quadratically convergent to x if $\{\left\|\exp^{-1}(x^{-1} \cdot x_n)\right\|\}$ is quadratically convergent to* 0; *that is, $\{x_n\}_{n\geq 0}$ is convergent to x and there exists a constant q and an natural number K such that*

$$\left\|\exp^{-1}(x^{-1}) \cdot x_{n+1})\right\| \leq q \left\|\exp^{-1}(x^{-1} \cdot x_n)\right\|^2 \quad \textit{for all} \quad n \geq K.$$

Note that convergence of a sequence $\{x_n\}_{n\geq 0}$ in G to x in the sense of Definition 1 above is equivalent to that $\lim_{n\to+\infty} d(x_n, x) = 0$.

In the remainder of this paper, let $f : G \to g = T_eG$ be a C^1 - mapping. The differential of f at a point $x \in G$ is a linear map $f_x^{'} : T_xG \to g$ defined by

$$f_x'(\triangle_x) = \frac{d}{dt} f(x \cdot \exp(t((d_{x-1})_x)(\triangle_x))) \,|_{t=0} \quad \text{for any} \quad \triangle_x \in T_xG. \tag{2}$$

The differential f_x' can be expressed via a function $df_x : g \to g$ given by

$$df_x = (f \circ L_x)_e' = f_x' \circ (dL_x)_e.$$

Thus, by (8), it follows that

$$df_x(u) = f_x'((dL_x)_e(u)) = \frac{d}{dt} f(x \cdot \exp(tu) \,|_{t=0} \quad \text{for any} \quad u \in g.$$

Therefore the following lemma is clear.

Lemma 2 *Let $x \in G$, $u \in g$ and $t \in \mathbb{R}$. Then*

$$\frac{d}{dt} f(x \cdot \exp(-tu)) = -df_{x\cdot\exp(-tu)}(u) \tag{3}$$

and

$$f(x \cdot \exp(tu)) - f(x) = \int_0^t df_{x\cdot\exp(su)}(u)ds. \tag{4}$$

As in [10] Newton's method for f with initial point $x_0 \in G$ is defined as follows

$$x_{n+1} = x_n \cdot \exp(-df_{x_n}^{-1} \circ f(x_n)) \quad (n \geq 0). \tag{5}$$

We also define the modified Newton's method by

$$x_{n+1} = x_n \exp(-df_{x_0}^{-1} f(x_n) \quad (n \geq 0). \tag{6}$$

3 Local convergence analysis of Newton's method (5)

We will use the following definition involving Lipschitz conditions:

Definition 3 *Let $r > 0$, and let $x_0 \in G$ be such that $df_{x_0}^{-1}$ exists. Then $df_{x_0}^{-1} df$ is said to satisfy: the center Lipschitz condition with constant $\ell_0 > 0$ in $C(x_0, r)$ if*

$$\left\| df_{x_0}^{-1}(df_{x_0\exp(u)} - df_{x_0}) \right\| \leq \ell_0 \left\| u \right\|, \quad \textit{for each } u \in g \textit{ with } \left\| u \right\| \leq r; \tag{7}$$

the center Lipschitz condition with constant ℓ in $C(x_0, r)$ if

$$\left\| df_{x_0}^{-1}(df_{x\cdot\exp(u)} - df_x) \right\| \leq \ell \left\| u \right\| \tag{8}$$

holds for any $u, v \in g$ and $x = x_0 \exp(\tau)$ with $\|u\| + \|v\| \leq r$.

Remark 4 . *In general*

$$\ell_0 \leq \ell, \tag{9}$$

holds, and $\frac{\ell}{\ell_0}$ *can be arbitrarily large [1],[4].*

We can show the following local convergence results for Newton's method (5).

Theorem 5 *Assume that* G *is un Abelian group. Choose* $r \in (0, \frac{2}{2\ell_0+\ell})$, *and let* $x^* \in G$ *such that* $f(x^*) = 0$ *and* $df_{x^*}^{-1}$ *exists. Moreover assume* $df_{x^*}^{-1} df$ *satisfies condition (8).*

Then sequence $\{x_n\}$ *generated by Newton's method (5) is well defined, remains in* $C(x^*, r)$ *for all* $n \geq 0$, *and converges quadratically to* x^* *provided that* $x_0 \in C(x^*, r)$, *with ratio* α *given by*

$$\alpha = \frac{\ell}{2(1 - \ell_0 \|u_0\|}, \tag{10}$$

where $u_0 \in g$ *with* $\|u_0\| \leq r$, *and* $x_0 = x^* \exp(u_0)$.

Proof. Set $\alpha_0 = \alpha \|u_0\|$. In view of (10) $\alpha_0 \in [0, 1)$. We shall show using induction that for each $n \geq 0$, x_n is well-defined, remains in $C(x^*, r)$, and there exists $u_n \in g$ with $\|u_n\| \leq r$ such that

$$x_n = x^* \exp(u_n), \quad \text{and} \quad \|u_{n+1}\| \leq \alpha \|u_n\|^2 \leq \alpha_0^{2^{n+1}-1} \|\upsilon_0\|. \tag{11}$$

Estimates (11) hold true for $n = 0$ by the initial conditions. Assume estimates (11) hold true for $n \leq k$, x_n is well defined and there exist $U_n \in g$ with $\|u_n\| \leq r$ such that (11) hold. Using (7) we get

$$\left\| df_{x^*}^{-1}(df_{x^* \exp(u_k)} - df_{x^*}) \right\| \leq \ell_0 \|u_k\| < 1. \tag{12}$$

It follows from the Banach Lemma [8] $df_{x_k}^{-1}$ exists and

$$\left\| df_{x_k}^{-1} df_{x^*} \right\| \leq \frac{1}{1 - \ell_0 \|u_k\|}. \tag{13}$$

That is x_{k+1} is well defined. Set

$$u_{k+1} = u_k - df_{x_k}^{-1}(f(x_k)). \tag{14}$$

In view of (13) and (14) we get in turn

$$\begin{aligned} \|u_{k+1}\| &= \left\| u_k - df_{x_k}^{-1}(f(x_k) - f(x^*)) \right\| \qquad (15) \\ &\leq \left\| df_{x_k}^{-1} df_{x^*} \right\| \int_0^1 \left\| df_{x^*}^{-1}(df_{x_k} - df_{x^* \exp(tu_k)})u_k \right\| dt \\ &\leq \frac{1}{1 - \ell_0 \|u_k\|} \int_0^1 (1-t) l \|u_k\|^2 dt \\ &= \frac{2}{2(1 - \ell_0 \|u_k\|)} \|u_k\|^2 \leq \alpha \|u_k\|^2 \leq \alpha_0^{2^{k+1}-1} \|u_0\| = r, \end{aligned}$$

which establishes the quadratic convergence and $\|u_{k+1}\| \leq \tau$. Moreover since G is an Abelian group

$$x_{k+1} = x^* \exp(u_k) \exp[-df_{x_k}^{-1} f(x_k)] = x^* \exp(u_{k+1}). \tag{16}$$

That completes the induction and the proof of the theorem. ■

Remark 6 *If $\ell_0 = \ell$, 5 reduces to Theorem 3.1 in [9]. Otherwise it is an improvement. Indeed let τ_{WL} be the corresponding radius of convergence in [9] selected in $(0, \frac{2}{3l})$. Then since $(0, \frac{2}{3l}) \subseteq (0, \frac{2}{2l_0+l})$ it follows our radius r_A is such that*

$$r_{WL} < r_A. \tag{17}$$

Hence, our approach allows a wider choice of initial guesses x_0. Moreover the ratio is smaller than the corresponding one $\overline{\alpha}$ in [9] (simply let $\ell_0 = \ell$ in (10) to obtain $\overline{\alpha}$),since we have

$$\alpha < \overline{\alpha}. \tag{18}$$

The uniqueness of the solution x^* is discussed next.

Proposition 7 *Let $r \in (0, \frac{2}{l_0})$. Assume $f(x^*) = 0$ and $df_{x^*}^{-1} df$ satisfies (7) in $U(x^*, r)$. Then x^* is the unique zero of f in $U(x^*, r)$.*

Proof. Let y^* be a zero of f in $U(x^*, \tau)$. It follows that there exists $u \in g$ so that $y^* = x^* \exp(u)$ and $\|u\| \leq \tau$. We can have in turn

$$\begin{aligned} \|u\| &= \left\| -df_{x^*}^{-1}(f(y^*) - f(x^*)) + u \right\| \\ &= \left\| -df_{x^*}^{-1} \int_0^1 df_{x^* \exp(tu)}(u)dt + u \right\| \\ &= \left\| -df_{x^*}^{-1} \int_0^1 (df_{x^* \exp(u)} - df_{x^*})udt \right\| \\ &\leq \int_0^1 t\ell_0 \|u\|^2 dt = \frac{\ell_0}{2} \|u\|^2 . \end{aligned} \tag{19}$$

In view of (19) we deduce $\|u\| \geq \frac{2}{\ell_0}$. Hence, we arrived at a contradiction.

That completes the proof of the Proposition. ■

4 Semilocal convergence analysis of Newton's method (5)

Our semilocal convergence analysis of method (5) depends on the scalar sequence $\{s_n\}$ $(n \geq 0)$ introduced by us in [1], [4]:

$$s_0 = 0,\ s_1 = n,\ s_{n+2} = s_{n+1} + \frac{L(s_{n+1} - s_n)^2}{2(1 - L_0 s_{n+1})}\ (n \geq 0) \tag{20}$$

for some $L_0 > 0$, $L > 0$ with $L_0 \leq L$ and $\eta > 0$. Sufficient convergence conditions for majorizing sequence $\{s_n\}$ we given in [1],[4]. Here we summarize the conditions:

$$h_\delta = (L + \delta L_0)\eta \leq \delta,\ \delta \in [0,1]. \tag{21}$$

or

$$h_\delta \leq \delta,\ \delta \in [0,2), \tag{22}$$

$$\frac{2L_0\eta}{2-\delta} \leq 1 \tag{23}$$

and

$$\frac{L_0\delta^2}{2-\delta} \leq L \tag{24}$$

or

$$h_\delta \leq \delta,\ \delta \in [\delta_0, 2) \tag{25}$$

where,

$$\delta_0 = \frac{-b\sqrt{b^2+8b}}{2},\ b = \frac{L}{L_0}. \tag{26}$$

Under any of the above conditions $\{s_n\}$ converges (increasingly) to some $s^* \in (0, \frac{2\eta}{2-\delta}]$. Iteration $\{s_n\}$ coincides for $L_0 = L$ with iteration $\{t_n\}$ used in [9]:

$$t_0 = 0,\ t_1 = n,\ t_{n+2} = t_{n+1} + \frac{L(t_{n+1} - t_n)}{2(1 - Lt_{n+1})},\ \ (n \geq 0) \tag{27}$$

and has been compared favorably with it when $L_0 < L$. Indeed we showed in [1],[4]:

$$s_n < t_n\ (n \geq 2), \tag{28}$$

$$s_{n+1} - s_n < t_{n+1} - t_n,\ (n \geq 2), \tag{29}$$

$$s^* \leq t^* = \frac{1 - \sqrt{1-2h}}{L}, \tag{30}$$

and

$$s^* - s_n \leq t^* - t_n,\ \ (n \geq 0), \tag{31}$$

provided that any of (21) or (22)-(24) or (25)-(26) and the famous Newton-Kantorovich condition [8]

$$h = 2L\eta \leq 1 \tag{32}$$

hold. Note that,

$$h \leq 1 \Longrightarrow h_1 \leq 1 \tag{33}$$

but not vice versa unless if $L_0 = L$.

We need definitions corresponding to Definition 3 above. Let us first introduce the metric closed ball of radius $r > 0$ about $y \in G$ denoted by

$$\overline{U}(y, \tau) = \{z \in G : d(z, y) \leq r\}. \tag{34}$$

Note that

$$C(y, r) \subseteq U(y, r). \tag{35}$$

Definition 8 *Let $r > 0$, and let $x_0 \in G$ be such that $df_{x_0}^{-1}$ exists. Then $df_{x_0}^{-1} df$ is said to satisfy: the center Lipschitz condition which constant L_0 in $U(x_0, r)$ if*

$$\left\| df_{x_0}^{-1}(df_x - df_{x_0}) \right\| \leq L_0 d(x_0, x), \quad \textit{for all } x \in U(x_0; r); \tag{36}$$

the Lipschitz condition in the inscribed sphere with constant L in $U(x_0, r)$ if

$$\begin{aligned} &\left\| df_{x_0}^{-1}(df_y - df_x) \right\| \leq L d(x, y) \textit{ holds for all } x, y \in U(x_0, r) \textit{ with} \\ &d(x_0, x) + d(x, y) \leq r. \end{aligned} \tag{37}$$

We can show the main semilocal convergence result for Newton's method (5):

Theorem 9 *. Let $x_0 \in G$ be such that $df_{x_0}^{-1}$ exists and set $n = \left\| df_{x_0}^{-1}(f(x_0)) \right\|$. Assume that either (21) or (22)-(24) or condition (25) hold. Moreover, assume $df_{x_0}^{-1} df$ satisfies (35) and (36). Then sequence $\{x_n\}$ generated by Newton's method (5) is well defined, remains in $U(x_0, \tau, s^*)$ for all $n \geq 0$ and converges to n zero s^* of f in $\overline{U}(x_0, s^*)$.Moreover, the following estimates hold for all $n \geq 0$:*

$$d(x_{n+1}, x_n) \leq s_{n+1} - s_n, \tag{38}$$

and

$$d(x_n, x^*) \leq s^* - s_n. \tag{39}$$

Furthermore, if G is an Abelian group, then there is n zero s^* of f in $C(x_0, s^*)$ such that for all $n \geq 0$, there exists $u_n \in g$ such that $x_n = x^* \exp(u_n)$, and for al $n \geq 1$

$$\|u_n\| \leq \frac{L(s^* - t_{n-1})}{2(1 - L_0 t_{n-1})} \left(\frac{\|u_n\|}{s^* - t_{n-1}} \right)^2. \tag{40}$$

Proof. We shall show

$$d(x_{n+1}, x_n) \leq \|v_n\| \leq t_{n+1} - t_n, \tag{41}$$

where, $v_n = -df_{x_n}^{-1} f(x_n)$, $(n \geq 0)$.

Let us define the curve $c_0(t) = x_0 \exp(tv_0)$, $t \in [0, 1]$. Then c_0 is smooth and connects x_0 to x_1 with *leng* $(c_0) = \|v_0\|$. That is, $d(x_1, x_0) \leq$ *leng* $(c_0) = \|v_0\|$. That is, $d(x_1, x_0) \leq leng(c_0) = \|vd\|$. In view of $\|v_0\| = \left\| -df_{x_0}^{-1} f(x_0) \right\| \leq \eta \leq s_1 - s_0$, (40) holds true for $n = 0$. We assume (40) to hold true for $n = 0, 1, ..., k-1$.

It follows

$$d(x_k, x_0) \leq \sum_{i=0}^{k-1} d(x_{i+1}, x_i) \leq \sum_{i=0}^{k-1} \|v_i\| \leq s_k - s_0 = s_k < s^*. \tag{42}$$

That is $x_k \in U(x_0, s^*)$. As in (13) but using (35) instead of (7) we deduce $df_{x_k}^{-1}$ exists and

$$\left\| df_{x_k}^{-1} df_{x_0} \right\| \leq \frac{1}{1 - L_0 s_k}. \tag{43}$$

In view of (5), x_{k+1} is well defined. Using (5), (36), and (42) we obtain in turn:

$$
\begin{aligned}
\left\|df_{x_0}^{-1}f(x_k)\right\| &\leq \int_0^1 \left\|df_{x_0}^{-1}[df_{x_{k-1}}\exp(tv_{k-1})-df_{x_{k-1}}]\right\|\left\|v_{k-1}\right\|dt \\
&\leq \int_0^1 Ld(x_{k-1},x_{k-1}\exp(tv_{k-1}))\left\|v_{k-1}\right\|dt \\
&\leq \int_0^1 L\left\|tv_{k-1}\right\|\left\|v_{k-1}\right\|dt \\
&\leq \frac{L}{2}(s_k-s_{k-1})^2,
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
\left\|v_k\right\| &= \left\|df_{x_k}^{-1}df_{x_0}df_{x_0}^{-1}f(x_k)\right\| \\
&\leq \left\|-df_{x_k}^{-1}df_{x_0}\right\|\left\|df_{x_0}^{-1}f(x_k)\right\| \\
&\leq \frac{L(s_k-s_{k-1})^2}{2(1-L_0-s_{k-1})}=s_{k+1}-s_k,
\end{aligned} \tag{45}
$$

which also shows (37).We define the curve c_k (as c_0 above) by $c_k(t)=x_k\exp(tv_k)t\in[0,1]$.As above we have $d(x_{k+1},x_k)\leq leng(c_k)=\left\|v_k\right\|$. That completes the induction for (40). It follows that sequence $\{x_n\}$ is Cauchy and as such it converges to some $x^*\in\overline{U}(x_0,s^*)$ (since $\overline{U}(x_0,s^*)$ is a closed set). By letting $k\to\infty$ in (43) we obtain $f(x^*)=0$. Moreover (38) follows from (37) by using standard majorizations techniques. Define

$$
u_n=-\sum_{k=n}^{\infty}v_k\ (n\geq 0). \tag{46}
$$

It follows by (40) that

$$
\left\|u_n\right\|\leq s^*-s_n\ (n\geq v). \tag{47}
$$

Let $x^*=x_0\exp(-u_0)$. Then we have $x^*\in C(x_0,s^*)$. Moreover, we get

$$
x_k=x_0\prod_{i=0}^{k-1}\exp(v_i)=x_0\exp\left(\sum_{i=0}^{k-1}v_i\right).
$$

It follows that clearly $x_n=x^*\exp(u_n)$. That is sequence $\{x_n\}$ converges to x^* which is a zero of f in $C(x_0,s^*)$.To complete the proof we must show (39). Estimate (39) holds for $n=0$ by the initial conditions. Assume that (39) holds for $n \leq k$. We can have in turn

$$\begin{aligned} \|u_{k+1}\| &= \left\|u_k - df_{x_k}^{-1} f(x_k)\right\| \\ &= \left\| df_{x_k}^{-1} df_{x_0} df_{x_0}^{-1} \int_0^1 \left[df_{x_0}^{-1} (df_{x_k} - df_{x^* \exp(tu_k)}) u_k dt \right\| \\ &\leq \frac{1}{1 - L_0 s_k} \int_0^1 L(1-t) \|u_k\|^2 dt \\ &= \frac{L(s^* - s_k)}{2(1 - L_0 s_k)} \left(\frac{\|_k\|}{s^* - s_k} \right)^2 . \end{aligned} \tag{48}$$

That completes the proof of the Theorem. ■

Remark 10 *In view of (35) and (36) it follows that*

$$L_0 \leq L \tag{49}$$

holds in general and $\frac{L}{L_0}$ *can be arbitrarily large. If* $L_0 = 2$ *our results can be reduced to the corresponding ones in [9] (simply replace* $\frac{2n}{2-5}$ *by* $\frac{1+\sqrt{1-2h}}{L}$*). Otherwise according to the discussion above 8 the constitute an improvement in the sense that under weaker (or the same hypotheses; as the hypotheses in [9] but simply replace sequence* $\{t_n\}$ *by* $\{s_n\}$ *we provide finer error estimates on the distances* $d(x_{n+1}, x_n)$, $d(x_n, x^*)$ $n \geq 0$ *and an at least as precise information on the location of the solution. Note also that the above advantages are obtained under the same computational cost since in practice computing L requires computing* L_0.

5 Convergence of the modified Newton method

Let us consider the convergence of modified method (6). Using only (7) and simply replacing $h = 2L\eta \leq 1$, $t^*, t^{**} = \frac{1+\sqrt{1-2h}}{L}, L$ by $h_A = 2L_0\eta \leq 1$, $s^*, \frac{2\eta}{2-\delta}, L_0$ respectively in the proofs we obtain respectively the corresponding improvements of Proposition 4.1, Lemma 4.1 and Theorem 4.2 given in [9]:

Proposition 11 *Let* $t_0 \in \left[0, \frac{2\eta}{2-\delta}\right]$ *.The following statements hold true:*

(a) *for each* $n \geq 0$, $t_n \in [0, s^*]$ *if* $t_0 \in [0, s^*]$ *if* $t_0 \in [0, s^*]$, *and* $t_n \in [s^*, \frac{2\eta}{2-\delta})$ *and*

(b) *Sequence* $\{s_n\}$ *converges (increasingly) to* s^*.

Lemma 12 *. Let G be an Abelian group Assume there exists* $x_0 \in G$ *such that* $df_{x_0}^{-1}$ *exists and*

$$h_A = 2L_0\eta \leq 1. \tag{50}$$

Moreover assume condition (7) holds converges to a zero x^* *of* f *in* $U(x_0, s^*)$. *Then sequence* $\{x_n\}$ *generated by modified Newton method (6) converges to a zero* x^* *of* f *in* $U(x_0, s^*)$.

Theorem 13 *Assume that G is an Abelian group. Assume conditions (7) and (50) hold true. Let $r \in [s^*, \frac{2\eta}{2-\delta})$ if $h_A < 1$ and $r = s^*$ if $h_A = 1$. Then there exists a unique zero of f in $\overline{U}(x_0, r)$.*

Remark 14 *. The results obtained here are immediately extended to the Hölder case with exponent $\gamma \in (0,1)$. We showed in [3, Lemma 2]: If there exist parameters $L \geq 0$, $L_0 \geq 0, \eta \geq 0$, $\gamma \in [0,1)$, and $q \in [0,1)$ with η and not zero at the same time with*

$$h_\gamma = \left[L + \frac{\overline{\delta} L_0}{(1-q)^\gamma}\right] \eta^\gamma \leq \overline{\delta},\ \overline{\delta} = (1+\gamma)q, \tag{51}$$

then majorizing sequence $\{\omega_n\}$ $(n \geq 0)$ given by

$$w_0 = 0, \omega_1 = n,\ \omega_{n+2} = \omega_{n+1} + \frac{L}{(1+\gamma)[1 - L_0 \omega_{n+1}^\gamma]} (\omega_{n+1} - \omega_n)^\gamma, \tag{52}$$

is nondecreasing, bounded above by $\omega^{**} = \frac{\eta}{1-q}$ and converges to some $\omega^* \leq \omega^{**}$. It then follows that the semilocal convergence results obtained in Section 3 for Newton's method (5) hold true if $\{s_n\}$, (21) (or (22)-(24) ore (25), (39), s^*, $\frac{2\eta}{2-\delta}$ are replaced by (51), (52),$\frac{L(\omega^*-\omega_{n-1})}{(1+\gamma)(1-L_0\omega_{n-1}^\gamma)}$ $\frac{\|u_n\|^{1+\gamma}}{\omega^*-\omega_{n-1}}$), ω^*, ω^{**} respectively. Similarly for the results in Section 4, where $h_A, s^*, \frac{2\eta}{2-\delta}$ are replaced by $h_{A,\gamma} = 2L_0 n^\gamma, \omega^*, \omega^{**}$ respectively. In the local case the results of section 2 hold true for $r \in \left(0, \frac{2}{2\ell_0+\ell}\right)$, convergence order 2, α, $r \in \left(0, \left(\frac{1+\lambda}{\ell+(1+\lambda)\ell_0}\right)^{\frac{1}{\lambda}}\right)$, order $1+\lambda$, $\frac{\ell}{(1+\lambda)(1-\ell_0\|u_0\|)^\lambda}$, $r \in \left(\frac{1+\lambda}{\ell_0}\right)$ respectively.

The advantages of our approach in this Section over Section 4 in [9] have already been explained in Section 3 (for $L_0 < L$). Moreover for $L_0 = L$, our results can also be reduced to the corresponding ones in [9].

References

[1] Argyros, I.K., A unifying local-semilocal convergence analysis for two-point Newton-Like methods in Banach space, J. Math. Anal. Applic. 298, (2004), 374-397.

[2] Argyros, I.K., On the Newton-Kantorovich method in Riemannian manifolds, Adv. Non. Var. Ineq. 8, 2, (2005), 81-85.

[3] Argyros, I.K., Concerning the "terra incognita" between convergence regions of Newton's methods, Non. Anal. 62, (2005), 179-194.

[4] Argyros, I.K., Approximate solution of operator equations with applications, World Scientific Publ. Comp. Hackenscack, New Jersey, 2005, USA.

[5] DoCarmo, M.P., Riemannian Geometry, Birkhauser, 1992, Boston, USA.

[6] Ferreira ,O.P. and Svaiter, B.F., Kantorovich's theorem on Newton's method in Riemannian manifolds, J. Complexity, 18, (2002), 304-329.

[7] Gutierrez, J.M., Hernandez, M.A. and Salanova, M.A., Accesssibility of solutions by Newton's method, Intern. J. Comput. Math. 57, (1995), 239-247.

[8] Kantorovich, L.V. and Akilov, G.P., Functional analysis in normed spaces, Pergamon Press, Oxford, 1982.

[9] Li, C. and Wang, J.H., Kantorovich's theorem for Newton's method on Lie groups, to appear in the Journal of Science A and B, Zheijiang University.

[10] Owren, B. and Welfert, B., The Newton Iteration on Lie Groups, BIT, 40, 2, (200), 121-145.

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ON THE SEMI-LOCAL CONVERGENCE OF A NEWTON-TYPE METHOD IN BANACH SPACES UNDER A GAMMA-TYPE CONDITION

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Abstract

The semi-local convergence of Newton-type method used to solve nonlinear sequations in a Banach space is studied. Recently, [3], [12] an interest has been shown for this type of methods [8], [10]. Here we show under gamma-type condition that the R-order of convergence of the method is $1+\sqrt{2}$. Numerical Examples are provided, including a nonlinear integral equation attributed to S. Chandrasekhar, a Nobel prize winner in astrophysics (1983).

AMS, (MOS) Subject Classification Codes: 65G99, 65K10, 47H17, 49M15.

Key Words: Newton-type method, Newton's method Banach space, semi-local convergence, Lipschitz condition, Fréchet-derivative, γ-condition

1 Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a twice-Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

We use the Newton-type method given for $x_0, y_0 \in D$ by

$$\begin{aligned} x_{n+1} &= x_n - F'(z_n)^{-1}F(x_n), \quad z_n = \frac{x_n + y_n}{2}, \ (n \geq 0), \\ y_{n+1} &= x_n - F'(z_n)^{-1}F(x_{n+1}) \end{aligned} \tag{1.2}$$

to generate a sequence $\{x_n\}$ $(n \geq 0)$ approximating x^*.

Let us illustrate how this method is conceived:
We start with the identity

$$F(x) - F(y) = \int_0^1 F'(y + t(x-y))dt(x-y), \quad \text{for all } x, y \in D. \tag{1.3}$$

If x^* is a solution of equation (1.1), then identity (1.3) gives

$$F(x) = \int_0^1 F'(x + t(x^* - x))dt(x^* - x). \tag{1.4}$$

The linear operator in (1.4) can be approximated in different ways [1]–[12]:
If for example

$$\int_0^1 F'(x + t(x^* - x)dt = F'(x), \text{ for all } x \in D, \tag{1.5}$$

then (1.4) suggests the famous quadratically convergent Newton's method [1]-[12]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad (n \geq 0). \tag{1.6}$$

Another choice is given by

$$\int_0^1 F'(x + t(F'(x + t(x^* - x) - x))dt = F'\left(\tfrac{x^*+x}{2}\right), \text{ for all } x \in D, \tag{1.7}$$

which leads to the implicit iteration:

$$x_{n+1} = x_n - F'\left(\tfrac{x_n + x_{n+1}}{2}\right)F(x_n), \quad (n \geq 0). \tag{1.8}$$

Unfortunately iterates in (1.8) can only be computed in very restrictive cases. Hence, we arrive at method (1.2). This method shall be shown to be of R-order $1 + \sqrt{2}$. That is method (1.2) is faster than Newton's given by (6).

However the semilocal convergence of method (1.2) not studied before especially under the γ-condition, is presented. The advantages of this approach over the Newton-Kantorovich have been explained in [3], [11].

2 Semilocal convergence of method (1.2)

Let $\beta \geq 0$, and $\gamma > 0$ be fixed. It is convenient for us to define function f on $\left[0, \frac{1}{\gamma}\right)$ by

$$f(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t}; \tag{2.1}$$

constants α, t^*, and t^{**} by

$$\alpha = \beta\gamma, \tag{2.2}$$

$$t^* = \frac{1 + \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}, \tag{2.3}$$

$$t^{**} = \frac{1+\alpha+\sqrt{(1+\alpha)^2-8\alpha}}{4\gamma}; \tag{2.4}$$

and scalar sequences $\{t_n\}$, $\{s_n\}$ $(n \geq 0)$ for $t_0 \geq 0$ and by some

$$s_0 \geq t_0 \tag{2.5}$$

$$\begin{aligned} t_{n+1} &= t_n - f'(r_n)^{-1} f(t_n),\ r_n = \tfrac{t_n+s_n}{2}, \\ s_{n+1} &= t_{n+1} - f'(r_n)^{-1} f(t_{n+1}). \end{aligned} \tag{2.6}$$

If

$$\alpha \leq 3 - 2\sqrt{2}, \tag{2.7}$$

then f has t^* and t^{**} as real roots, and

$$\beta \leq t^* \leq \left(1+\tfrac{1}{\sqrt{2}}\right)\beta \leq \left(1-\tfrac{1}{\sqrt{2}}\right)\frac{1}{\gamma} \leq t^{**},\ \ [11]. \tag{2.8}$$

Using (2.1), we have for all $t \in [0, t^*)$

$$f(t) > 0,\ f'(t) = \frac{1-2(1-\gamma t)^2}{(1-\gamma t)^2} < 0, \tag{2.9}$$

$$f''(t) = \frac{2\gamma}{(1-\gamma t)^3} > 0,\ \text{and}\ f'''(t) = \frac{6\gamma^2}{(1-\gamma t)^4} > 0. \tag{2.10}$$

We need the following lemma on majorizing sequences $\{t_n\}$, $\{s_n\}$ $(n \geq 0)$:

Lemma 2.1 *Under hypothesis (2.7), scalar sequences $\{t_n\}$, $\{s_n\}$ generated by (2.6) are well defined for all $n \geq 0$, and converge monotonically to t^* with*

$$0 \leq t_n \leq s_n \leq t_{n+1} < t^*. \tag{2.11}$$

Proof. We shall show that for all $k \geq 0$

$$0 \leq t_k \leq s_k \leq t_{k+1} < t^* \tag{2.12}$$

using induction. Estimate (2.12) holds true for $k = 0$ from the initial conditions, and t_1 is well defined. Assume (2.12) holds for all $k = 0, 1, ..., n-1$. Then there exists $u_n \in [t_{n+1}, t^*]$ such that

$$f(t_{n+1}) - f(t^*) = f'(u_n)(t_{n+1} - t^*). \tag{2.13}$$

In view of (2.6), and (2.13) we get

$$s_{n+1} - t^* = \tfrac{\gamma[(t_n-u_n)+(s_n-u_n)][(1-\gamma u_n)+(1-\gamma r_n)](t_{n+1}-t^*)}{2f'(r_n)[1-\gamma r_n]^2[1-\gamma u_n]^2} < 0, \tag{2.14}$$

which implies $s_{n+1} < t^*$. Clearly, we have, $s_{n+1} \leq t_{n+2}$. As in (2.14), we show $t_{n+1} < t^*$.

That completes the induction for estimate (2.11), and the proof of the Lemma. ■

We also need the following Lemma on the R-order of convergence:

Lemma 2.2 *[3], [12] Let* $0 < a_0,\ a_1 < 1,\ p > 1,\ q \geq 0,$ *and* $c \geq 0$ *be fixed. If a scalar sequence* $\{a_n\}$ *satisfies for all* $n \geq 0$

$$o < a_{n+1} \leq c a_n^p a_{n-1}^q, \tag{2.15}$$

then it converges to zero with R-order of convergence

$$\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}. \tag{2.16}$$

In order for us to establish the order of convergence of scalar sequences $\{t_n\}$, $\{s_n\}$ we need a result by W. Werner [12, p.337]:

Theorem 2.3 *Let* $F : D \subseteq X \to Y$ *be a twice Fréchet-differentiable operator. Assume:*

$$F'(x)^{-1} \in L(Y, X), \quad \textit{for all } x \in D;$$
$$\sup_{x \in D} \left\| F'(x)^{-1} \right\| \leq \Gamma; \tag{2.17}$$

for all $x, y \in D$

$$\|F'(x) - F'(y)\| \leq L_1 \|x - y\|, \tag{2.18}$$

$$\|F''(x) - F''(y)\| \leq L_2 \|x - y\|. \tag{2.19}$$

Set

$$A = \frac{1}{2}\Gamma L_1, \ \textit{and} \ \ B = \frac{\Gamma L_2}{24}.$$

Denote by r_0, r_1 *the unique solutions of equations*

$$B\rho_0^2 + A\rho_1 = 1$$
$$2A\rho_0^2 + A\rho_0\rho_1 = \rho_1$$

on the interval $\left(0, \frac{1}{A}\right)$.

Choose $x_0 \in U(x^*, r_0),\ y_0 \in U(x^*, r_1),$ *and assume*

$$U\left(x^*, \tfrac{1}{A}\right) \subseteq D, \ \textit{for } x^* \textit{such that } F(x^*) = 0$$

Set

$$a_n = \|x_n - x^*\|, \ \textit{and } b_n = \|x_n - x^*\|. \tag{2.20}$$

Then sequences $\{x_n\},\ \{y_n\}$ *generated by method (1.2) are well defined for all* $n \geq 0$, *and converge to* x^*.

Moreover the following estimates hold true:

$$b_{n+1} \leq A(2 + \tfrac{r_1}{r_0}) a_{n+1} a_n,$$
$$a_{n+1} \leq B a_n^2 a_{n-1} + A a_n b_n,$$

and

$$a_{n+1} \leq c a_n^2 a_{n-1} \ \textit{for some } c > 0.$$

Furthermore the R-order of convergence is $1 + \sqrt{2}$.

By (2.6) we obtain the identities

$$t_{n+1} - t^* = f'(r_n)^{-1}[f'\left(\tfrac{t_n+t^*}{2}\right)(t_n - t^*) - f(t_n) + f(t^*)] \\ + f(r_n)^{-1}\left[f'(r_n) - f'\left(\tfrac{t_n+t^*}{2}\right)\right](t_n - t^*)$$

and

$$s_{n+1} - t^* = f'(r_n)^{-1}[f'(r_n)(t_{n+1} - t^*) - f(t_{n+1}) + f(t^*)].$$

In view of the above identities we arrive at the corollary:

Corollary 2.4 *Let $\delta > \frac{1}{2}$, be fixed and restrict function f on $\left[0, \left(1 - \sqrt{\frac{\delta}{2\delta-1}}\right)\frac{1}{\gamma}\right] = D_0$.*

Set $X = Y = \mathbf{R}$, $D = D_0$,

$$\Gamma = \delta,\ L_1 = 4\gamma\sqrt{2},\ \text{and } L_2 = 24\gamma^2.$$

Assume condition (2.7) holds true.

Then scalar sequences $\{t_n\}$, $\{s_n\}$ are well defined, and converge to t^* with R-order of convergence $1 + \sqrt{2}$.

Proof. Using the definition of scalar function f on D_0 given by (2.1), we can easily see that (2.17)-2.19 hold true for the above choices of Γ, L_1, and L_2. Hence, the conclusions follow according to Theorem 2.3 and Lemma 2.2, since $p = 2$, and $q = 1$.

That completes the proof of the corollary. ■

We shall use the following definition of the γ-condition:

Definition 2.5 *Let $\gamma > 0$, x_0 and $y_0 \in D$ be fixed, and such that $F'(z_0)^{-1} \in L(Y, X)$. Assume that operator is thrice Fréchet-differentiable on D. We say that operator F satisfies the γ-condition if:*

$$\left\|F'(z_0)^{-1}F(x_0)\right\| = \beta = s_0 \tag{2.21}$$

$$\left\|F'(z_0)^{-1}F''(z_0)\right\| \leq 2\gamma \tag{2.22}$$

$$\left\|F'(z_0)^{-1}F'''(x)\right\| \leq \frac{6\gamma^2}{(1 - \gamma\left\|x - z_0\right\|)^4} \tag{2.23}$$

for all $x \in U(z_0, \overline{r})$,where,

$$\overline{r} = \left(1 - \tfrac{1}{\sqrt{2}}\right)\frac{1}{\gamma}, \tag{2.24}$$

and

$$U(z_0, \overline{r}) \subseteq D. \tag{2.25}$$

Note that function f defined by (2.1) satisfies the conditions of Definition 2.5.

We need the Lemmas:

Lemma 2.6 *Suppose F satisfies γ-condition and $y_0 \in U(z_0, \overline{r})$. Then the following estimates hold true:*

$$\left\|F'(z_0)^{-1}F''(x)\right\| \leq f''(\|x - z_0\|), \tag{2.26}$$

$$F'(x)^{-1} \text{ exists, and } \left\|F'(x)^{-1}F'(z_0)\right\| \leq -\frac{1}{f'(\|x - z_0\|)}. \tag{2.27}$$

Proof. Using (2.22) and (2.23) we get:

$$\begin{aligned}
&\left\|F'(z_0)^{-1}F''(x)\right\| \leq \\
&\leq \left\|F'(z_0)^{-1}F''(z_0)\right\| + \left\|F'(z_0)^{-1}F''(x) - F'(z_0)^{-1}F''(z_0)\right\| \\
&= \left\|F'(z_0)^{-1}F''(z_0)\right\| + \left\|\int_0^1 F'(z_0)^{-1}F'''(z_0 + t(x - z_0))(x - z_0)dt\right\| \\
&\leq 2\gamma + \int_0^1 f'''(t\,\|x - z_0\|)\,\|x - z_0\|\,dt \\
&= 2\gamma + f''(\|x - z_0\|) - f''(0) \\
&= f''(\|x - z_0\|),
\end{aligned}$$

which implies (2.26). Moreover by (2.9), and (2.22) we can have:

$$\begin{aligned}
\left\|F'(z_0)^{-1}F'(x) - I\right\| &= \left\|F'(z_0)^{-1}[F'(x) - F'(z_0)]\right\| \\
&= \left\|F'(z_0)^{-1}\int_0^1 F''(z_0 + t(x - z_0))(x - z_0)dt\right\| \\
&\leq \int_0^1 f''(t\,\|x - z_0\|)\,\|x - z_0\|\,dt \\
&= f'(\|x - z_0\|) - f'(0) \\
&= f'(\|x - z_0\|) + 1 < 1.
\end{aligned}$$

By the Banach Lemma on invertible operators [9] we conclude $F'(x)^{-1}$ exists, and $\left\|F'(x)^{-1}F'(z_0)\right\| \leq \frac{1}{1 - \|F'(z_0)^{-1}F'(x) - I\|} = \frac{1}{f'(\|x - z_0\|)}$.

That completes the proof of Lemma 2.6. ■

In the next Lemma following our ideas in [4], [5] we provide an Ostrowski-type representation for $F(x_n)$.

Lemma 2.7 *Suppose X and Y are the Banach spaces, D is an open convex of the Banach space X, $F : D \subset X \to Y$ has thrice order continuous Fréchet derivatives. Moreover, assume sequences $\{x_n\}$, $\{y_n\}$ generated by (1.2) are well defined.*

Then, for all $n \geq 0$ the following identity holds true:

$$
\begin{aligned}
&F(x_{n+1}) = \qquad (2.28)\\
&= \int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(x_{n+1}-y_n)^2\\
&+ \tfrac{1}{2}\int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(x_{n+1}-y_n)(y_n-x_n)\\
&+ \tfrac{1}{2}\int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(y_n-x_n)(x_{n+1}-y_n)\\
&+ \tfrac{1}{4}\int_0^1\int_0^1 F'''\left(\tfrac{(1+t)x_n}{2} + \tfrac{(1-t)y_n}{2} + st(x_{n+1}-x_n)\right)t(1-t)dsdt(x_{n+1}-x_n)(y_n-x_n)^2.
\end{aligned}
$$

Proof. Using (1.2) we can have in turn:

$$
\begin{aligned}
F(x_{n+1}) &= F(x_{n+1}) - F\left(\tfrac{x_n+y_n}{2}\right) - F'\left(\tfrac{x_n+y_n}{2}\right)\left(x_{n+1} - \tfrac{x_n+y_n}{2}\right)\\
&\quad + F\left(\tfrac{x_n+y_n}{2}\right) + F'\left(\tfrac{x_n+y_n}{2}\right)\left(x_{n+1} - \tfrac{x_n+y_n}{2}\right)\\
&= \int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt\left(x_{n+1} - \tfrac{x_n+y_n}{2}\right)^2\\
&\quad + F\left(\tfrac{x_n+y_n}{2}\right) + F'\left(\tfrac{x_n+y_n}{2}\right)\left(x_{n+1} - \tfrac{x_n+y_n}{2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&F\left(\tfrac{x_n+y_n}{2}\right) + F'(x_n)\left(x_{n+1} - \tfrac{x_n+y_n}{2}\right) =\\
&= F\left(\tfrac{x_n+y_n}{2}\right) + F'\left(\tfrac{x_n+y_n}{2}\right)\left(x_{n+1} - x_n - \tfrac{y_n-x_n}{2}\right)\\
&= F\left(\tfrac{x_n+y_n}{2}\right) - F(x_n) - F'\left(\tfrac{x_n+y_n}{2}\right)\tfrac{y_n-x_n}{2}\\
&= -\frac{1}{4}\int_0^1 F''\left(\tfrac{x_n+y_n}{2} - t(\tfrac{y_n-x_n}{2})\right)(1-t)dt(y_n-x_n)^2.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
F(x_{n+1}) &= \int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(x_{n+1}-y_n)^2 \\
&+ \frac{1}{2}\int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(x_{n+1}-y_n)(y_n-x_n) \\
&+ \frac{1}{2}\int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(y_n-x_n)(x_{n+1}-y_n) \\
&+ \frac{1}{4}\int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(y_n-x_n)^2 \\
&- \frac{1}{4}\int_0^1 F''\left(\tfrac{x_n+y_n}{2} - t(\tfrac{y_n-x_n}{2})\right)(1-t)dt(y_n-x_n)^2 \\
&= \int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(x_{n+1}-y_n)^2 \\
&+ \frac{1}{2}\int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(x_{n+1}-y_n)(y_n-x_n) \\
&+ \frac{1}{2}\int_0^1 F''\left(\tfrac{x_n+y_n}{2} + t(x_{n+1} - \tfrac{x_n+y_n}{2})\right)(1-t)dt(y_n-x_n)(x_{n+1}-y_n) \\
&+ \frac{1}{4}\int_0^1\int_0^1 F'''\left(\tfrac{(1+t)x_n}{2} + \tfrac{(1-t)y_n}{2}) + st(x_{n+1}-x_n)\right)t(1-t)ds\ dt \\
&(x_{n+1}-x_n)(y_n-x_n)^2.
\end{aligned}
$$

That completes the proof of Lemma 2.7. ■

We can show the following semilocal convergence result for method (1.2).

Theorem 2.8 *Under the γ-condition, further assume (2.7), and $\overline{U}(z_0, t^*) \subseteq D$ hold true for $t_0 \geq \|\frac{z_0-y_0}{2}\|$, and s_0 satisfying (2.5). Then, the sequences $\{x_n\}$, $\{y_n\}_{n\geq 0}$ generated by (1.2) are well defined, $x_n \in \overline{U(z_0, t^*)}$ and converge to the unique solution x^* in $U\left(z_0, (1-\frac{1}{\sqrt{2}})\frac{1}{\gamma}\right)$, with $\|x_n - x^*\| \leq t^* - t_n$.*

Proof. We shall that prove by induction that the following estimates hold

$$\|x_n - z_0\| \leq t_n; \tag{2.29}$$

$$\|y_n - x_n\| \leq s_n - t_n; \tag{2.30}$$

$$\|y_n - z_0\| \leq s_n; \tag{2.31}$$

$$\left\|F'\left(\tfrac{x_n+y_n}{2}\right)^{-1} F'(z_0)\right\| \leq -g'\left(\tfrac{t_n+s_n}{2}\right)^{-1}; \tag{2.32}$$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n. \tag{2.33}$$

Estimate (2.29) holds true for $n = 0$ by the initial conditions.

Assuming (2.29) holds true for $k = 0, 1, ..., n$ we get

$$\|x_{n+1} - z_0\| \leq \|x_{n+1} - x_n\| + \|x_n - z_0\| \leq t_{n+1} - t_n + t_n = t_{n+1}.$$

By Lemma 2.7 we can have in turn:

$$
\begin{aligned}
&\left\|F'(z_0)^{-1}F(x_{n+1})\right\| && (2.34)\\
&\leq \int_0^1 f''\left(\tfrac{t_n+s_n}{2}+t(t_{n+1}-\tfrac{t_n+s_n}{2})\right)(1-t)dt(t_{n+1}-s_n)^2\\
&\int_0^1 f''\left(\tfrac{t_n+s_n}{2}+t(t_{n+1}-\tfrac{t_n+s_n}{2})\right)(1-t)dt\\
&(t_{n+1}-s_n)(s_n-t_n)\\
&+\frac{1}{4}\int_0^1\int_0^1 f'''\left(\tfrac{(1+t)t_n}{2}+\tfrac{(1-t)s_n}{2}+st(t_{n+1}-t_n)\right)t(1-t)ds\ dt\\
&(t_{n+1}-t_n)(s_n-t_n)^2\\
&=f(t_{n+1}).
\end{aligned}
$$

Hence, we get by (1.2)

$$
\begin{aligned}
\|y_{n+1}-x_{n+1}\| &= \left\|-F'(z_{n+1})^{-1}F(x_{n+1})\right\|\\
&\leq \left\|-F'(z_{n+1})^{-1}F'(z_0)\right\|\left\|F'(z_0)^{-1}F(x_{n+1})\right\|\\
\text{and } &\leq -f'(r_{n+1})^{-1}f(t_{n+1}) = s_{n+1}-t_{n+1},\\
\|y_{n+1}-z_0\| &\leq \|y_{n+1}-x_{n+1}\|+\|x_{n+1}-z_0\| = s_{n+1},
\end{aligned}
$$

since by Lemma 2.6 we get

$$
\left\|F'\left(\tfrac{x_{n+1}+y_{n+1}}{2}\right)^{-1}F(z_0)\right\| \leq -f'\left(\tfrac{t_{n+1}+s_{n+1}}{2}\right)^{-1} \tag{2.35}
$$

By (1.2), (2.35) and (2.35) we get

$$
\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq \left\|F'(z_{n+1})^{-1}F'(z_0)\right\|\cdot\left\|F'(z_0)^{-1}F(x_{n+1})\right\|\\
&\leq t_{n+2}-t_{n+1}.
\end{aligned}
$$

Hence, the sequences $\{x_n\},\{y_n\}_{n\geq 0}$ are well defined, $x_n, y_n \in \overline{U(z_0,t^*)}$ and $\{x_n\}$, $\{y_n\}$ converge to the solution $x^* \in \overline{U(z_0,t^*)}$ of the equation (1.1). To show uniqueness, let us suppose y^* is a solution of the equation (1.1) on $U(z_0,\overline{r})$.

Using (2.22):

$$
\begin{aligned}
&\left\| F'(z_0)^{-1} \int_0^1 F'(x^* + t(y^* - x^*))dt - I \right\| \\
&\le \left\| F'(z_0)^{-1} \int_0^1 \left[F'(x^* + t(y^* - x^*)) - F'(z_0) \right] dt \right\| \\
&\le \left\| F'(z_0)^{-1} \int_0^1 \int_0^1 F''(z_0 + s(x^* - z_0 + t(y^* - x^*)))ds\ dt(x^* - z_0 + t(y^* - x^*)) \right\| \\
&\le \int_0^1 \int_0^1 f''(s \left\| x^* - z_0 + t(y^* - x^* \right\|)ds\ dt \left\| x^* - z_0 + t(y^* - x^*) \right\| \\
&= \int_0^1 f'(\left\| x^* - z_0 + t(y^* - x^* \right\|)dt - f'(0) \\
&= \int_0^1 f'(\left\| (1-t)(x^* - z_0) + t(y^* - z_0) \right\|)dt + 1 < 1.
\end{aligned}
$$

By the Banach Lemma the inverse of $\int_0^1 F'[x^* + t(y^* - x^*)]dt$ exists. In view of the identity.

$$F(y^*) - F(x^*) = \int_0^1 F'[x^* + t(y^* - x^*)]dt(y^* - x^*),$$

we deduce $y^* = x^*$. For $m > n$, we have

$$\|x_m - x_n\| \le \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + ... + \|x_{n+1} - x_n\| \le t_m - t_n.$$

By letting $m \to \infty$, we get

$$\|x_n - x^*\| \le t^* - t_n.$$

That completes the proof of Theorem 2.8. ■

We complete the study with two numerical examples. In the first example we show that our results can apply, where the famous Newton-Kantorovich hypothesis [3], [9]:

$$h = 2l\overline{\beta} \le 1, \ \left\| F'(x_0)^{-1} F(x_0) \right\| \le \overline{\beta} \tag{2.36}$$

with ℓ being the Lipschitz constant in hypothesis:

$$\left\| F'(x_0)^{-1}[F'(x) - F'(y)] \right\| \le \ell \|x - y\| \ \text{ for all } x, y \in D \tag{2.37}$$

(which is the sufficient convergence condition for Newton's method (1.6)) is violated. In the second example we revisit a nonlinear integral equation appearing in the theory of radiative transfer, neutron transport, and in the kinetic theory of gasses [3], [6] studied also by us in [5].

Example 2.9 *Let* $X = Y = \mathbf{R}$, $D = [w, 2-w]$, $w \in \left[0, \frac{1}{2}\right)$, $x_0 = 1$, $y_0 = 1.01$, *and define function* F *on* D *by*

$$F(x) = x^3 - w. \tag{2.38}$$

Let $\gamma = 1$. *Using (2.2), (2.21)-2.23, 2.37, and (2.38) we obtain,*

$$\overline{\beta} = \frac{1}{3}(1-w), \ \alpha = \beta = \frac{(1-p)}{3.030075}, \ \ell = 2(2-w). \tag{2.39}$$

The Newton-Kantorovich hypothesis is violated since

$$h = 2\ell\beta = \frac{3}{4}(1-w)(2-w) > 1 \text{ for all } p \in \left[0, \frac{1}{2}\right). \tag{2.40}$$

Hence, there is no guarantee that Newton's method starting from $x_0 = 1$ *converges to* $x^* = \sqrt[3]{w}$.

However condition (2.7)

$$\alpha = \beta\gamma = \frac{1 = w}{e.030075} \leq 3 - 2\sqrt{2}$$

holds true for all

$$p \in \left[.48012132, \frac{1}{2}\right) = I.$$

Note that $.48012132 < \frac{1}{2}$, *i.e* $I \neq \emptyset$.

That is our Theorem 2.8 guarantees converge of method (1.2) to x^* *for all* $w \in I$.

We complete this study with another numerical example involving the famous Chandrasekhar's integral equation appearing in radiative transfer in connection with the birth and death of planetary stars [3]-[6], [9].

Example 2.10 *Let* $X = Y = C[0,1]$, $\lambda = .25$, $x_0(s) = 1$, $y_0\ (s) = 1.0000001$, *and define operator* F *on* X *by*

$$F(x(s)) = \lambda x(s) \int_0^1 \frac{s}{s+t} x(t) dt - x(s) + 1. \tag{2.41}$$

We have

$\left\| F'(z_0(s))^{-1} \right\| \leq b_0 = 1.53039421, and$

$\left\| F'(z_0(s))^{-1} F(x_0(s)) \right\| \leq b_0 \left\| F(x_0(s)) \right\| \leq b_0 \ln 2 |\lambda|$ *[3], [5]. Then, let* $\gamma = \beta = b_0 \ln 2 |\lambda|$. *It can easily be seen that all hypotheses of Theorem 2.8 are satisfied, since*

$$\alpha = \beta\gamma = .070329506 \leq 3 - 2\sqrt{2} = .17157287.$$

Hence, according to Theorem 2.8, Chandrasekhar's integral equation (2.41) has a solution $x^*(s)$ *in* $U(z_0(s), t^*)$, *which is found by (1.2), a method faster than Newton's.*

References

[1] Amat, S. and Busquier, S., *Convergence and numerical analysis of a family of two-step Steffensen's method,* Comput. and Math. with Applic. 49, (2005), 13-22.

[2] Argyros, I.K., *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space,* J. Math. Anal. Applic. 298, (2004), 374-397.

[3] Argyros, I.K., *Approximate solution of operator equations with applications,* World Scientific Publ. Co. Ptl. Ltd., Hackensack, N.J., 2005, U.S.A.

[4] Argyros, I.K. and Chen, D., *On the midpoint method for solving equations in Banach spaces,* Appl. Math. Letters, Vol. 5, N0.4, (1992), 7-9.

[5] Argyros, I.K., and Chen, D., *On the midpoint iterative method for solving nonlinear operator equations and applications to the solution of integral equations,* Revue d'Analyse Numerique et de Theorie de l'Approximation, Tome 23, fasc.2, (1994), 139-152.

[6] Chandrasekhar, S., *Radiative Transfer,* Dover Publ., New York, 1960.

[7] Gutierrez, J.M., Hernandez, M.A. and Salanova, M.A. *Accesibility of solutions by Newton's method, J. Comput. Math. 57, (1995), 239-247.*

[8] Homeir, H.H.H., *A modified Newton method for rootfinding with cubic convergence,* J. Comput. Appl. Math., 157, (2003), 227-230.

[9] Kantorovich, L.V. and Akilov, G.P. *Functional Analysis in normed spaces,* Pergamon Press, Oxford, 1982.

[10] Özban, A.Y. *Some new variants of Newton's method,* Appl. Math. Letters, 17, (2004), 677-682.

[11] Wang, X.H., *Convergence on the iteration of Halley family in weak condition,* Chinese Science Buletin, 42, 7, (1997), 552-555.

[12] Werner, W. *Über ein Verfahren der Ordnung* $1+\sqrt{2}$ zür Nullstellenbestimmung, Numer. Math. 32, (1979), 333-342.

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Neural Network Representation of Stochastic Fuzzy Multi-Objective Optimization Problems with A Known Distribution

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Abstract

This paper presents an application of neural network to stochastic fuzzy linear and nonlinear programming problem. After converting the proposed stochastic fuzzy programming problem to a deterministic problem. Neural network approach is applied to find the compromise solution. Assuming the coefficients of the right-hand-side parameters in the constraints to be normal distribution random variables, a methodology is presented to convert the probabilistic problem to a deterministic problem. The method leads to an efficient solution as well as an optimal compromise solution.

Keywords: Neural network; Stochastic programming; Multi-objective programming; Stochastic fuzzy programming; Nonlinear programming.

1 Introduction

Fuzzy stochastic or probabilistic programming deals with situations where some or all the parameters of a mathematical programming problem are described by stochastic variables rather than by deterministic quantities. Several models have been presented in the field of stochastic programming [12]. Contini [2] has developed an algorithm for stochastic linear programming when the random variables are normally distributed with known means and variances. He transformed the stochastic problem to an equivalent deterministic quadratic programming problem, where the objective functions consisted of maximizing the probability of a vector of R.H.S. lying in the confidence region of a predefined size. Sullivan and Fitzsimmoms [13] suggested an algorithm using probabilistic programming based on the concept of chance constraints of Charnes and Cooper [1] where the goals can be stated in terms of probability of satisfying the aspiration levels. Teghem et al. [14] and Leclercq [9] have presented interactive methods in fuzzy stochastic programming. Two major approaches to stochastic programming [3,7] are recognized as Chance constrained programming. The chance constrained programming (CCP) technique is one which can be used to solve problems involving chance constraints i.e., constraints having finite probability of being violated. The CCP was originally developed by Charnes and Cooper [1] and has now in recent years been generalized in several Erections an as various applications. In the recent past, fuzzy stochastic programming has been applied to the problems having multiple, conflicting and non-commensurable objectives where generally there does not exist a single solution which can maximize (minimize) all the objectives. There exists a set of alternative solutions out of which an "appropriate alternative solution" also termed as "compromise solution" has to be singled out. But in a multiple criteria decision-making system, the decision-maker generally satisfies of criteria rather than maximization (minimization) of objectives. These problems, however, become more complex when the parameters are stochastic and/or fuzzy. Zimmermann [15] presented an application of fuzzy linear programming to the linear vector-maximum problem. He stated that an "optimal" solution obtained by the fuzzy linear programming is always efficient. Hanan [4] and Narasimhan [10] have also presented methods to goal programming problems where the goals are stated imprecisely and the decision environment is fuzzy. Leberling [8] showed that by using fuzzy min-operator together with linear as well as non-linear membership functions, the obtained solutions are always a compromise solution of the original multi-objective problem. In this paper, we consider only some fuzzy multi-objective linear programming problems with some random variables. After converting into a deterministic model, neural network used to find the compromise solution. The paper is organized as follows. In Section 2, Multi-objective Chance constrained programming technique with a joint constraint. In section 3, Neural networks for deterministic nonlinear equations. In Section 4, A few membership functions are presented. In section 5, Given solution procedure for stochastic fuzzy multi-objective programming, Section 6, Gives the numerical example. In section 7, Gives the conclusion of this paper. .

2 Chance constrained programming technique with a joint constraint [14].

A multi-objective chance constrained programming problem with a joint probability constraint can be stated as

$$\min Z^{(k)}(x) = \sum_{j=1}^{n} c_j^{(k)} x_j, \quad k = 1, 2, ..., K \tag{1}$$

$$\text{Subject to: } Prob\ [\sum_{j=1}^{n} a_{ij} x_j \geq b_i] \geq 1 - \alpha, i = 1, ..., m, \tag{2}$$

$$x_j \geq 0, \quad j = 1, ..., n, \quad \alpha \in (0, 1). \tag{3}$$

where b_i's are independent Uniform random variables with known means and variances. Equation (2) is a joint probabilistic constraint and $0 < \alpha < 1$ is a specified probability. We assume that the decision variables x_j's are deterministic.

2.1 Deterministic model

Let b_i, i = 1,...,m are mutually independent random variables, have uniform distributions,$b_i \sim U(c_i, d_i)$, i.e. the probability denisty function of b_i is:

$$f_b i(x) = \begin{cases} \frac{1}{d_i - c_i} & \text{for} \quad c_i \leq b_i \leq d_i \\ 0 & \text{for} \quad otherwise \end{cases} \tag{4}$$

with cumulative distribution function given by

$$F_b i(x) = \begin{cases} \frac{x - c_i}{d_i - c_i} & \text{for} \quad c_i \leq b_i \leq d_i \\ 1 & \text{for} \quad d_i \leq b_i \end{cases} \tag{5}$$

Thus the stochastic problem is equivalent to

$$\min Z^{(k)}(x) = \sum_{j=1}^{n} c_j^{(k)} x_j, \quad k = 1, 2, ..., K \tag{6}$$

$$\text{Subject to: } \sum_{j=1}^{n} a_{ij} x_j \leq K_\alpha i \tag{7}$$

$$x_j \geq 0, \quad j = 1, ..., n, \quad \alpha \in (0, 1), \quad i = 1, ..., m. \tag{8}$$

Where K_{α_i} is the solution of $\alpha_i = \frac{K_{\alpha_i} - c_i}{d_i - c_i} i.e., K_{\alpha_i} = c_i + \alpha_i (d_i - c_i)$Thus, the probabilistic linear programming problem stated in$Eqations, (1) - (3)$ is equivalent to the following deterministic linear programming problem

$$\min Z^{(k)}(x) = \sum_{j=1}^{n} c_j^{(k)} x_j, \quad k = 1, 2, ..., K \tag{9}$$

$$\text{Subject to: } \sum_{j=1}^{n} a_{ij} x_j \leq c_i + \alpha_i (d_i - c_i) \quad i = 1, ..., p \tag{10}$$

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad i = p+1, ..., m \tag{11}$$

$$x_j \geq 0, \quad j = 1, ..., n, \quad \alpha \in (0,1). \tag{12}$$

3 Neural networks for Deterministic Nonlinear Equations

Let us consider the following deterministic nonlinear programming problem:

$$\min Z^{(k)}(x) = \sum_{j=1}^{n} c_j^{(k)} x_j, \ k = 1, 2, ..., K \tag{13}$$

$$\text{subject to:} \quad f_i(x) \leq 0 \quad \text{where} \quad i = 1, 2, ..., p \tag{14}$$

$$\tag{15}$$

and $D_i(x) = 0$ where $i = p+1, p+2, ..., m$

where $x \in R^{n\times 1}$ is the vector of the independent variables, $Z^{(k)}(x) : R^{n\times m} \to R$ is the objective function, and functions $f_i(x), D_i(x) : R^{n\times 1} \to R$ represent constraints. To simplify the derivations of the algorithms we will assume that both the objective function and the constraints are smooth differentiable functions of independent variables.

$$f_i(x) = \sum_{j=1}^{n} a_{ij}x_j \leq c_i + \alpha_i(d_i - c_i) \quad i = 1, ..., p, \tag{16}$$

$$D_i(x) = \sum_{j=1}^{n} a_{ij}x_j - b_i = 0, \quad i = p+1, ..., m, \tag{17}$$

$$x_j \geq 0, \quad j = 1, ..., n. \tag{18}$$

Using surplus variables, the inequality constraints can be converted to equality constraints. Similarly, each of the equality constraints can be converted to a pair of inequality constraints according to

$$D_i(x) = 0 \Longleftrightarrow D_i(x) \leq 0 \quad \text{and} \quad D_i(x) \geq 0 \tag{19}$$

3.1 Neural Networks for Penalty Function for System of Nonlinear Equations

This Method using penalty functions make an attempt to transform the system of nonlinear equations (SNE) to an equivalent unconstrained optimization problem, or to a sequence of constrained optimization problems. This transformation is accomplished through modification of the objective function so that it includes terms that penalize every violation of the constraints. In general, the modified objective function takes the following form:

$$S_A(x) = \sum_{i=1}^{p} H_i^{(1)} \Phi_i^{(1)}[D_i(x)] + \sum_{i=p+1}^{m} H_i^{(2)} \Phi_i^{(2)}[f_i(x)] \tag{20}$$

Functions $\Phi_i^{(1)}$ and $\Phi_i^{(2)}$ are called penalty functions, and they are designed to increase the value of the modified objective function $S_A(x)$ whenever the vector of independent variables violates a constraint, or in other words whenever it is outside the feasible region. Penalty functions are commonly selected as at least one-time differentiable function satisfying the following requirements:

1. For equality constraints

$$\Phi_i^{(1)} \begin{cases} > 0 & \text{for} \quad D_i(x) \neq 0 \\ = 0 & \text{for} \quad D_i(x) = 0 \end{cases} \tag{21}$$

2. For inequality constraints

$$\Phi_i^{(2)} \begin{cases} > 0 & \text{for} \quad f_i(x) > 0 \\ = 0 & \text{for} \quad f_i(x) \leq 0 \end{cases} \tag{22}$$

For example, the typical modified objective function for this problem can written as

$$S_A(x) = \sum_{i=1}^{p} \frac{H_i^{(1)}}{\rho_1} |D_i(x)|^{\rho_1} + \sum_{i=p+1}^{m} H_i^{(2)} \max\{0, f_i(x)\}^{\rho_2} \tag{23}$$

Where $\rho_1, \rho_2 \geq 0$. Parameters $H_i^{(1)}, H_i^{(2)} \geq 0$ are commonly referred to as penalty parameters or penalty multipliers, and in (22) we have assumed that a separate penalty parameter is associated with each of the penalty functions. In practice this is rarely the case, and commonly there is only one parameter mulitiplying the entire penalty term, that is,

$$S_A(x) = H \left[\sum_{i=1}^{p} \frac{1}{\rho_i} |D_i(x)|^{\rho_1} + \sum_{i=p+1}^{m} \max\{0, f_i(x)\}^{\rho_2} \right] = kp(x) \tag{24}$$

where $p(x)$ represents the penalty term.

There are two fundamental issues that need to be addressed in the practical application of penalty functions. First, we need to be aware that (24) represents merely an approximation of the original problem in (13) through (15). The question is. How close is the approximation? The second issue involves a design of a computationally efficient neural network algorithm that can successfully solve the unconstrained problem in a timely manner. From the form of the augmented objective function in (24), it should be obvious that the solution resides in the region where the value of the penalty function $P(x)$ is small. As a matter of fact, if K is increased toward infinity, the solution of the unconstrained problem will be forced into the feasible region of the original NP problem. Remember that if the point is in the feasible region, all the constraints are satisfied and the penalty function equals zero. In the limiting case, when $k \to \infty$, the two problems become equivalent. Applying the steepest descent approach, we can generate the update equations in accordance with

$$x(k+1) = x(k) - \mu \frac{\partial f_A(x)}{\partial x} \tag{25}$$

where $\mu > 0$ is the learning rate parameter and the gradient term on the right hand side of (25) depends on the penalty function selection. For example, when the form of the energy function is as given in (24), with $\rho_1 = 2$ and $\rho_2 = 1$,we have

$$\frac{\partial f_A(x)}{\partial x} = H \left[\sum_{i=1}^{p} \frac{\partial D_i(x)}{\partial x} D_i(x) + \sum_{i=p+1}^{m} \frac{\partial}{\partial x} \max\{0, f_i(x)\} \right] \tag{26}$$

After substituting (26) into (25), we have for the learning rule

$$x(k+1) = x(k) - \mu \left[H \sum_{i=1}^{p} \frac{\partial D_i(x)}{\partial x} D_i(x) + H \sum_{i=p+1}^{m} \frac{\partial}{\partial x} \max\{0, f_i(x)\} \right] \tag{27}$$

The neural network architecture realization of this process is presented in figure 1. Note that only a portion of the network for computing a single component of the independent-variable vector is presented. Also note that the network corresponds to the general case of this problem.

4 Membership functions

In this section, a few membership functions are presented to solve the multiobjective linear and nonlinear programming problems.
(a)Linear membership function: The linear membership function [15] for a vector-maximum problem is given by

$$\mu_k(x) = \begin{cases} 0 & \text{for} \quad z_k \leq L_k \\ \frac{z_k(x) - L_k}{U_k - L_k} & \text{for} \quad L_k \leq z_k \leq U_k \\ 1 & \text{for} \quad z_k \geq U_k \end{cases} \tag{28}$$

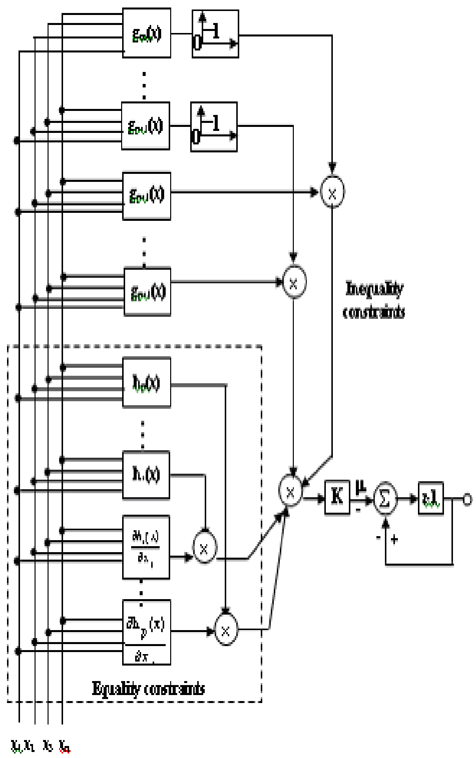

Figure 1: Discrete-tim network for deterministic nonlinear constraints implementing penalty function method, implementation of Equation (27).

where $\mu_k(x)$ is the membership function of the k^{th} objective function, U_k and L_k are the upper and lower bounds of $z_k(x)$. It is assumed that L_k is not equal to U_k.
(b) Non-linear membership function using product-operator [15]. This is given by

$$\mu(x) = \prod_{i=1}^{m} \mu_k(x). \tag{29}$$

where $\mu_k(x)$ is the membership function of the k^{th} objective function and is given by

$$\mu_k(x) = \frac{z_k(x) - L_k}{U_k - L_k}, L_k \leq z_k \leq U_k. \tag{30}$$

(c) Hyperbolic membership function: The hyperbolic type of mempbership function [8] is given as

$$\mu_k(x) = 1/2(tanh[z_k(x) - \frac{(U_k + L_k)}{2}] + 1), \tag{31}$$

where U_k and L_k are respectively, the upper and lower bound of $z_k(x)$ and $\alpha \geq 0$ is a constant and $\alpha = \frac{6}{(U_k - L_k)}$

5 Solution Procedure for Stochastic Fuzzy Multi-objective Programming

We now presented the methodology to solve stochastic programming problem using fuzzy programming approach.
Step 1: First,convert the given stochastic programming problem into an equivalent deterministic programming problem by chance constrained programming technique as discussed in Section 3.
Step 2: Solve the system of nonlinear equations (16)-(17). Let $x^{(1)}, x^{(2)}, ..., x^{(k)}$ be the respective ideal solutions of this system and substitute by this values in the objective function.
Step 3: From Step 2, obtain the upper and lower bounds (U_k *and* L_k) for the objective functions.
Step 4: Using a linear membership function, formulate a crisp model. By introducing an augmented variable formulate single objective non-linear programming problem. Hence, the model can be formulated as

1. Using a linear membership function.

$$\min \lambda$$

subject to

$$\Psi_i(x) = \begin{cases} z_k(x) + (U_k - L_k)\lambda \leq U_k & \text{for} \quad k = 1, ..., K \\ \sum_{j=1}^{n} a_{ij}x_j \leq \alpha(d_i - c_i), & \text{for} \quad i = 1, ..., p \end{cases} \tag{32}$$

and

$$\xi_i(x) = \sum_{j=1}^{n} a_{ij}x_j - b_i = 0, \text{for} i = p + 1, ..., m. \tag{33}$$

Using surplus variables, the inequality constraints can be converted to equality constraints. Similarly, each of the equality constraints can be converted to a pair of inequality constraints according to

$$\xi_i(x) = 0 \leftrightarrow \xi_i(x) \leq 0 \quad and \quad \xi_i(x) \geq 0. \tag{34}$$

2. Using a non-linear membership function with product-operator.

$$\text{Minimize} \prod_{k=1}^{K} \mu_k(x) \tag{35}$$

subject to

$$\Psi_i(x) = \sum_{j=1}^{n} a_{ij}x_j \leq c_i + \alpha(d_i - c_i), \quad i = 1, ..., p \tag{36}$$

$$\xi_i(x) = \sum_{j=1}^{n} a_{ij}x_j - b_i = 0, \quad i = p+1, ..., m \tag{37}$$

3.Using hyperbolic membership function.

$$\text{Minimize} \quad \lambda \tag{38}$$

Subject to:

$$\mu_k(x) = \lambda - 1/2(tanh[z_k(x) - \frac{(U_k + L_k)}{2}\alpha_k] + 1) \leq \ 0.5 \tag{39}$$

$$\Psi_i(x) = \sum_{j=1}^{n} a_{ij}x_j \leq c_i + \alpha(d_i - c_i), \quad i = 1, ..., p, \tag{40}$$

$$\xi_i(x) = \sum_{j=1}^{n} a_{ij}x_j - b_i = 0, \quad i = p+1, ..., m, \tag{41}$$

$$x_j, \lambda \geq 0, \quad j = 0, ..., n. \tag{42}$$

5.1 Neural Networks for Penalty Function FDNP Methods

The modified objective function for this problem can written as

$$R_A(x) = \lambda + \sum_{i=1}^{p} H_i^{(1)}\varphi_i^{(1)}|\xi_i(x)| + \sum_{i=p+1}^{m} H_i^{(2)}\varphi_i^{(2)}[\Psi_i(x)] \tag{43}$$

In practice this is rarely the case, and commonly there is only one parameter multiplying the entire penalty term, that is,

$$R_A(x) = \lambda + H\left[\sum_{i=1}^{p} \frac{1}{\rho_1}|\xi_i(x)|^{\rho_1} + \sum_{i=p+1}^{m} \max\{0, \Psi_i(x)\}^{\rho_2}\right] = \lambda + Hp(x) \tag{44}$$

where $p(x)$ represents the penalty term.
Applying the steepest descent approach, we can generate the update equations in accordance with

$$x(k+1) = x(k) - \mu\frac{\partial R_A(x)}{\partial x} \tag{45}$$

. The energy function is as given in (44), with $\rho_1 = 2$ and $\rho_2 = 1$,we have

$$\frac{\partial R_A(x)}{\partial x} = \lambda + H\left[\sum_{i=1}^{p} \frac{\partial \xi_i(x)}{\partial x}\xi_i(x) + \sum_{i=p+1}^{m} \frac{\partial}{\partial x}\max\{0, \Psi_i(x)\}\right] \tag{46}$$

After substituting (46) into (45), we have for the learning rule

$$x(k+1) = x(k) - \mu \left[\lambda + H \sum_{i=1}^{p} \frac{\partial \xi_i(x)}{\partial x} \xi_i(x) + H \sum_{i=p+1}^{m} \frac{\partial}{\partial x} \max\{0, \Psi_i(x)\} \right] \tag{47}$$

The neural network architecture realization of this process is presented in figure 2.

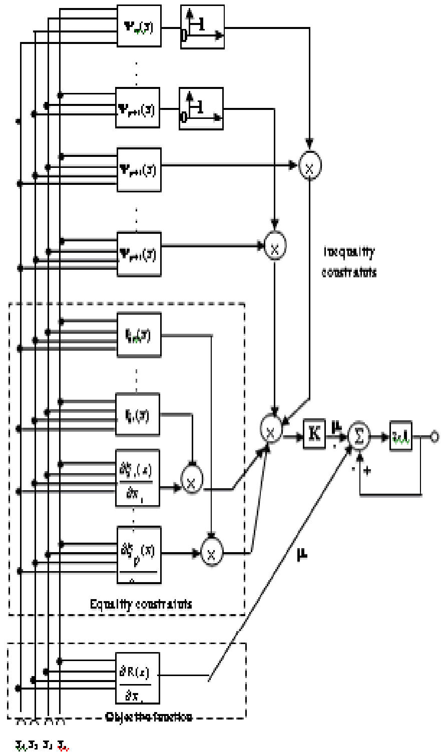

Figure 2: Discrete-tim network for deterministic nonlinear constraints implementing penalty function method, implementation of Equation (47).

6 Numerical example

$$\begin{aligned}
\text{min}: \quad & z^{(1)} = -x_1 - 2x_2, \\
\text{min}: \quad & z^{(2)} = -3x_1 - x_2, \\
\text{subject to:} \quad & x_1 + x_2 = 6, \\
& Prob[2x_1 + x_2 \leq b_1] \geq 0.4, \\
& Prob[2x_1 \leq b_2] \geq 0.3, \\
& Prob[x_2 \leq b_3] \geq 0.5, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

where b_2, b_3,and b_4 are give as $b_2 \sim U(8.6, 9.6)$, $b_3 \sim U(3, 6.33)$,$b_4 \sim U(4, 6)$

We obtain the equivalent deterministic programming problem for the above multi-objective stochastic programming problem by using Eqs. (8)-(10).

$$\text{min}: \quad z^{(1)} = -x_1 - 2x_2,$$

$$\begin{aligned}
\min: \quad & z^{(2)} = -3x_1 - x_2, \\
\text{subject to:} \quad & x_1 + x_2 = 6, \\
& 2x_1 + x_2 \leq 9, \\
& 2x_1 \leq 4, \\
& x_2 \leq 5, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

The above problem is obviously an linear programming problem with inequality constraints. The modified penalty function can be formed as:

$$F_A(x) = (x_1 + x_2 - 6) + H\frac{\partial}{\partial x}\max\{0, (3x_1 + 2x_2 - 18)\}$$

By using the steepest descent ethod, the update equations can be computed as

$$\begin{aligned}
x_1(k+1) &= x_1(k) - \mu H(x_1 + x_2 - 6) + H(max(0,3)) \\
x_2(k+1) &= x_2(k) - \mu H(x_1 + x_2 - 6) + H(max(0,2))
\end{aligned}$$

The neural network architecture shown in figure 1 was used to determine the solution of the NP problem. Parameters of the network were chosen as $H = 1$, and $\mu = 0.001$, and initial solution was set as $x = [0 \quad 3]$. The network converged in approximately 300 iterations, and the optimal solution given by.

$$x^{(1)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \tag{48}$$

(a) Linear membership function: Using linear membership function and applying the fuzzy programming technique we formulate the problem as

$$\begin{aligned}
\max: \quad & \lambda \\
& \text{subject to:} \\
& x_1 + 2x_2 + 3\lambda \geq 11, \\
& 3x_1 + x_2 + 7\lambda \geq 8, \\
& x_1 + x_2 = 6, \\
& 2x_1 + x_2 \leq 9, \\
& 2x_1 \leq 4, \\
& x_2 \leq 5, \\
& x_1, x_2, \lambda \geq 0.
\end{aligned}$$

The modified penalty function for this problem can be formed as:

$$R_A(x) = \lambda + (x_1 + x_2 - 6) + H\frac{\partial}{\partial x}\max\{0, (7x_1 + 5x_2 + 10\lambda - 37)\}$$

By using the steepest descent method, the update equations can be computed as

$$\begin{aligned}
x_1^*(k+1) &= x_1^*(k) - \mu(\lambda + H(x_1 + x_2 - 6) + H(\max(0,7))) \\
x_2^*(k+1) &= x_2^*(k) - \mu(\lambda + H(x_1 + x_2 - 6) + H(\max(0,5))) \\
\lambda(k+1) &= \lambda(k) - \mu(1 + H(\max(0,10)))
\end{aligned}$$

To solve this SMOFLP problem, the neural network in Figure 2 is simulated. The parameters of the network were chosen as $\mu = 0.01$, and $H = 1$.

The network converges in approximately 100 iterations. The optimal solution to the SMONLP is given as

$$x_1^* = \begin{pmatrix} 2.54167 \\ 3.91667 \end{pmatrix}, \quad \text{and} \quad \lambda = 0.20833 \tag{49}$$

(b) Hyperbolic membership function: Using hyperbolic membership function and applying the fuzzy programming technique we formulate the problem as

$$\begin{aligned}
\max: \quad & \lambda \\
\text{subject to:} \quad & \\
& x_1 + 2x_2 + 0.5\lambda \geq 15.2, \\
& 3x_1 + x_2 + 1.16\lambda \geq 43.5, \\
& x_1 + x_2 = 6, \\
& 2x_1 + x_2 \leq 9, \\
& 2x_1 \leq 4, \\
& x_2 \leq 5, \\
x_1, x_2, \lambda \geq 0. &
\end{aligned}$$

The modified penalty function for this problem can be formed as:

$$R_A(x) = \lambda + (x_1 + x_2 - 6) + H\frac{\partial}{\partial x}\max\{0, (7x_1 + 5x_2 + 1.66\lambda - 76.62)\}$$

By using the steepest descent ethod, the update equations can be computed as

$$\begin{aligned}
x_1^*(k+1) &= x_1^*(k) - \mu(\lambda + H(x_1 + x_2 - 6) + H(\max(0,7))) \\
x_2^*(k+1) &= x_2^*(k) - \mu(\lambda + H(x_1 + x_2 - 6) + H(\max(0,5))) \\
\lambda(k+1) &= \lambda(k) - \mu(1 + H(\max(0,1.66)))
\end{aligned}$$

To solve this SMOFLP problem, the neural network in Figure 2 is simulated. The parameters of the network were chosen as $\mu = 0.05$, and H = 1. The network converges in approximately 500 iterations. The solution: Infeasible solution

(c) Nonlinear membership function using product operator: Using the product type of membership, we formulate the non-linear programming problem as maximize:

$$\begin{aligned}
\max: \quad & \lambda_1\lambda_2 \\
\text{subject to:} \quad & \\
& x_1 + 2x_2 + 3\lambda \geq 11, \\
& 3x_1 + x_2 + 7\lambda \geq 8, \\
& x_1 + x_2 = 6, \\
& 2x_1 + x_2 \leq 9, \\
& 2x_1 \leq 4, \\
& x_2 \leq 5, \\
& x_1, x_2, \lambda_1\lambda_2 \geq 0.
\end{aligned}$$

The modified penalty function for this problem can be formed as:

$$R_A(x) = \lambda_1\lambda_2 + (x_1 + x_2 - 6) + H\frac{\partial}{\partial x}\max\{0, (7x_1 + 5x_2 + 3\lambda_1\lambda_2 - 37)\}$$

By using the steepest descent ethod, the update equations can be computed as

$$\begin{aligned}
x_1^*(k+1) &= x_1^*(k) - \mu(\lambda_1\lambda_2 + H(x_1 + x_2 - 6) + H(\max(0,7))) \\
x_2^*(k+1) &= x_2^*(k) - \mu(\lambda_1\lambda_2 + H(x_1 + x_2 - 6) + H(\max(0,5))) \\
\lambda_1(k+1) &= \lambda_1(k) - \mu(\lambda_2 + H(\max(0,3))) \\
\lambda_2(k+1) &= \lambda_2(k) - \mu(\lambda_1 + H(\max(0,7)))
\end{aligned}$$

Solve this problem, the neural network in Figure 2 is simulated. The parameters of the network were chosen as $\mu = 0.001$, and $H = 0.1$. The network converges in approximately 1000 iterations. The compromise solution is obtained as.

$$x^{*(1)} = \begin{pmatrix} 2.513 \\ 3.4 \end{pmatrix}, \tag{50}$$

$$\lambda^{*(1)} = \begin{pmatrix} 0.562 \\ 0.294 \end{pmatrix}, \tag{51}$$

7 Conclusions

We have proposed a recurrent neural network for solving fuzzy stochastic multi-objective programming problems with general nonlinear constraints. The proposed neural network has a simpler structure and a lower complexity for implementation than the existing neural networks for solving such problems. It is shown here that the proposed neural network is stable in the sense of Lyapunov and globally convergent to an optimal solution. Compared with the existing convergence results, the present results do not require Lipschitz continuity condition on the fuzzy stochastic multi-objective function.

References

[1] A. Charnes and W. Cooper, " Chance constrained programming ", Management Sci. 6 (1959) 73-79.

[2] Fredric M. Ham " Principles of Neurocomputing for Science and Engineering " McGraw-Hill Higher Education, International Edition 2001

[3] H. Li, B. S. Manjunath, and S. K. Mitra, "Multisensor image fusion using the wavelet transform," Graph. Models Image Process., vol. 57, no. 3, pp. 235-245, 1995.

[4] J.P.Leclercq, "Stochastic programming: an interactive multicriteria approach", Eur. J. Oper. Res. 10 (1982)33-41.

[5] J. Jr. Teghem, D. Dufrance, M. Thauvoye and P. Kunch, Strange, "An interactive method for multi-objective linear programming under uncertainty", Eur. J. Oper. Res. 26 (1986) 65-82.

[6] M.P. Kennedy and L. O. Chua, "Neural networks for nonlinear programming," IEEE Transactions on Circuits and Systems, vol. 35, no. 5, 554-562, 1988.

[7] M. El-Sayed Wahed " Neural Network Represention of Stochastic Quadratic Programming Problems with bi's and cj's Follow Goint Distribution" Jour. Inst. Maths. & Computer Sciences, Vol1. 15, No.2, pp. 263-279, (2004).

[8] M. El-Sayed Wahed " Neural Network For Stochastic Linear Programming Problems " Egyption Informatics Journal , Vol. 5/ No.2, pp. 28-42, (2004).

[9] M. El-Sayed Wahed " Neural Network For Stochastic Multi-objective Optimization Problems " JOURNAL OF CONCRETE AND APPLICABLE MATHEMATICS,Vol. 5/ No.3, pp. 219-229, (2007).

[10] M. R. Hestens, "Multiplier Gradient Methods," Journal of Optimization Theory and Applications, vol. 4, pp. 303-320, (1969).

[11] M. Wahed, "On some solution for solving a stochastic programming problem with known probability distribution", M. Sc. Thesis, Department of Computer Science, Institute of Statistical Studies and Research, Cairo University, (1991).

[12] N.S. Kambo, "Mathematical Programming Techniques", (Affiliated East-West Press Pvt. Ltd., 1984).

[13] Q. Tao, J. D. Cao, M. S. Xue, and H. Qiao, "A high performance neural network for solving nonlinear programming problems with hybrid constraints," Phys. Lett. A, vol. 288, no. 2, pp. 88-94, 2001.

[14] S. S. Rao, "Optimization Theory and Applications", (Wiley Eastern Limited, New Delhi, 2nd edn,1984).

[15] W. Mohamed, "Fuzzy programming approach to multi-objective stochastic linear programming problems with a known distribution", The Journal of Fuzzy Mathematics Vol. 7, No. 3, (1999)

[16] W. Mohammed, "Chance constrained fuzzy goal programming with right-hand side uniform random variable coefficients", Fuzzy sets and systems 109 (2000) 107-110.

[17] W.H. Press, B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, Numerical Recipes in C, Cambridge University Press, Cambridge.

[18] Y.S. Xia and J. Wang, "A general methodology for designing globally convergent optimization neural networks," IEEE Trans. Neural Netw., vol. 9, no. 6, pp. 1331-1343, Nov. 1998.

[19] Y.S. Xia and G. Feng "A Modified Neural Network for Quadratic Programming with Real-Time Applications", Neural Information Processing-Letters and Reviews, vol. 3, no.3, pp. 69-76, 2004.

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Project Management with Capital Budgeting in Model-Driven Architecture

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Project Management with Capital Budgeting in Model-Driven Architecture

Abstract

Because project management is part of investment decisions, it should be coordinated with capital budgeting. Under the framework of capital budgeting, we should allow the flexibility that can be represented by future actions. There are several options for investment decisions, such as an abandonment option, a deferral option, an expansion option, a shrinkage option, and a switching option.

In addition, because project management is part of enterprise management, it should also be coordinated with enterprise architecture. A complicated system of merging, importing, and sharing resource related data across multiple project files is needed. We have integrated both project management and capital structure into Enterprise Architecture. This is a Unified Model Language based case tool.

Keywords: Project Management, Capital Budgeting, Investment Options, UML, MDA

1. Introduction

Modern projects, ranging from high-tech R&D researches to general engineering management, are amazingly large, complex, and costly. Completing such projects on time and within budget is not an easy task. To facilitate this, there are several questions to be answered: (1) What is the expected project completion date? (2) What is the potential variability in this date? (3) What activities are critical in the sense that they must be completed exactly as scheduled? (4) How might resources be concentrated most effectively on activities in order to speed up project completion? (5) What controls can be exercised on the flows of expenditures for the various activities throughout the duration of the project so that the overall budget can be maintained?

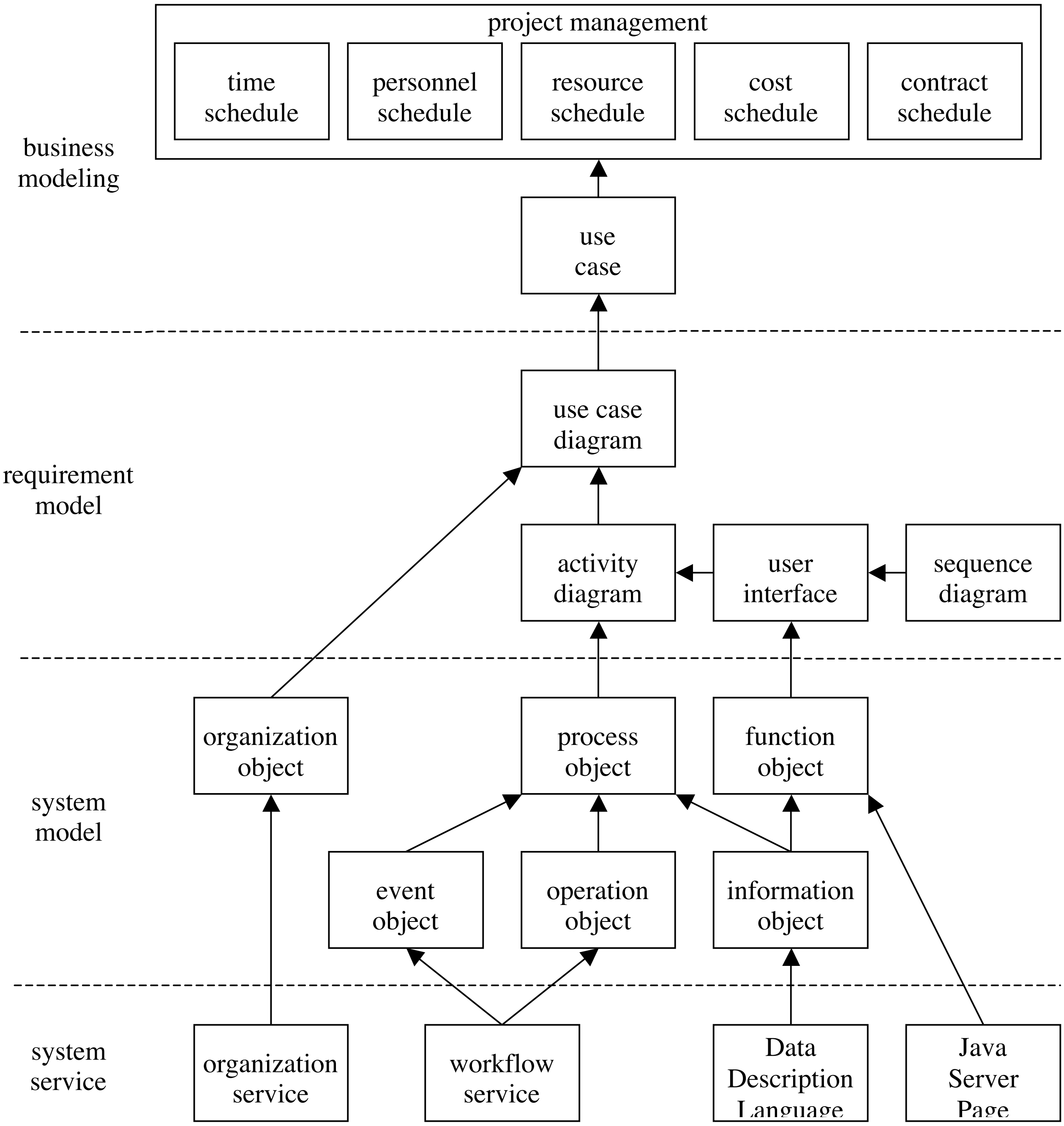
Whether a project can succeed or not depends not only on its cash outflows, which is generally the focus of project management, but also on its cash inflows. Until now, capital budgeting was not been an issue in project management. However, if we cannot meet the cash outflows with appropriate cash inflows, the company may have the problems meeting its liabilities. Once a company cannot meet its liabilities, there will be a serious problem causing by lenders lack of confidence, resulting in a domino effect. We should therefore integrate capital budgeting into project management. In addition, top managers allocate resources across different projects, and software that considers a project to be a complete entity cannot cope with this complexity. A complicated system of merging, importing, and sharing resource related data across multiple project files is needed. We have therefore developed a free flowing and modifiable project hierarchy in a modified Unified Modeling Language (UML) based case tool.

2. Blueprint of Enterprise Architecture

Although Enterprise Resource Planning (ERP) is the most common software for enterprises, there have been many negative comments on its rigidity and implementation cost. On the contrary, a UML model can be either platform-independent or platform-specific, and the conversion step is highly automated. We can direct it to generate calls using whatever interfaces and protocols are required, even if these run a cross platforms. Model-Driven Architecture (MDA) / UML applications are future-proof, i.e., MDA-enabled tools can be updated to include new items. Using XML Metadata Interchange, we can transfer the proposed UML model from one tool into a repository, or into another tool for refinement or to the next step in our chosen development process. With these benefits of flexibility and standardization, we integrate project management into a modified MDA/UML enterprise system.

There are four layers in the enterprise architecture, as shown in Figure 1. The business requirement model is composed of use case diagram, activity diagram, user interface and sequence diagram. The system design model is composed of organization object, process object, function object and information object. The activity diagram is an event-driven model. Organization service is composed of data structure and application program interface.

Figure 1. The Four Layer Mapping in MDA



3. Project Management and Capital Budgeting

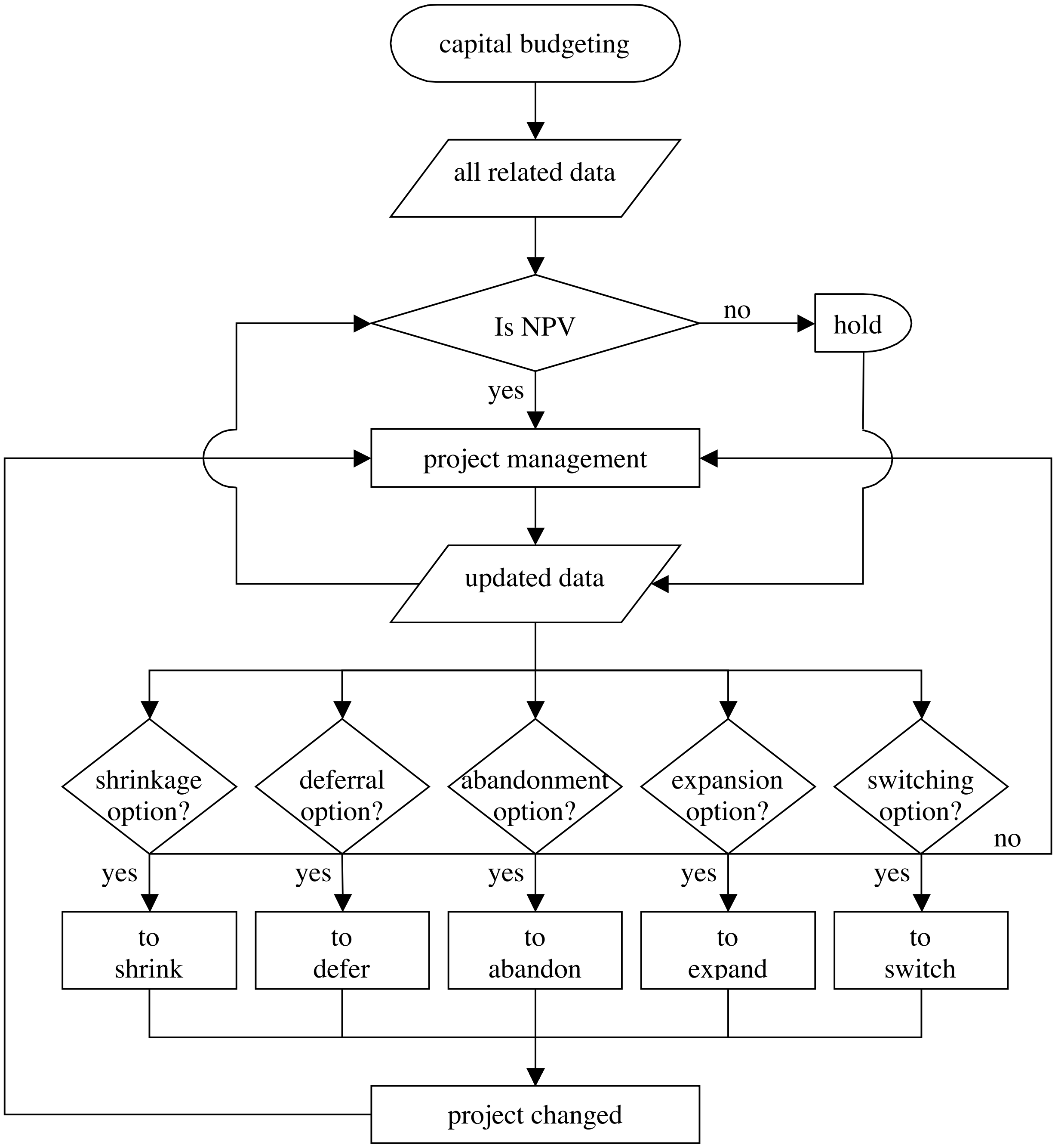
To evaluate a project, we must consider all of the cash outflows and the cash inflows. In contrast, for project management, we generally consider only the cash outflows, and we ignore many important related issues, such as the time value of money. If a project lasts for a long time, the compounding factor will significantly affect the performance of the project.

Project management should be considered as part of the overall capital investment decisions, and the following important concepts should be applied: (1) stand-alone principle, (2) incremental cash flows principle, (3) putting aside sunk costs principle, (4) opportunity costs principle, and (5) including side effects principle. The net present value (NPV) approach is the basic approach for evaluating investments. It involves asking what-if questions and includes many options, such as a shrinkage option, a deferral option, an abandonment option, an expansion option, and a switching option (a shut-down / reopen decision).

The basic form of what-if analysis is called scenario analysis. There are a number of possible scenarios that can be considered, such as the worst-case scenario and the best-case scenario. The second form is sensitivity analysis, which is a variation on scenario analysis. It freezes all of the variables except one and then sees how sensitive the estimate of NPV is to changes in that one variable. Simulation analysis is a combination of these two forms.

In the framework of capital budgeting, we can do the following break-even analysis: (1) accounting break-even which occurs when income is zero, (2) cash break-even which occurs when operating cash flow is zero, and (3) financial break-even which occurs when the NPV of the project is zero. We can also analyze operating leverage of the project. The degree of operating leverage is defined as the percentage change in operating cash flow relative to the percentage change in demand. Operating leverage is the degree to which a project is committed to fixed production costs. Generally speaking, projects with a relatively heavy investment in plant and equipment will have a relatively high degree of operating leverage. Such projects are said to be capital investment. The capital budgeting flowchart is shown in Figure 2.

Figure 2. Capital Budgeting Flowchart



4. Project Management and Software

Project management serves as a basis for communication and coordination with external entities in a company's inbound and outbound supply chain. Based on project management, commitments are made to subcontractors to deliver materials, support activities are planned, and due dates are set. During project execution, however, project activities are subject to considerable uncertainty.

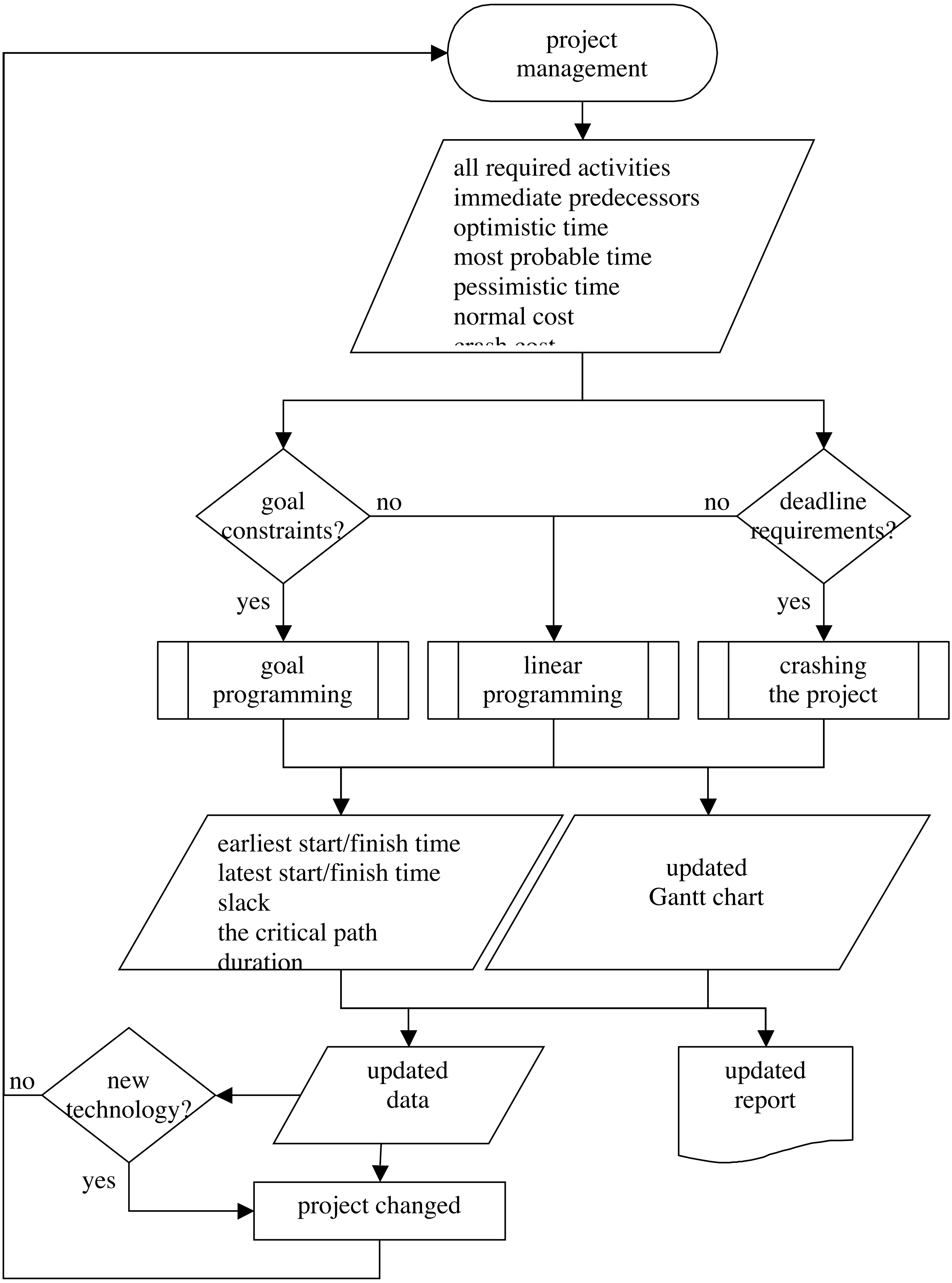
In the literature, surveyed by Herroelen and Leus (2005), there are five approaches to deal with uncertainty: (1) Reactive scheduling revises or re-optimizes the project schedule when an unexpected event occurs. (2) Stochastic scheduling concentrates on the so-called stochastic resource-constrained project scheduling. (3) Scheduling under fuzziness recommends the use of fuzzy numbers for modeling activity durations, rather than stochastic variables. Instead of probability distributions, these quantities make use of membership functions. (4) Proactive (robust) scheduling uses numerous techniques. (5) Sensitivity analysis addresses what-if questions, such as given a specific change of a parameter, what is a new optimal solution?

Basically, it is difficult to execute a project under such risky, uncertain or fuzzy model, so the first and last models are more realistic. Except fuzzy model, we estimate the expected value of the activity time under a particular probability distribution. The most commonly used probability distribution is the beta distribution. It is capable of assuming a wide variety of shapes. We use mode rather than mean in project management. Mode is the most likely value under normal circumstances.

By importing updated data and using advanced software, we can effectively cope with the uncertainty. For example, Intellisys has the following functions: (1) web-enabled collaboration, concurrent access collaboration, and office operation collaboration, (2) time tracking function, (3) security function, (4) hierarchical projects function, (5) spawning new project from existing project, archiving project, and restoring project, (6) assignment of resource allocation, (7) printing Gantt/PERT charts.

Gantt charts are useful tools for planning and scheduling projects in the following aspects: (1) to assess how long a project should take, (2) to lay out the order in which tasks need to be carried out, (3) to help manage the dependencies between tasks, (4) to determine the resources needed, (5) to monitor progress, (6) to see how remedial action may bring the project back on course. By the aid of advanced software, we can set up a project management flowchart, as shown in Figure 3.

Figure 3. Project Management Flowchart



5. Example

This is a hypothetical construction project. The activity list is in Table 1, and the time-cost trade-off data is in Table 2. Under the objective function to minimize the expected time to complete all the activities and the subjective constraints to follow the precedence relationships among the activities, we can get the earliest start time (ES), the earliest finish time (EF), the latest start time (LS), the latest finish time (LF), and the slack for each activity in Table 3. In addition, we can identify the critical path 2-1-3-4-6-9 and the total project cost $288,000. We can also get the Gantt chart, which is used as a record-keeping device for following the progression in time of the subtasks of a project. We can see which individual tasks are on or behind schedule. We can also check the planned cost as well as the actual cost.

Table 1. Activity List

activity number	activity	immediate predecessors	most probable time (weeks)
1	foundation, ceiling ,and walls	2	5
2	excavation	-	3
3	reinforcing bars	1	2
4	timbers, sheathing ,and shingles	3	3
5	electrical wiring	1	4
6	elevator pit	4	8
7	exterior siding	8	5
8	windows	1	2
9	paint	6, 7, and 10	2
10	inside wall board	5 and 8	3

Table 2. Time-Cost Trade-off Data

activity number	maximum crash time (weeks)	normal cost ($1,000)	crash cost ($1,000)	cost per crash time ($1,000/week)
1	2	50	72	(72-50)/2=11
2	1	20	30	(30-20)/1=10
3	1	15	30	(30-15)/1=15
4	2	8	20	(20-8)/2=6
5	0	30	30	-
6	4	13	21	(21-13)/4=2
7	4	45	65	(65-45)/4=5
8	1	45	52	(52-45)/1=7
9	0	40	40	-
10	1	22	34	(34-22)/1=12

Table 3. Spreadsheet for the Project Critical Path Analysis

activity number	ES	EF	LS	LF	slack
1	3	8	3	8	0
2	0	3	0	3	0
3	8	10	8	10	0
4	10	13	10	13	0
5	8	12	14	18	6
6	13	21	13	21	0
7	10	15	16	21	6
8	8	10	14	16	6
9	21	23	21	23	0
10	12	15	18	21	6

In order to set up the event-driven system, we use XML to describe conditions, filters, and codes. The even-driven system is in Figure 4. In the condition block, we describe the events that the system will be triggered to operate. In the filter block, we describe the logical conditions to operate. In the code block, there is the code generator. We simulate some event-driven models in Table 4.

Figure 4. Event-driving System

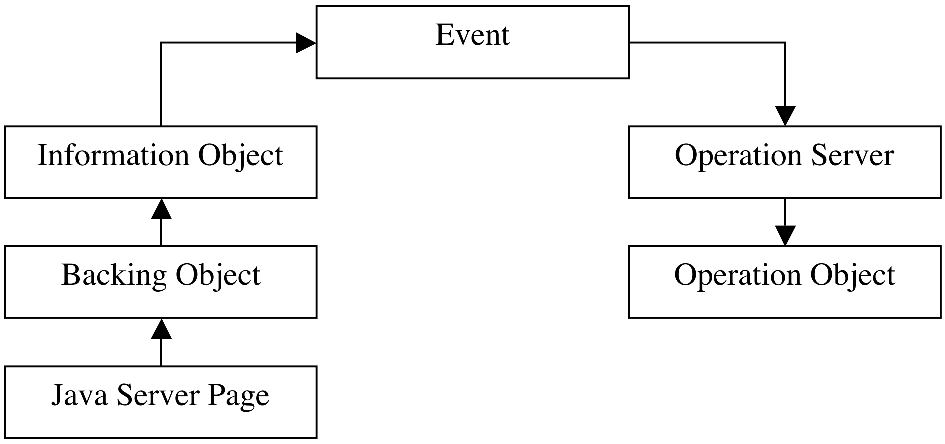


Table 4. Event-driven Models

event	subroutine	result
(1) crashing requirement:	linear programming:	duration=16 weeks
$EF_9 <= 16$	Min $11Y_1 + 10Y_2 + 15Y_3$	total crash cost=$318,000
	$+6Y_4 + 2Y_6 + 5Y_7$	the critical path
	$+7Y_8 + 12Y_{10}$	= 2- 1- 3- 4- 6- 9
	s. t. $ES_1 - ES_2 >= 3 - Y_2$	$Y_2^* = 1$
	$ES_5 - ES_1 >= 5 - Y_1$	$Y_4^* = 2$
	$ES_8 - ES_1 >= 5 - Y_1$	$Y_6^* = 4$
	$ES_3 - ES_1 >= 5 - Y_1$	
	$ES_4 - ES_3 >= 2 - Y_3$	
	$ES_6 - ES_4 >= 3 - Y_4$	
	$ES_{10} - ES_5 >= 4 - Y_5$	
	$ES_{10} - ES_8 >= 2 - Y_8$	
	$ES_7 - ES_8 >= 2 - Y_8$	
	$ES_9 - ES_6 >= 8 - Y_6$	
	$ES_9 - ES_7 >= 5 - Y_7$	
	$ES_9 - ES_{10} >= 3 - Y_{10}$	

Table 4. Event-driven Models (continued)

event	subroutine	result
	$EF_9 = ES_9+2-Y_9$	
	$EF_9 <= 16$	
	$Y_1 <= 2$	
	$Y_2 <= 1$	
	$Y_3 <= 1$	
	$Y_4 <= 2$	
	$Y_6 <= 4$	
	$Y_7 <= 4$	
	$Y_8 <= 1$	
	$Y_{10} <= 1$	
	$Y_5 = Y_9 = 0$	
	all variable>=0	
	Y_i: time crashed	
	For activity i	

Table 4. Event-driven Models (continued)

event	subroutine	result
(2) new technique offered: multi-functional windows for traditional windows most probable time = 3 weeks normal cost=$48,000	linear programming: Min EF_9 s. t. $ES_1-ES_2 >= 3$ $ES_5-ES_1 >= 5$ $ES_8-ES_1 >= 5$ $ES_3-ES_1 >= 5$ $ES_4-ES_3 >= 2$ $ES_6-ES_4 >= 3$ $ES_{10}-ES_5 >= 4$ $ES_{10}-ES_8 >= 2$ $ES_7-ES_8 >= 2$ $ES_9-ES_6 >= 8$ $ES_9-ES_7 >= 5$ $ES_9-ES_{10} >= 3$ $EF_9 = ES_9 + 2$ all variables >=0	duration= 23 weeks total normal cost = $291,000 the critical path = 2- 1- 3- 4- 6- 9

Table 4. Event-driven Models (continued)

event	subroutine	result
(3) the absolute priority ($P_1, P_2, \ldots$) for the goal constraints: P_1: $EF_9 <= 12$ weeks P_2: total crash cost <= \$340,000	goal programming: Min $P_1V_1 + P_2V_2$ s. t. $EF_9+U_1-V_1= 12$ $(11Y_1+10Y_2+15Y_3 +6Y_4+2Y_6+5Y_7+7Y_8 +12Y_{10})+U_2-V_2 =340,000$ and, all the system constraints in the model of crashing requirement aforementioned	duration= 13 weeks total crash cost = \$355,000 the critical path = 2- 1- 3- 4- 6- 9 $V_1^*= 1$ $V_2^*= 15,000$
(4) The total expected discounted cash outflow equals \$287,352 and the total expected discounted cash inflow equals \$380,000 at the beginning of the project (the discounted rate equals .0002 per week).	Capital budgeting	Because the updated NPV is less than zero, it is suggested to abandon or defer the project.

Table 4. Event-driven Models (continued)

event	subroutine	result
After one week, the updated expected cash inflow equals \$280,000 and the expected cash outflow equals \$280,683. Both are discounted to the end of the first week excluding the sunk cost in the first week.		

6. Concluding Remarks

Project management is part of capital budgeting under uncertainty in a multi-period setting. Thus, we should allow the flexibility that can be represented by future actions. To identify potential operating flexibility and strategic factors, project management should cope with capital budgeting. In addition, because project management is part of an enterprise's operations, a complicated system of merging, importing, and sharing resource related data across multiple project files is needed. We therefore use a UML based case tool to integrate project management into enterprise architecture.

References

C. Chapman, S. Ward, Constructively simple estimating: a project management example, *Journal of the Operational Research Society*, 54, 1050-1058 (2003).

R. L. Daniels, P. Kouvelis, Robust scheduling to hedge against processing time uncertainty in single-stage production, *Management Science*, 41, 363-376 (1995).

E. Demeulemeester, M. Vanhoucke, W. Herroelen, RanGen: a random generator for activity-on-the-node networks, *Journal of Scheduling*, 6, 17-38 (2003).

S. E. Elmaghraby, On the fallacy of averages in project risk management. European, *Journal of Operational Research*, 165, 307-313 (2005).

A. A. Fernandez, R. L. Armacost, J. Pet-Edwards, 1996. The role of the non-anticipativity constraint in commercial software for stochastic project scheduling, *Computers and Industrial Engineering*, 31, 233-236 (1996).

D. Golenko-Ginzburg, A. Gonik, Stochastic network project scheduling with non-consumable limited resources, *International Journal of Production Economics*, 48, 29-37 (1997).

W. Herroelen, R. Leus, Project scheduling under uncertainty: survey and research potentials, *European Journal of Operational Research*, 165, 289-306 (2005).

A. Marucheck, R. Pannesi, C. Anderson, An exploratory study of the manufacturing strategy process in practice, *Journal of Operations Management*, 9, 101-123 (1990).

S. V. Mehta, R. M. Uzsoy, Predictable scheduling of a job shop subject to breakdowns, *IEEE Transactions on Robotics and Automation*, 14, 365-378 (1998).

J. R. Wang, A fuzzy set approach to activity scheduling for product development, *Journal of the Operational Research Society*, 50, 1217-1228 (1999).

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Spectral Properties of Numerical Differentiation

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Abstract

We study the numerical differentiation formulae for functions given in grids with arbitrary number of nodes. We investigate the case of the infinite number of points in the formulae for the calculation of the first and the second derivatives. The spectra of the corresponding weight coefficients sequences are obtained. We examine the first derivative calculation of a function given in odd-number points and analyze the spectra of the weight coefficients sequences in the cases of both finite and infinite number of nodes. We derive the one-sided approximation for the first derivative and examine its spectral properties.

Mathematics Subject Classification: Primary 65D25; Secondary 65T50

Introduction

The problem of numerical differentiation is a long-standing issue. There are plenty of published works devoted to the generation of finite difference formulae in both one and multi dimensional lattices (see, e.g., Ref. [1]). However, many of those methods require preliminary construction of an interpolating polynomial, and hence are very awkward. Moreover, the majority of the previous techniques are valid in the case of a function given in the limited number of nodes.

The finite difference formulae for the calculation of any order derivative in a one dimensional grid with arbitrary spacing were discussed in Refs. [2, 3]. However, only recursion relations for the weight coefficients have been derived. The explicit formulas for the derivatives calculation were recently derived in Ref. [4] on the basis of the generalized Vandermonde determinant.

It should be noted that the low order derivatives (the first and the second ones) as well as equidistant lattices are of the major importance in many problems of applied mathematics and physics. The first and the second numerical derivatives in the equidistant one dimensional grid were studied in Ref. [5]. The finite difference formulae for the central derivatives of a function given on a lattice with arbitrary number of elements have been derived in that work. It is important that these formulae have been obtained in the explicit form. This method enabled one to examine the spectral properties of weight coefficients sequences as well as to analyze the accuracy of the numerical differentiation.

In the present paper we continue to study the numerical differentiation formulae for functions given in grids with arbitrary number of nodes. On the basis of the results of Ref. [5] in Sec. 1 we investigate the case of the infinite number of points in the formulae for the calculation of the first and the second derivatives. The spectra of the corresponding weight coefficients sequences are also obtained. Then, in Sec. 2 we examine the first derivative calculation of a function given in odd-number points. We also analyze the spectra of the weight coefficients sequences in the cases of both finite and infinite number of nodes. In Sec. 3 we derive the one-sided approximation for the first derivative and examine its spectral properties. It is worth noticing that the derivations of the finite difference formulae in all cases are performed for the arbitrary number of points. Finally, in Sec. 4 we resume our results.

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1 Spectral properties of the first and the second derivatives for infinite number of points

Let us study the function $f(x)$ given in the equidistant points x_m, $f(x_m) = f_m$, where $m = 0, \ldots, \pm n$. It was found in Ref. [5] that the first and the second derivatives are approximated as

$$f'(0) \approx \frac{1}{2h} \sum_{m=1}^{n} \alpha_m^{(1)}(n)(f_m - f_{-m}). \tag{1.1}$$

and

$$f''(0) \approx \frac{1}{h^2} \sum_{m=1}^{n} \alpha_m^{(2)}(n)(f_m - 2f(0) + f_{-m}), \tag{1.2}$$

where h is the distance between nodes. The coefficients $\alpha_m^{(1)}(n)$ and $\alpha_m^{(2)}(n)$ can be calculated explicitly for arbitrary n (see Ref. [5]).

The spectral properties of the sequences $\alpha_m^{(1)}(n)$ and $\alpha_m^{(2)}(n)$ in Eqs. (1.1) and (1.2) in the case of finite number of interpolation points were carefully examined in Ref. [5]. We found out that the more points we involved in the sequence $\alpha_m^{(1)}(n)$ the more close to linear the corresponding spectrum was. Thus the considered sequence produces more accurate first derivative of a function in the case of great number of points. As for the sequence $\alpha_m^{(2)}(n)$, it was also shown in Ref. [5] that its spectrum approached to parabola if $n > 1$. We expect that the corresponding spectra will be exactly linear and parabolic ones if $n \to \infty$.

Let us consider the spectral characteristics of the sequences $\alpha_m^{(1)}(n)$ and $\alpha_m^{(2)}(n)$ in the case of infinite number of interpolation points. First we remind the result for the $\alpha_m^{(1)}(n)$ in the limit $n \to \infty$ (see Ref. [5])

$$\alpha_m^{(1)} = \lim_{n\to\infty} \alpha_m^{(1)}(n) = (-1)^{m+1}\frac{2}{m}. \tag{1.3}$$

The Fourier transform of a function $f(x)$ can be presented in the form (see, e.g., Ref. [6])

$$c(\omega) = h \sum_x e^{-i\omega x} f(x) = h \sum_{m=-\infty}^{+\infty} e^{-i\omega mh} f(mh). \tag{1.4}$$

The inverse Fourier transformation is given by the following expression:

$$f(x) = \int_{-\pi/h}^{\pi/h} \frac{d\omega}{2\pi} c(\omega) e^{i\omega x}, \quad x = kh,$$

and has the cutoff at high frequencies, $|\omega| \leq \pi/h$.

Now we can calculate the spectrum of the sequence $\alpha_m^{(1)}$,

$$\beta_1(\omega) = h \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} e^{-i\omega mh}\alpha_m^{(1)} = -4ih \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\sin(m\omega h)}{m} = -2i\omega h^2, \tag{1.5}$$

where we use Eqs. (1.3) and (1.4). Note that Eq. (1.5) is valid if $0 \leq \omega < \pi/h$. The first derivative of the function can be expressed via the spectra $\beta_1(\omega)$ and $c(\omega)$,

$$f'(x) = \frac{1}{2h} \int_{-\pi/h}^{\pi/h} \frac{d\omega}{2\pi h} \beta_1^*(\omega) c(\omega) e^{i\omega x}, \quad x = kh. \tag{1.6}$$

Using the result for the calculation of $\beta_1(\omega)$ presented in Eq. (1.5) we readily find that

$$f'(x) = \int_{-\pi/h}^{\pi/h} \frac{d\omega}{2\pi} (i\omega) c(\omega) e^{i\omega x}. \tag{1.7}$$

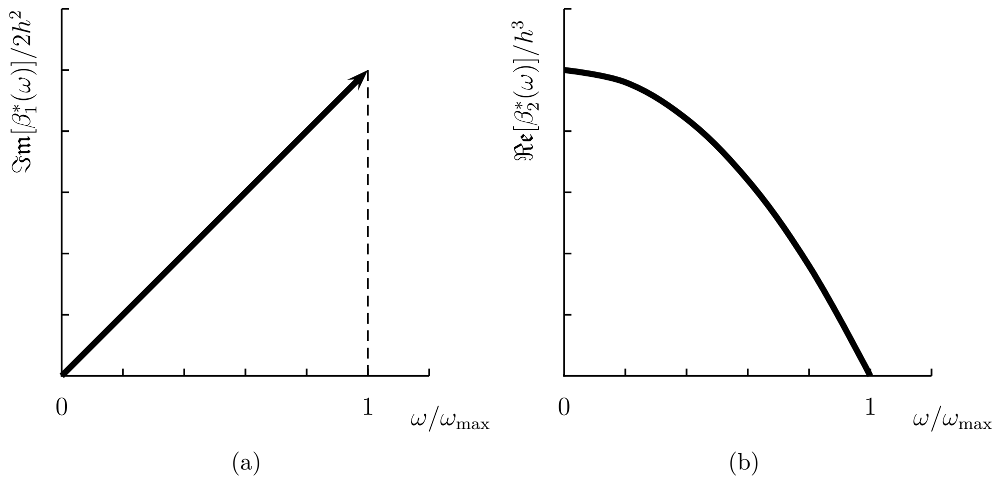


Figure 1: The spectra of differentiating filters, (a) $\alpha_m^{(1)}$ and (b) $\alpha_m^{(2)}$, in the case of infinite number of points.

Eq. (1.7) shows that the first derivative calculation with help of the sequence $\alpha_m^{(1)}$ gives the exact value of the derivative in the case of infinite number of interpolation points for all frequencies except $\omega_{\max} = \pi/h$. The fact that the first derivative computation does not give correct results at $\omega = \omega_{\max}$ also follows from Fig. 1(a). However, it can be verified directly with help of Eq. (1.3) for the function $f_m = (-1)^m = \cos(\omega_{\max} mh)$.

In Ref. [5] we showed that the use of the coefficients $\alpha_m^{(1)}$ give exact value for the first derivative of the function $y(x) = \sin(\omega_{\max} x/2)$. However, Eq. (1.7) [see also Fig. 1(a)] indicates that this method will give the correct results not only for $\omega = \omega_{\max}/2$ but also for all frequencies $\omega < \omega_{\max}$.

The second derivative calculation in the case of infinite number of interpolation points can be analyzed in the similar manner as we have done it for the first derivative. The explicit form of the sequence $\alpha_m^{(2)}$ is

$$\alpha_m^{(2)} = \lim_{n\to\infty} \alpha_m^{(2)}(n) = (-1)^{m+1}\frac{2}{m^2}.$$

For the spectrum of the sequence $\alpha_m^{(2)}$ we obtain

$$\beta_2(\omega) = -\omega^2 h^3 + \frac{\pi^2}{3}h. \tag{1.8}$$

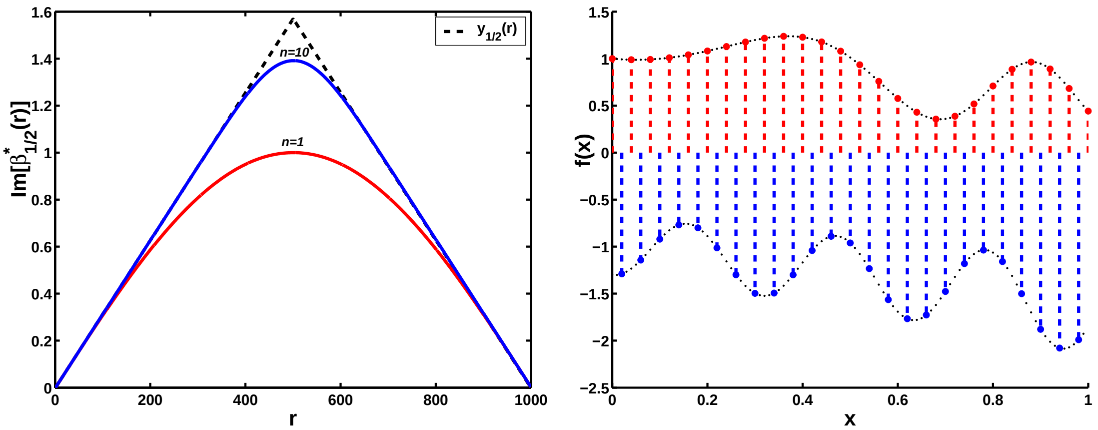
It should be noted that Eq. (1.8) is valid for all frequencies $0 \le \omega \le \pi/h$. The expression for the second derivative takes the form

$$f''(x) = \frac{1}{h^2}\int_{-\pi/h}^{\pi/h} \frac{d\omega}{2\pi h}\left[\beta_2^*(\omega) - \beta_2^*(0)\right] c(\omega) e^{i\omega x} = \int_{-\pi/h}^{\pi/h} \frac{d\omega}{2\pi}(-\omega^2) c(\omega) e^{i\omega x}, \quad x = kh. \tag{1.9}$$

Eq. (1.9) demonstrates that the computation of the second derivative with the use of the sequence $\alpha_m^{(2)}$ gives the exact results in the case of infinite number of interpolation points for all frequencies even including the maximal one. The spectrum $\beta_2(\omega)$ is depicted in Fig. 1(b).

2 The first derivative computation of a function given in odd-number points

In this section we discuss the calculation of the first derivative in the case of a function given in odd-number nodes. Then we discuss the spectral properties of the derived weight coefficients sequences in the case of both finite and infinite number of nodes.



(a) (b)

Figure 2: (a) The spectra of various sequences $\alpha_m^{(1/2)}(n)$ at $N = 2000$. (b) The example of the use of the sequence $\alpha_m^{(1/2)}(n)$. A rapidly oscillating function corresponds to the dashed lines, its envelope functions are shown by the dotted lines.

It follows from Fig. 1(a) that the computation of the first derivative gives unsatisfactory results at high frequencies near $\omega_{\max}$. In order to introduce the numerical differentiation of such rapidly oscillating functions we consider the modified sequence

$$\alpha_{2m+1}^{(1/2)}(n) = \frac{1}{(2m+1)\pi_m^{(1/2)}(n)}, \quad m = 0, \dots, n-1, \tag{2.1}$$

where

$$\pi_m^{(1/2)}(n) = \prod_{\substack{k=0 \\ k \neq m}}^{n-1} \left(1 - \frac{(2m+1)^2}{(2k+1)^2}\right),$$

and $\alpha_{2m}^{(1/2)}(n) = 0$.

The coefficients in Eq. (2.1) can be formally derived if we consider the first derivative calculation of a function given in the odd-number points only

$$f'(0) \approx \frac{1}{2h} \sum_{m=0}^{n-1} \alpha_{2m+1}^{(1/2)}(n)(f_{2m+1} - f_{-2m-1}). \tag{2.2}$$

Note that originally the function $f(x)$ was given in $2n+1$ points.

It is worth noticing that the results for the computation of the weights with help of Eq. (2.1) in some particular cases (namely for $n = 3, 5, 7$ and 9) coincide with those presented in Ref. [2] for the centered approximations at a 'half-way' point. However, the method for the central derivatives calculation elaborated in our paper enables one to get the expressions for the weight coefficients in the explicit form for any number of nodes.

We consider the spectral properties of the obtained sequence $\alpha_m^{(1/2)}(n)$. Using the technique developed in Ref. [5] one can compute the spectrum of the sequence in question,

$$\beta_{1/2}(r) = \sum_{m=0}^{N-1} \alpha_m^{(1/2)}(n) \exp\left(-i\frac{2\pi}{N}mr\right).$$

The spectra of the sequences $\alpha_m^{(1/2)}(n)$ are depicted in Fig. 2(a) for the various values of n at $N = 2000$. The function $y_{1/2}(r)$ has the form

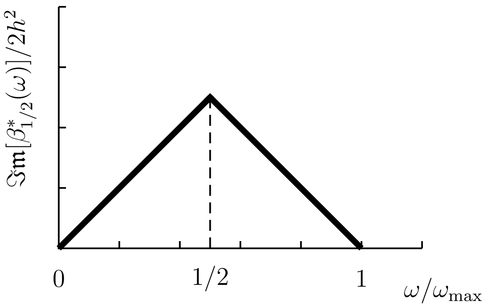


Figure 3: The spectrum of the sequence $\alpha_m^{(1/2)}$ in the case of infinite number of points.

$$y_{1/2}(r) = \begin{cases} 2\pi r/N, & \text{if } 0 \le r \le N/4, \\ \pi - 2\pi r/N, & \text{if } N/4 \le r \le N/2. \end{cases}$$

It follows from this figure that for $n = 1$ the imaginary part of the spectrum is the function $\sin(2\pi r/N)$. The linearity condition is satisfied only in the vicinity of zero and $N/2$. However at $n = 10$ linearity condition remains valid for $r \lesssim 350$ and $r \gtrsim 650$.

We examine the details of the differentiation procedure performed with help of the sequence $\alpha_m^{(1/2)}(n)$. If a function is slowly varying, then Eq. (2.2) gives the approximate value of the first derivative. It also results from Fig. 2(a). However, if a function is rapidly oscillating (e.g., at $\omega \lesssim \omega_{\max}$), we can consider its upper and lower envelope functions. Therefore the use of the sequence $\alpha_m^{(1/2)}(n)$ will produce the first derivative of envelope functions since in Eq. (2.2) we calculate the sum in odd-number points only. Envelope functions can be treated as "smooth" in this case. Fig. 2(b) schematically illustrates this process.

Now let us discuss the case of infinite number of the interpolation points. We can treat the sequence $\alpha_m^{(1/2)}(n)$ in the similar way as it was done in Ref. [5]. Indeed, proceeding to the limit $n \to \infty$ in Eq.(2.1) we find that

$$\alpha_{2m+1}^{(1/2)} = \lim_{n\to\infty} \alpha_{2m+1}^{(1/2)}(n) = (-1)^m \frac{4}{\pi(2m+1)^2}. \tag{2.3}$$

In Eq. (2.3) we used the known value of infinite product,

$$\prod_{k=0}^{\infty} \left(1 - \frac{x^2}{(2k+1)^2}\right) = \cos\left(\frac{\pi x}{2}\right).$$

With help of Eq. (2.3) it is possible to obtain the spectrum of the sequence $\alpha_m^{(1/2)}$

$$\beta_{1/2}(\omega) = -2ih \times \begin{cases} \omega h, & \text{if } 0 \le \omega \le \pi/2h, \\ (\pi - \omega h), & \text{if } \pi/2h \le \omega \le \pi/h. \end{cases} \tag{2.4}$$

The imaginary part of the spectrum $\beta_{1/2}(\omega)$ is presented in Fig. 3. As it follows from Eq. (2.4) (see also Fig. 3) the sequence $\alpha_m^{(1/2)}$ performs the differentiation of a function in question if $\omega \le \omega_{\max}/2$, or its envelope functions if $\omega \ge \omega_{\max}/2$. The of case $\omega = \omega_{\max}/2$ was also considered in details in Ref. [5].

3 One-sided approximation for the first derivative

In this section we derive the weight coefficients for the one-sided approximation of the first derivative and then we analyze the spectral characteristics of the weight coefficients sequence. It should be noted that the derivation of the weight coefficients is analogous to case the of the central derivatives which was carefully examined in Ref. [5].

Without restriction of generality we suppose that we approximate the first derivative in the zero point. Let us consider the function $f(x)$ given in the equidistant nodes $x_m = mh > 0$, where $m = 0, \dots, n$, and h is the constant value. We can pass the interpolating polynomial of the nth power through these points,

$$P_n(x) = \sum_{k=0}^{n} c_k x^k.$$

The values of the function in the nodes $x_m = mh$, $f_m = f(x_m)$, should coincide with the values of the interpolating polynomial in these points,

$$f_m = \sum_{k=0}^{n} c_k h^k m^k. \tag{3.1}$$

In order to find the coefficients c_k, $k = 0, \dots, n$, we receive the system of inhomogeneous linear equations with the given free terms f_m. It will be shown below that this system has the single solution.

We will seek the solution of the system (3.1) in the following way:

$$c_k = \frac{1}{h^k} \sum_{m=0}^{n} f_m a_m^{(k)}(n),$$

where $a_m^{(k)}(n)$ are the undetermined coefficients satisfying the condition,

$$\sum_{m=0}^{n} a_m^{(l)}(n) m^k = \delta_{lk}, \quad l, k = 0, \dots n. \tag{3.2}$$

It is worth to be noted that, if we set $k = 0$ and $l \neq 0$ in Eq. (3.2), we obtain the constraint which should be imposed on the coefficients $a_m^{(l)}(n)$

$$\sum_{m=0}^{n} a_m^{(l)}(n) = 0, \quad l = 1, \dots n. \tag{3.3}$$

Analogous relation between the weight coefficients was established in Ref. [2]. In deriving of Eq. (3.3) (as well as in all subsequent similar formulae) we suppose that $m^0 = 1$ if $m = 0$.

Let us resolve the system of equations (3.2) according to the Cramer's rule

$$a_m^{(l)}(n) = \frac{\Delta_m^{(l)}(n)}{\Delta_0(n)}, \tag{3.4}$$

where

$$\Delta_0(n) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & n \\ 0 & 1 & 2^2 & \dots & n^2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2^n & \dots & n^n \end{vmatrix} = n! \prod_{1 \le i < j \le n} (j - i) \neq 0, \tag{3.5}$$

and

$$\Delta_m^{(l)}(n) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & m-1 & 0 & m+1 & \dots & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2^l & \dots & (m-1)^l & 1 & (m+1)^l & \dots & n^l \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2^n & \dots & (m-1)^n & 0 & (m+1)^n & \dots & n^n \end{vmatrix}. \tag{3.6}$$

In Eq. (3.5) we use the formula for the calculation of the Vandermonde determinant. From Eq. (3.5) it follows that the determinant of the system of equations (3.2) is not equal to zero, i.e. the system of equations (3.1) has the single solution.

The most simple expression for $\Delta_m^{(l)}(n)$ is obtained in the case of $l = 1$ that corresponds to the calculation of the first-order derivative

$$\Delta_m^{(1)}(n) = (-1)^{m+1} \left(\frac{n!}{m}\right)^2 \prod_{\substack{1 \le i < j \le n \\ i,j \ne m}} (j - i), \quad m = 1, \ldots, n. \tag{3.7}$$

From Eq. (3.4) as well as taking into account Eqs. (3.5) and (3.7) we get the expression for the coefficients $a_m^{(1)}(n)$

$$a_m^{(1)}(n) = (-1)^{m+1} \frac{1}{m} \binom{n}{m}, \quad m = 1, \ldots, n, \tag{3.8}$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!},$$

are the binomial coefficients. It is remarkable to note that the coefficient $a_1^{(1)}(n) = n$. To simplify numerical calculations (especially the analysis of the spectra of the derived sequences) Eq. (3.8) should be rewritten in the form

$$a_m^{(1)}(n) = \frac{1}{m p_m(n)}, \quad m = 1, \ldots, n,$$

where

$$p_m(n) = \prod_{\substack{k=1 \\ k \ne m}}^{n} \left(1 - \frac{m}{k}\right).$$

In order to find the coefficient $a_0^{(1)}(n)$ we use Eq. (3.3) rather than compute the determinant (3.6). Thus we receive the following expression for this coefficient,

$$a_0^{(1)}(n) = -\sum_{m=1}^{n} a_m^{(l)}(n) = -\sum_{m=1}^{n} (-1)^{m+1} \frac{1}{m} \binom{n}{m} = -\sum_{m=1}^{n} \frac{1}{m}. \tag{3.9}$$

Eqs. (3.8) and (3.9) provide the weight coefficients for the one-sided approximation of the first derivative of the function $f(x)$ given in $n+1$ equidistant nodes,

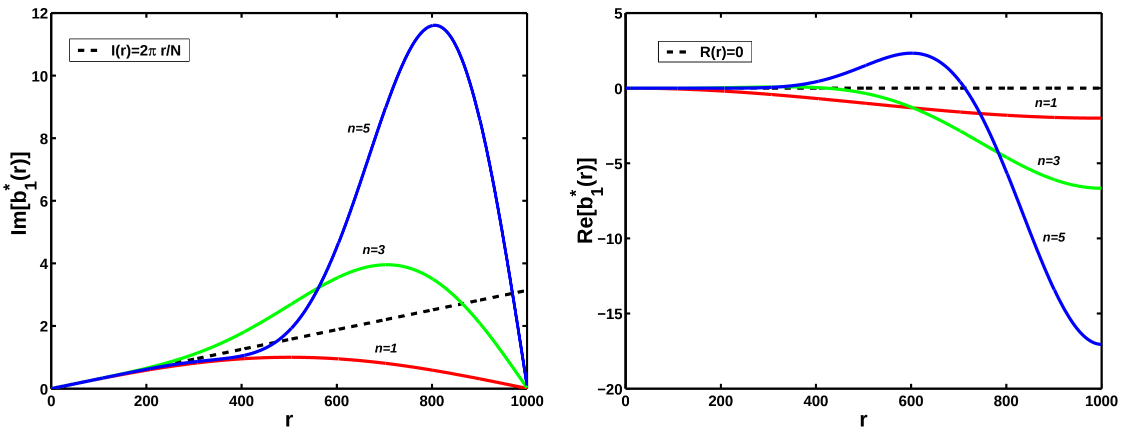
$$f'(0) \approx \frac{1}{h} \sum_{m=0}^{n} f_m a_m^{(1)}(n).$$

The results for the computation of the weights in some particular cases (namely for $n = 1, 2, \ldots, 8$) coincide with those presented in Ref. [2]. However, the technique for derivatives calculation developed in the present work allows one to obtain the expressions for the weight coefficients in the explicit form for any n.

Without derivation we mention that on the basis of Eqs. (3.4)-(3.6) one can find the coefficients $a_m^{(n)}(n)$ that correspond to the computation of the nth-order derivative

$$a_m^{(n)}(n) = (-1)^{m+n} \frac{1}{n!} \binom{n}{m}. \quad m = 0, \ldots, n. \tag{3.10}$$

It should be noted that Eq. (3.10) is consistent with Eq. (3.3).



(a) (b)

Figure 4: The imaginary parts (a) and the real parts (b) of the spectra of various sequences $a_m^{(1)}(n)$ at $N = 2000$.

Now we consider the spectral properties of the derived sequence $a_m^{(1)}(n)$. Using the results of the previous section (see also Ref. [5]) we readily find the expression for the spectrum of the considered sequence,

$$b_1(r) = \sum_{m=0}^{N-1} a_m^{(1)}(n) \exp\left(-i\frac{2\pi}{N}mr\right).$$

The imaginary parts, $\mathfrak{Im}\left[b_1^*(r)\right]$, of the spectra of the sequences $a_m^{(1)}(n)$ for the various values of n at $N = 2000$ as well as the linearly growing sequence $I(r) = 2\pi r/N$ are presented in Fig. 4(a). It follows form the this figure that the imaginary parts are close to the linear sequence only in the vicinity of zero ($r \lesssim 200$) even at $n = 5$. It points out that the one-sided approximation of the first derivative has worse accuracy in comparison with the central derivatives. This fact was also mentioned in Ref. [7]. Therefore, the application of the sequence $a_m^{(1)}(n)$ for the calculation of the one-sided first derivative will give reliable results only for slowly varying functions.

The spectrum $b_1(r)$ has not only imaginary part, but also nonzero real part since $a_0^{(1)}(n)\neq 0$. The real parts, $\mathfrak{Re}\left[b_1^*(r)\right]$, of the spectra of the sequences $\alpha_m^{(1)}(n)$ for the various values of n at $N = 2000$ as well as the constant sequence $R(r) = 0$ are shown in Fig. 4(b). It can be seen from this figure that the real parts of the spectra are close to zero if $r \lesssim 100$ at $n = 1$, and if $r \lesssim 300$ at $n = 3$ and $n = 5$. The deviation from zero is especially great if $r \gtrsim 700$ at $n = 3$, and if $r \gtrsim 500$ at $n = 5$. Such a behavior of the real parts of the spectra also reveals the limited level of accuracy of the one-sided first derivative.

4 Conclusion

In conclusion we note that in our paper we have studied the numerical differentiation formulae for functions given on grids with arbitrary number of nodes. In Sec. 1 we have investigated the case of the infinite number of points in the formulae for the calculation of the first and the second derivatives. The spectra of the corresponding weight coefficients sequences have been obtained. It has been revealed that the calculation of the first derivative with help of the derived formulae gave reliable results for all spacial frequencies except $\omega_{\max}$. As for the calculation of the second derivative we have shown that the corresponding formulae were valid for all spacial frequencies including $\omega_{\max}$. In Sec. 2 we have examined the first derivative calculation of a function given in odd-number points. We have also analyzed the spectra of the weight coefficients sequences in the cases of both finite and infinite number of nodes. It has been found out that the obtained formulae perform the differentiation of the considered function if $\omega \leq \omega_{\max}/2$, and its envelope functions if $\omega \geq \omega_{\max}/2$. In Sec. 3 we have derived the one-sided approximation for the first derivative and examined its spectral properties. The accuracy of the one-sided first derivative has been discussed. On the basis of the spectral properties of the weight coefficients sequences it has been shown that the accuracy of the one-sided approximation for the first derivative was essentially lower compared to the computation of the central derivatives. This our result is in agreement with previous works (see, e.g. Ref. [7]). Nevertheless, the obtained one-sided first derivative formulae could be of use in solving differential equations by means of numerical methods. It is also possible to apply the elaborated technique of the numerical differentiation in construction of quantum field theory models of unified interactions.

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington D. C., 1964.

[2] B. Fornberg, Generation of Finite Difference Formulas on Arbitrary Spaced Grids, *Math. Comp.*, 51(184), 699-706, (1988).

[3] B. Fornberg, Calculation of Weights in Finite Difference Formulas, *SIAM Rev.*, 40(3), 685-691, (1998).

[4] J. Li, General Explicit Difference Formulas for Numerical Differentiation, *J. Comp. & Appl. Math.*, 183, 29-52, (2005).

[5] M. Dvornikov, Formulae of Numerical Differentiation, *JCAAM*, 5, 77-88, (2007) [e-print arXiv: math.NA/0306092].

[6] G. Münster and M. Walzl, Lattice Gauge Theory – A Short Primer, in: *Proceedings of the Summer School on Phenomenology of Gauge Interactions* (D. Graudenz and V. Markushin, eds.), PSI, Villigen, Switzerland, 2000, pp. 127-160 [e-print arXiv: hep-lat/0012005].

[7] B. P. Demidovich and I. A. Maron, *Foundations of Computational Mathematics* (2nd ed.), Fiz. Mat. Lit., Moscow, 1963.

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Random fixed point theorems for asymptotically nonexpansive random operators

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Abstract

Let (Ω, Σ) be a measurable space with Σ a σ-algebra of subset of Ω, X a separable Banach space, C a nonempty closed bounded convex subset of X and $T : \Omega \times C \to C$ a random operator. We prove the random version of a deterministic fixed point theorem when T is an asymptotically nonexpansive random operator and X satisfy the uniform Opial's condition. Furthermore, we also prove that if T is an asymptotically nonexpansive random operator which satisfy the Maluta's constant $D(X) < 1$, and T is weakly random asymptotically regular on C. Then T has a random fixed point.

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Key words and phrases: random fixed point, asymptotically nonexpansive random operator, uniform Opial's condition, Maluta's constant.

1 Introduction

Probabilistic functional analysis has come out as one of the momentous mathematical disciplines in view of its requirements in dealing with probabilistic model in applied problem. In this direction, there have appeared various papers concerning random fixed point theorems for single-valued and set-valued random operator; (see, for instance, Itoh [6], Plubtieng and Kumam [16], Tan and Yuan [14], Xu [17] Yuan and Yu [20] and reference therein).

In 1972, Goebel and Kirk [2] prove that if the space X is assumed to be uniformly convex, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Recently, Lim and Xu [9] prove two fixed point theorems for asymptotically nonexpansive mappings which connect with Maluta's constant for a Banach space. Finally, in 1995, Xu [17] have demonstrated the existence of fixed points for asymptotically nonexpansive mappings in Banach spaces with uniform Opial's condition.

The pupose of this paper is to establish two foxed point theorems for asymptotically nonexpansive random operators. Firstly, we prove a random fixed point theorem for a Banach space with uniform Opial's condition. Moreover, we also state the random version of a fixed point result base on the Maluta's constant of Banach space due to Lim and Xu [9].

2 Preliminaries

Through this paper we will consider a measurable space (Ω, Σ) (where Σ is a $\sigma-$algebra of subsets of Ω) and (X, d) will be a metric space. We denote by $CL(X)(resp.CB(X), KC(X))$ the family of all nonempty closed (resp. closed bounded, compact) subsets of X, and by H the Hausdorff metric on $CB(X)$ induced by d, i.e.,

$$H(A, B) = \max \quad \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)$$

for $A, B \in CB(X)$, where $d(x, E) = \inf\{d(x, y)|y \in E\}$ is the distance from x to $E \subset X$.

A set-valued operator $T : \Omega \to 2^X$ is call $(\Sigma)-$ measurable if, for any open subset B of X,

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ. A mapping $x : \Omega \to X$ is said to be a *measurable selector* of a measurable set-valued operator $T : \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. Let M be a nonempty closed subset of X. An operator $T : \Omega \times M \to 2^X$ is call a random operator if, for each fixed $x \in M$, the operator $T(\cdot, x) : \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in M : x \in T(\omega, x)\}.$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega, \cdot) : M \to 2^X, F(\omega)$ may be empty. A measurable operator $x : \Omega \to M$ is said to be a *random fixed point of a operator* $T : \Omega \times M \to 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times M \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T(\omega, \cdot) : M \to 2^X$ is continuous.

Let C be a closed bounded convex subset of a Banach space X. A random operator $T : \Omega \times C \to C$ is said to be *nonexpansive* if, for fixed $\omega \in \Omega$ the map $T(\omega, \cdot) : C \to C$ is nonexpansive. We will say that T is *asymptotically nonexpansive* if there exists a sequence of function $k_n : \Omega \to [1, +\infty)$ such that for each fixed $\omega \in \Omega, \lim_{n\to\infty} k_n(\omega) = 1$ and

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq k_n(\omega)\|x - y\|$$

for all $x, y \in C$ and integer $n \geq 1$. A random operator T is said to be *weakly random asymptotically regular* on C, if for each $\omega \in \Omega$, $w-\lim_n \|(T^n(\omega, x) - T^{n+1}(\omega, x))\| = 0$ for all $x \in C$. There $T^n(\omega, x)$ is the value at x of the nth iterate of the map $T(\omega, \cdot)$, i.e., $T^n(\omega, x) = T(\omega, T^{n-1}(\omega, x))$.

We recall some definitions about the properties satisfied by a Banach space X.

Let C be a nonempty bounded closed convex subset of a Banach space X, and let $diam(A) = \sup\{\|x-y\| : x, y \in C\}$ be the diameter of C. For each $x \in C$, let $r(x, C) = \sup\{\|x-y\| : y \in C\}$ and let $r(C) = \inf\{r(x, C) : x \in C\}$, the Chebyshev radius of C relative to itself.

Let X be a Banach spaces. Then recall that Maluta's constant $D(X)$ of X is defined by

$$D(X) = \sup \quad \frac{\limsup_n d(x_{n+1}, co(x_1, ..., x_n))}{diam(x_n)}$$

where the supremum is taken over all bounded nonconstant sequence $\{x_n\}$ in X.

A Banach space X is said to satisfy *Opial's condition* if for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \neq x$.

A Banach space X is said to satisfy the *uniform Opial condition* if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x + x_n\|$$

for every $x \in X$ with $\|x\| \geq c$. and each sequence $\{x_n\}$ in X such that $w - \lim x_n = 0$ and $\liminf_n \|x_n\| \geq 1$. We now define *Opial's modulus* of X, denote by r_X, as follows

$$r_X(c) = \inf \left\{ \liminf_{n\to\infty} \|x + x_n\| - 1 \right\},$$

where $c \geq 0$ and the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and sequence $\{x_n\}$ in X such that $w - \lim x_n = 0$ and $\liminf \|x_n\| \geq 1$. It is easy to see that the function r_X is nondecreasing and that X satisfies the uniform Opial condition if and only if $r_X(c) \geq 0$ for all $c > 0$.

Let C be a nonempty bounded closed subset of Banach spaces X and $\{x_n\}$ bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$\begin{aligned} r(C, \{x_n\}) &= \inf\{r(x, \{x_n\}) : x \in C\}, \ where\{r(x, \{x_n\})\} = \limsup_n \|x_n - x\|, \\ A(C, \{x_n\}) &= \{x \in C : \{r(x, \{x_n\})\} = r(C, \{x_n\})\}. \end{aligned}$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf\{\sup\{\|x-y\| : y \in D\} : x \in C\}.$$

Definition 2.1. *Let $\{x_n\}$ and C be a nonempty bounded closed subset of Banach spaces X. Then $\{x_n\}$ is called regular with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.*

For the convenience of the reader, we will list the following results related to the concept of measurability.

Theorem 2.2. (Wagner cf. [15]). *Let (X, d) be a complete separable metric spaces and $F : \Omega \to CL(X)$ a measurable map. Then F has a measurable selector.*

Theorem 2.3. (Tan and Yuan cf. [14]). *Let X be a separable metric spaces and Y a metric spaces. If $f : \Omega \times X \to Y$ is a measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \to Y$ is measurable.*

Proposition 2.4. *Let C be a closed convex separable subset of a Banach space X and (Ω, Σ) be a measurable space. Suppose $f : \Omega \to C$ is a function that is w-measurable, i.e., for each $x^* \in X^*$, the dual space of X, the numerically-valued function $x^* f : \Omega \to (-\infty, \infty)$ is measurable, then f is measurable.*

Lemma 2.5. (Domínguez Benavidel and Lopez Acedo cf.[4]) *Suppose C is a weakly closed nonempty separable subset of a Banach space $X, F : \Omega \to 2^X$ a measurable with weakly compact values, $f : \Omega \times C \to \mathbb{R}$ is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \to \mathbb{R}$ defined by*

$$r(\omega) := \inf_{x \in F(x)} f(\omega, x)$$

and the marginal map. $R : \Omega \to X$ defined by

$$R(\omega) := \{x \in F(x) : f(\omega, x) = r(\omega)\}$$

are measurable.

Lemma 2.6. ([18]). *Let T be an asymptotically nonexpansive mapping on a nonempty weakly compact convex subset of a Banach space X. Then there are a closed convex nonempty subset K of C and $\rho \geq 0$ such that:*

(i) *if $x \in X$, then every weak limit point of $\{T^n x\}$ is contained in K;*

(ii) *$\rho_x(y) = \rho$ for all $x, y \in K$, where ρ_x is the functional defined by*

$$\rho_x(y) = \limsup_{n \to \infty} \|T^n x - y\|, \;\; y \in X.$$

3 The results

Our purpose in this section is to prove the randomization of the corresponding deterministic theorem appeared in [10] for asymptotically nonexpansive mapping.

To show next Theorem we need the following Lemma.

Lemma 3.1. *Let C be a nonempty weakly compact convex separable subset of a Banach space X satisfying Opial's condition and let $T : \Omega \times C \to C$ be an random asymptotically nonexpansive mapping. Let $\{x_n(\omega)\}$ be a sequence in C which satisfies the following condition, for $\omega \in \Omega$,*

$$w - \lim_{n\to\infty} T^m(\omega, x_n(\omega)) = z_m(\omega), \quad \forall m \geq 0.$$

Then

$$\lim_{m\to\infty} b_m(\omega) = \inf\{b_m(\omega)\} : m \geq 0,$$

where $b_m(\omega) = \limsup_{n\to\infty} \|T^m(\omega, x_n(\omega)) - z_m(\omega)\|, \ \forall \omega \in \Omega.$

Proof. Let $x_n(\omega)$ be a sequence in C for $\omega \in \Omega$, and for any $\varepsilon > 0$, there exist N such that for each $\omega \in \Omega$, $k_n(\omega) < 1 + \varepsilon$ wherever $n \geq N$. For $m \geq 0$ and $j \geq N$, by Opial's condition, for $\omega \in \Omega$ we have

$$\begin{aligned}
b_{m+j}(\omega) &= \limsup_{n\to\infty} \|T^{m+j}(\omega, x_n(\omega)) - z_{m+j}(\omega)\| \\
&\leq \limsup_{n\to\infty} \|T^{m+j}(\omega, x_n(\omega)) - T^j(\omega, z_m(\omega))\| \\
&\leq k_j(\omega) \limsup_{n\to\infty} \|T^m(\omega, x_n(\omega)) - z_m(\omega)\| \\
&= k_j(\omega) b_m(\omega) \\
&\leq (1+\varepsilon) b_m(\omega).
\end{aligned}$$

This implies that $\lim_{n\to\infty} b_m(\omega) = \inf\{b_m(\omega) : m \geq 0\}$. □

The following is the random version of Theorem 5.1 in [10].

Theorem 3.2. *Let X be a Banach space which satisfying the uniform Opial's condition, C is a nonempty weakly compact convex separable subset of X, and $T : \Omega \times C \to C$ be a random asymptotically nonexpansive operator. Then T has a random fixed point.*

Proof. Let K, ρ_x and ρ be as in Lemma 2.6. Let x be any fixed element in K and consider the measurable function $x(\omega) \equiv x$ and for each $\omega \in \Omega$. Define a map $G : \Omega \to CB(C)$ by

$$G(\omega) := w - cl\{T^n(\omega, x(\omega))\}, \quad \omega \in \Omega.$$

(Here $w - cl$ denote the closure under the weak topology of X.) Then $G : \Omega \to CB(C)$ is w- measurable. By Lemma 2.2, G has a w- measurable selector $z : \Omega \to C$. Since C is separable $\{z\}$ is actually measurable by Proposition 2.4. By the definition of G, we note that $z(\omega)$ is a weak cluster point of $\{T^n(\omega, x(\omega))\}$ for each $\omega \in \Omega$. Hence, for a fix $\omega \in \Omega$, there exists a subsequence $\{i_n\}$ of positive integer $\{n\}$ such that $\{T^{i_n}(\omega, x(\omega))\}$ converging weakly to $z(\omega)$. Passing to subsequence and using the diagonal method, we may assume that $\{T^{i_n+m}(\omega, x(\omega))\}$ converges weakly for every for every $m \geq 0$ say $w - \lim_{n\to\infty} T^{i_n+m}(\omega, x(\omega)) = z_m(\omega)$. Let $b_m(\omega) = \limsup_{n\to\infty} \|T^{i_n+m}(\omega, x(\omega)) - z_m(\omega)\|$. By Lemma 3.1, $\{b_m(\omega)\}$ converges to $b(\omega) = \inf\{b_m(\omega) : m \geq 0\} \geq 0$. Note that $z_m(\omega) \in K$ for each $\omega \in \Omega, m \geq 0$, by Lemma 2.6 (i) and

$$\|z_m(\omega) - z_{m'}(\omega)\| \leq \limsup_{n\to\infty} \|z_m - T^{i_n+m'}(\omega, x(\omega))\| \leq \rho.$$

Hence, $diam(\overline{co}\{z_m(\omega) : m \geq 0\}) \leq \rho$, we claim that

(1) for any $\varepsilon > 0, \omega \in \Omega$ there exist $y(\omega) \in K$, for $\omega \in \Omega, m' > 0$ and $N > 0$ such that $\|T^n(\omega, y(\omega)) - z_{n+m'}(\omega)\| \leq \varepsilon$ wherever $n > N$; and

(2) $\rho = 0$.

To prove (1), we distinguish two cases.

case 1. $\lim_{m\to\infty} b_m(\omega) = 0$. In this case, without loss of generality, we may assume that $\{k_j(\omega)\}$ is a decreasing sequence, for any $\varepsilon > 0$ there is $m' > 0$ such that if $m \geq m'$, then $b_m(\omega) \leq \varepsilon/2k_1(\omega)$. Thus for each $j \geq 1$,

$$\begin{aligned}\|z_{m'+j}(\omega) - T^j(\omega, z_{m'}(\omega))\| &\leq \limsup_n \|z_{m'+j}(\omega) - T^{i_n+m'+j}(\omega, x(\omega))\| \\ &\quad + \limsup_n \|T^{i_n+m'+j}(\omega, x(\omega)) - T^j(\omega, z_{m'}(\omega))\| \\ &\leq b_{m'+j}(\omega) + k_j(\omega) b_{m'}(\omega) \leq \varepsilon.\end{aligned}$$

If we choose $y(\omega) = z'_m(\omega)$ and $N = 1$, then (1) holds in this case.

case 2. For each $\omega \in \Omega$ $\lim_{m\to\infty} b_m(\omega) = b > 0$. Then, since X has the uniform Opial's property, for any $\varepsilon > 0$, there exist $\delta > 0$ and an integer $N > 1$ such that for any integer $m \geq N$ and $z(\omega) \in X, \omega \in \Omega$,

$$\limsup_{n\to\infty} \|T^{i_n+j}(\omega, x(\omega)) - z(\omega)\| \leq b + \delta \Rightarrow \|z(\omega) - z_m(\omega)\| \leq \varepsilon. \quad (3.1)$$

We may also assume that N is chosen so large that for all $n \geq N$, $b_m(\omega) \leq \sqrt{b(b+\delta)}$; and $k_m(\omega)\sqrt{(b+\delta)/\delta}$. Hence, for all $j \geq N$, we have

$$\begin{aligned}&\limsup_n \|T^{i_n+N+j}(\omega, x(\omega)) - T^j(\omega, z_N(\omega))\| \\ &\leq k_j(\omega) \limsup_n \|T^{i_n+N}(\omega, x(\omega)) - z_N(\omega)\| \\ &= k_j(\omega) b_N \leq b + \delta.\end{aligned}$$

If we choose $m' = N$ and $y(\omega) = z_N(\omega)$, then for any $j \geq N$, be (3.1), we have $\|T^j(\omega, y(\omega)) - z_{m+j}(\omega)\| \leq \varepsilon$, This prove (1).

To prove (2), by Lemma 2.6, it is enough to show that if $\rho > 0$, then there exist $z_0(\omega), y(\omega) \in K$ such that $\rho_y(z_0(\omega)) = \limsup_n \|z_0(\omega) - T^n(\omega, y(\omega))\| < \rho$. To this end,

case i. there is $N' > 0$ such that $diam(\overline{co}\{z_m(\omega) : m \geq N'\}) = \rho_0 < \rho$. By (1), there are $y(\omega) \in K, m', n \in \mathbb{N}$ such that

$$\|T^n(\omega, y(\omega)) - z_{m+n}(\omega)\| \leq \frac{\rho - \rho_0}{2} \quad \forall n > N',$$

so if $n > \max\{N, N'\}$, then

$$\begin{aligned}\|z_N(\omega) - T^n(\omega, y(\omega))\| &\leq \|z_N(\omega) - z_{m+n'}(\omega)\| + \|z_{m+n'}(\omega) - T^n(\omega, y(\omega))\| \\ &\leq \tfrac{\rho_0+\rho}{2} \leq \rho.\end{aligned}$$

case 2. $diam(\overline{co}\{z_m(\omega) : m \geq N\}) = \rho$, for all $N \in \mathbb{N}$. Since X satisfies the uniform Opial condition and hence the Opial's condition and C is weakly compact, C has normal structure. Hence, for each $\omega \in \Omega$, there exists $z_0(\omega) \in \overline{co}\{z_m(\omega) : m \in \mathbb{N}\}$, such that

$$\rho_0 = \sup_{m\in\mathbb{N}} \|z_0(\omega) - z_m(\omega)\| < diam(\overline{co}\{z_m(\omega) : m \in \mathbb{N}\}) = \rho.$$

By (1), there are $y(\omega) \in K$, $m', N \in \mathbb{N}$ such that $\|T^n(\omega, y(\omega)) - z_{n+m'}(\omega)\| \leq (\rho - \rho_0)/2$ wherever $n \geq N$. So if $n \geq N$, then

$$\begin{aligned} \|z_0(\omega) - T^n(\omega, y(\omega))\| &\leq \|z_0(\omega) - z_{n+m'}(\omega)\| + \|z_{n+m'}(\omega) - T^n(\omega, y(\omega))\| \\ &\leq \tfrac{\rho_0+\rho}{2} \leq \rho. \end{aligned}$$

This proves (2).

By (2), $K = x(\omega)$ and for each $\omega \in \Omega, \lim_{n\to\infty} \|T^n(\omega, x(\omega)) - x(\omega)\| = 0$. Therefore, $T(\omega, x(\omega)) = x(\omega)$ by the continuous of T since $T(\omega, x(\omega))$ is measurable function by Lemma 2.3. Hence $x(\omega)$ being the limit of a sequence of measurable is also measurable. So $x(\omega)$ is a random foxed point of the operator T. □

Corollary 3.3. *Let C be a nonempty closed bounded convex separable subset of a Banach spaces with uniform Opial condition and $T : \Omega\times C \to C$ be a random nonexpansive operator. Then T has a random fixed point.*

Proof. This follows since an random nonexpansive operator is of random asymptotically nonexpansive operator. □

Next, we are going to establish the stochasti version of theorem 4 of Lim and Xu [9].

Theorem 3.4. *Let C be a nonempty closed bounded convex separable subset of a Banach space with $D(X) < 1$, and $T : \Omega \times C \to C$ be a random asymptotically nonexpansive operator. Suppose in addition, that T weakly random asymptotically regular on C. Then T has a random fixed point.*

Proof. Fix a measurable function $x_0 : \Omega \to C$ and defined the function $f : \Omega \times C \to \mathbb{R}^+$ by

$$f(\omega, x) = \limsup_n \|T^n(\omega, x_0(\omega)) - x(\omega)\|, \ \ x \in X.$$

It is easily see that f is measurable function in $\omega \in \Omega$. Since f is continuous in $x \in X$ and convex, it is a weakly lower semicontinuous function. On the other hand, condition $D(X) < 1$ implies reflexivity and so C is weakly compact. Hence, by Theorem 2.5 the marginal functions $r(\omega) := \inf_{x\in C} f(\omega, x)$ and $R(\omega) := \{x \in C : f(\omega, x) = r(\omega)\}$ are measurable and $R(\omega)$ is a weakly compact convex subset of C. By Theorem 2.2, we can take a measurable selector $x(\omega)$ of $R(\omega)$. Define a map $G : \Omega \to CB(C)$ by

$$G(\omega) := w - cl\{T^n(\omega, x(\omega))\}, \quad \omega \in \Omega.$$

(Here $w - cl$ denote the closure under the weak topology of X.) Then $G : \Omega \to CB(C)$ is w- measurable. By Lemma 2.2, G has a w- measurable selector $x : \Omega \to C$. Since C is separable $\{x\}$ is actually measurable by Proposition 2.4. By the definition of G, we note that $x(\omega)$ is a weak cluster point of $\{T^n(\omega, x(\omega))\}$ for each $\omega \in \Omega$. Hence, for a fix $\omega \in \Omega$, there exists a subsequence $\{T^{n_i}(\omega, x(\omega))\}$ of $\{T^n(\omega, x(\omega))\}$ such that $\{T^{n_i}(\omega, x(\omega))\}$

converging weakly to $x(\omega)$ as $n_i \to \infty$. We claim that $x(\omega)$ is a random fixed point of T. According to the definition of $D(X)$, we have for $n \geq 1$

$$\limsup_i \|T^{n_i}(\omega, x(\omega)) - x(\omega)\| \leq D(X) diam(\{T^{n_i}(\omega, x(\omega))\}).$$

However, for any fixed $i > j$ and by weakly lower semicontinuous of $f(\omega, \cdot)$, we have

$$\begin{aligned}\|T^{n_i}(\omega, x(\omega)) - T^{n_j}(\omega, x(\omega))\| &\leq k_{n_j}(\omega)\|T^{n_i-n_j}(\omega, x(\omega)) - x(\omega)\| \\ &\leq k_{n_j}(\omega) \limsup_k \|T^{n_i-n_j}(\omega, x(\omega)) \\ &\quad -T^{n_k+(n_i-n_j)}(\omega, x(\omega))\| \\ &\leq k_{n_j}(\omega) k_{n_i-n_j}(\omega) \limsup_k \|x(\omega) \\ &\quad -T^{n_k}(\omega, x(\omega))\|,\end{aligned}$$

now, we taking upper limit $i, j \to \infty$ obtain

$$\limsup_i \|T^{n_i}(\omega, x(\omega)) - x(\omega)\| \leq D(X) \limsup_k \|x(\omega) - T^{n_k}(\omega, x(\omega))\|.$$

Since $D(X) < 1$, which implies that $\limsup_i \|T^{n_i}(\omega, x(\omega)) - x(\omega)\| = 0$, by $T(\omega, \cdot)$ is continuous and asymptotic regular we have $x(\omega) = T(\omega, x(\omega))$ which turn implies that $x(\omega)$ is a random fixed point of T. $\square$

Corollary 3.5. *Let C be a nonempty closed bounded convex separable subset of a Banach space with $D(X) < 1$, and $T : \Omega \times C \to C$ be a random nonexpansive operator. Suppose in addition, that T weakly random asymptotically regular on C. Then T has a random fixed point.*

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References

[1] J. M. Ayerbe Toledano, T. Domínguez Benavides, and G. López Acedo, "Measures of noncompactness in metric fixed point theory": Advances and Applications Topics in metric fixed point theory, Birkhuser-Verlag, Basel, **99** (1997).

[2] K. Goebel and W. A. Kirk, "Topic in metric fixed point theorem," Cambridge University Press, Cambridge (1990).

[3] T. Domínguez Benavides and P. Lorenzo Ramírez, "Structure of the fixed point set and common fixed points of asymptotically nonexpansive mapping", Proc. Amer. Math. Sco. 129 (2001), 3549-3557.

[4] T. Domínguez Benavides G. Lopez Acedo and H.-K Xu"Random fixed point of set-valued operator", Proc. Amer. Math. Soc.124 (1996), 838-838.

[5] T. Domínguez Benavides and H. K. Xu"A new geometrical coefficient for Banach spaces and its applications in fixed point theory", Nonlinear Anal. 25 No. 3 (1995), 311-325.

[6] S. Itoh, “Random fixed point theorem for a multivalued contraction mapping ”, Pacific J. Math. 68 (1977), 85-90.

[7] P. Lorenzo Ramírez,“Random fixed point of uniformly Lipschitzian mappings”, Nonlinear Anal. 57 (2004), 23-34.

[8] G. Li and B. Sims,“ fixed point theorems for mappings of asymptotically nonexpansive type”, Nonlinear Anal. 50 (2002), 1085-1091.

[9] T.C. Lim and H. K. Xu,“Fixed point theorem for asympotically nonexpansive mapping”, Nonlinear Anal. 22 (1994) No.11, 1345-1355.

[10] P.-K Lin, K.-K Tan and H. K. Xu,“Demiclosedness principle and asymptotic Behavior for asymptotically nonexpansive mappimgs”, Nonlinear Anal. 24 (1995) No6. 929-946.

[11] T. H. Kim and H. K. Xu,“Remark on asymptotically nonexpansive mappings”, Nonlinear Anal. 41(2000), 405-415.

[12] E. Maluta,“ uniform normal structure and relateed coefficients”, Pacific J. Math. 111 (1984), 357-369.

[13] N. Shahzad and S. Latif,“Random fixed points for several classes of 1-ball-contractive and 1-set-contractive random maps”, J. Math. Anal. Appl. 237 (1999), 83-92.

[14] K.-K. Tan and X.Z. Yuan, “Some Random fixed point theorem ”, in: K.-K. Tan (Ed), Fixed Point Theory and Applications, Wold Sciedtific, Singapro, 1992,334-345.

[15] D.-H. Wagner, “Survey of measurable selection theorems”, SIAM J. Control Optim. 15 (1977), 859-903.

[16] S. Plubtieng and P. Kumam ”Random fixed point theorems for multivalued nonexpansive non-self random operators”, J. Appl. Math. Stoch. Anal. Volume 2006 (2006), Article ID 43796.

[17] H. K. Xu,“Random fixed point theorems for nonlinear Uniform Lipschitzian mappings ”, Nonlinear Anal. 26 (1996) No.7, 1301-1311.

[18] H. K. Xu,“Existence and convergence for fixed point of mappings of asymptotically nonexpansive type”, Nonlinear Anal. TMA 16 (1991) 1139-1146.

[19] S. Reich,“Fixed point in locally convex spaces”, Math. Z. 125 (1972), 17-31.

[20] X. Yuan and J. Yu,“Random fixed point theorems for nonself mappings ”, Nonlinear Anal. 26 (1996)No.6, 1097-1102.

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Optimal relocation strategies for spatially mobile consumers

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Abstract

We develop a model of the behaviour of a dynamically optimizing economic agent who makes consumption-saving and spatial relocation decisions. We formulate an existence result for the model, derive the necessary conditions for optimality and study the behaviour of the economic agent, focusing on the case of a wage distribution with a single maximum.

Keywords: consumption decisions, spatial relocation, optimal control

2000 Mathematics Subject Classification: 91B42, 91B72, 49J15, 49K15

1. Introduction

The emergence of the literature on "new economic geography" in the 1990s has rekindled the interest in the spatial aspects of economics. The new generation of models makes heavy use of the standard economics toolkit and analyzes a number of issues from a dynamic perspective or from the perspective of optimizing agents. Interestingly, however, spatial models adopting the perspective of *dynamically optimizing* consumers remain in relative minority, despite the fact that they are standard fare in mainstream economic research. The models in [1], [2] and [3] are notable exceptions in this respect.

The present work develops a model that studies the behaviour of a dynamically optimizing economic agent who makes two types of interrelated choices: consumption-saving decisions and spatial relocation (migration) decisions. Unlike the constructs in [1], [2] and [3], the consumer in our model has a finite lifetime and a bequest incentive at the end of his life. This departure from classical Ramsey-type models enables richer global dynamics by allowing agents to inherit

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their ancestors' savings in a setup akin to that of overlapping generations models. It also offers the additional option of introducing heterogeneous agents whose economy-wide behaviour can be obtained through an explicit aggregation rule.

A second important difference with the above papers is that our consumer saves in nominal assets. This partly depends on our choice to center the model around the behaviour of (potentially different) individuals as opposed to that of a representative agent. More importantly, however, the nominal savings feature reflects our belief that pecuniary considerations play an important role in the choice of where to work and how much to consume.

Formally, we cast the model in the form of a continuous-time optimal control problem with a finite planning horizon. The assumptions of the model, while fairly standard in economics, create several mathematical challenges in the present setup. First, they preclude the direct application of the existence theorems for optimal control problems known to the authors. This requires an alternative approach to proving the existence of solutions of the model. In particular, unlike traditional existence proofs in the spirit of Theorem 4, §4.2 in [8], we prove an existence result that dispenses with convexity assumptions on the set of generalized speeds for the optimal control problem. Also, the functional forms employed in the model do not allow one to directly apply Pontryagin's maximum principle, since the transversality condition for one of the state variables is not defined at the point 0. To be able to use the maximum principle, we prove that for an optimal control-trajectory pair the terminal value of the particular state variable is strictly positive. Finally, economic considerations point to the fact that only a subset of the possible values for the other state variable in the model are of real interest. One way to take care of that issue is to constrain the values of this state variable to lie in a certain set – the interval $[0, 1]$ in our case – for each point in time. However, instead of using an explicit state constraint, which would complicate the use of the maximum principle, we introduce an additional correcting mechanism by suitably defining the wage distribution function $w(x)$ outside the interval $[0, 1]$. We claim this mechanism does not influence the other characteristics of the model, while being sufficient to ensure that the optimal state variable never leaves the set in question, and we prove that indeed this is the case.

The rest of the paper is organized as follows. Section 2 introduces the model and the assumptions we make. Section 3 proves the existence of a solution to the model under the above assumptions. Section 4 applies Pontryagin's maximum principle to obtain necessary conditions for optimality. Section 5 describes some convenient transformations of the system of necessary conditions and comments on the existence of solutions to this system. The analysis in section 6 characterizes the asymptotic behaviour of terminal assets for different sets of model parameters. This establishes facts that are useful for the study of relocation choices in section 7. The results in this section are also of independent economic interest as they shed light on the impact of intra- and intertemporal preferences on the saving decisions of an individual with a sufficiently long planning horizon. Finally, section 7 tackles the question of relocation behaviour in the basic case of a wage distribution having a single maximum (single-peaked wage distribution). The results obtained for this case are intuitive, if unsurprising: in most cases a consumer with a sufficiently long lifespan relocates toward the wage maximum. While the single-peak case offers easily predictable results, we consider it useful as a testing ground for the model before applying it to more interesting situations. Indeed, preliminary results by the authors on the case of a double-peaked wage distribution suggest that a host of complex situations, including multiple solutions and bifurcations, can arise.

2. The model

We employ a continuous-time model that deals with the case of a consumer who, given an initial location in space x_0 and asset level a_0, supplies inelastically a unit of labour in exchange for a location-dependent wage $w(x(t))$, and chooses consumption $c(t)$ and spatial location $x(t)$ over time. The consumer has a finite lifetime T at the end of which a bequest in the form of assets is left. This bequest provides utility to the consumer. More precisely, for ρ, r, η, ξ and p – positive constants, and $\theta \in (0,1)$, we look at the optimal control problem

$$\max_{c(t),z(t)\in\Delta} J(c(t),z(t)) := \int_0^T e^{-\rho t}\left(\frac{c(t)^{1-\theta}}{1-\theta} - \eta z^2(t)\right)dt + e^{-\rho T}\frac{a(T)^{1-\theta}}{1-\theta} \tag{1}$$

subject to

$$\dot{a}(t) = ra(t) + w(x(t)) - pc(t) - \xi z^2(t), \tag{2}$$

$$\dot{x}(t) = z(t), \tag{3}$$

$$a(0) = a_0 \geq 0,$$

$$x(0) = x_0 \in [0,1],$$

where $a(t)$, $x(t)$ are the state variables, assumed to be absolutely continuous, and $c(t)$, $z(t)$ are the control variables. The set of admissible controls Δ consists of all pairs of functions $(c(t), z(t))$ which are measurable in $[0,T]$ and satisfy the conditions

$$0 \leq c(t) \leq C, \tag{4}$$

$$|z(t)| \leq Z, \tag{5}$$

$$a(T) \geq 0. \tag{6}$$

The constants C and Z are such that

$$C^\theta > \max\left(1, \mu^{\frac{1}{\theta}}\right)\left(\frac{a_0 + T\sup_x |w(x)|}{p\frac{1-e^{-rT}}{r}}\right), \tag{7}$$

$$Z > \frac{T\sup_x |w'(x)|e^{rT}}{2\xi}, \tag{8}$$

where $\mu := \max_{t,t_0 \in [0,T]} e^{(r-\rho)(t-t_0)} > 0$.

Remark 1. The bounds we impose on the admissible controls through equations (4) and (5) are convenient from a technical viewpoint when proving the existence theorem in section 3. Conditions (7) and (8) ensure that these constraints are never binding. However, considerations of general nature – both economic and physical – make such constraints appealing.

In the above model $\rho > 0$ is a time discount parameter and $\theta \in (0, 1)$ is the utility function parameter. The control $c(t)$ represents physical units of consumption and the control $z(t)$ governs the speed of relocation in space. We assume that relocation in space brings about two type of consequences. First, relocation causes subjective disutility associated with the fact that there is habit formation with respect to the place one occupies. Second, changing one's location is associated with monetary relocation costs that have to be paid out of one's income or stock of assets. As a baseline case we choose to capture these phenomena by means of the speed of movement in space $\dot{x}(t)$ or, equivalently, $z(t)$, transformed through a quadratic function. The manner in which spatial relocation affects the consumer's utility and wealth can vary widely, however, therefore other functional forms are certainly admissible. The parameters $\eta, \xi \geq 0$ multiplying this function measure the subjective disutility from changing one's location in space and the relocation costs in monetary terms, respectively. The parameters $p > 0$ and $r > 0$ stand for the price of a unit of consumption and the interest rate, respectively.

The nonnegativity condition is imposed on terminal assets $a(T)$ both to have a well-defined objective functional and to capture the intuitive observation that, with a known lifetime, a debtor is unlikely to be allowed to leave behind outstanding liabilities to creditors. The condition $a(T) \geq 0$ also sheds light on the nonnegativity restriction for a_0, since in an environment where no debts are allowed at the end of one's lifetime, no debtor position can be inherited at birth.

For the purposes of our analysis we look at the basic case where economic space is represented by the real line. We are interested in only a subset of it, the interval $[0, 1]$. This is modelled by taking the initial location $x_0 \in [0, 1]$ and requiring the location-dependent wage, which is positive in $(0,1)$, to be negative outside $[0,1]$ and to satisfy additional assumptions. Namely, we have $w(x) > 0,\ x \in (0,1)$ and $w(x) < 0,\ x \notin [0,1]$, as well as $w'(x) > 0$, $x \in (-\infty, 0]$ and $w'(x) < 0$, $x \in [1, \infty)$. Later in the paper we formally verify the intuitive claim that an optimal trajectory for $x(t)$ will never leave the interval $[0,1]$ under the above conditions. We also assume that $w(x) \in C^2(\mathbb{R}^1)$ and $w(x)$, $w'(x)$ are bounded, i.e. $\sup_{x\in\mathbb{R}} |w(x)| < +\infty$ and $\sup_{x\in\mathbb{R}} |w'(x)| < +\infty$. We impose additional requirements on $w(x)$ to derive some of the results in section 7.

3. Existence of solutions

Next we investigate the issue of existence of a solution to the model. The proof requires two intermediate results, shown as lemmas below.

Lemma 1. *Let the functions x_i, $i = 1, 2, \dots,$ and $\bar{x}$ be defined on $[0,T]$ and take values in the interval $[a,b]$. Let x_i tend uniformly to $\bar{x}$ as $i \to \infty$ (denoted by $x_i \rightrightarrows \bar{x}$) and $w \in C^0[a,b]$. Then, in $[0,T]$, as $m \to \infty$ we have*

i) $\frac{1}{m}\sum_{i=1}^{m} x_i \rightrightarrows \bar{x}$,

ii) $w(x_m) \rightrightarrows w(\bar{x})$,

iii) $w(\frac{1}{m}\sum_{i=1}^{m} x_i) \rightrightarrows w(\bar{x})$.

Proof. The proof directly replicates the standard proofs of counterpart results on numerical sequences. ■

Lemma 2 (The Banach-Saks Theorem)**.** *Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of elements in a Hilbert space H which are bounded in norm: $\|v_n\| \le K = const$, $\forall n \in \mathbb{N}$. Then, there exist a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and an element $v \in H$ such that*

$$\left\| \frac{v_{n_1} + \cdots + v_{n_s}}{s} - v \right\| \to 0 \text{ as } s \to \infty.$$

Proof. See, for example, [5, pp.78-81]. ■

Theorem 1. *Under the assumptions stated in section 2, there exists a solution $(c(t), z(t)) \in \Delta$ of problem* (1)-(3)*.*

Proof. We start by noting that the set of admissible controls Δ is nonempty. To see this, choose controls $c(t) \equiv c_0 = const$ and $z(t) \equiv 0$. Then, any $c_0 \in (0, w(x_0)/p]$ will ensure that $a(T) \ge 0$.

Next, observe that $(c(t), z(t)) \in \Delta$ implies $c(t), z(t) \in L_\infty[0, T]$ and

$$0 \le a(T) \le const = e^{rT}(a_0 + T \sup_{x \in \mathbb{R}} |w(x)|). \tag{9}$$

(We note that (6) implies the following bounds,

$$p\|c(t)\|_{L_1[0,T]}, \xi\|z(t)\|^2_{L_2[0,T]} \le e^{rT}\left(a_0 + T \sup_x |w(x)|\right),$$

which do not depend on the constants C and Z.)

Through an application of Hölder's inequality one verifies that $\int_0^T c(t)^{1-\theta} e^{-\rho t} dt \le const(T)\|c(t)\|^{1-\theta}_{L_1}$.

Thus, for $(c(t), z(t)) \in \Delta$, the objective functional (1) is bounded. Consequently, $J_0 := \sup_{(c(t),z(t)) \in \Delta} J(c(t), z(t)) < \infty$. We can choose a sequence of controls $\{(c_k(t), z_k(t))\} \subset \Delta$ such that $J(c_k(t), z_k(t)) \to J_0$.

Let $a_k(t)$ and $x_k(t)$ be the state variables corresponding to the controls $(c_k(t), z_k(t))$. It is easy to verify that the functions $a_k(t)$ and $x_k(t)$ form a uniformly bounded and equicontinuous set. Then, by the Arzelà-Ascoli theorem (see, e.g., [8], Ch.4), there exists a subsequence $(a_{k_s}(t), x_{k_s}(t)) \rightrightarrows (\bar{a}(t), \bar{x}(t))$.

Then, if $c_{k_s}(t)$ and $z_{k_s}(t)$ are the controls corresponding to $(a_{k_s}(t), x_{k_s}(t))$, by Lemma 2 we can in turn choose subsequences $c_{k_{s_q}}(t)$ and $z_{k_{s_q}}(t)$ whose arithmetic means tend in $L_2[0, T]$ norm to some elements in $L_2[0, T]$, denoted $\bar{c}(t)$ and $\bar{z}(t)$, respectively. However, we do not claim that $\bar{a}(t)$ and $\bar{x}(t)$ correspond to $\bar{c}(t)$ and $\bar{z}(t)$. For brevity we introduce the notation $c_q(t) := c_{k_{s_q}}(t)$, $z_q(t) := z_{k_{s_q}}(t)$ etc., as well as $\tilde{c}_m(t) := \frac{1}{m}\sum_{q=1}^m c_q(t)$ and $\tilde{z}_m(t) := \frac{1}{m}\sum_{q=1}^m z_q(t)$.

Then, we have established that: (1) $(a_q(t), x_q(t)) \rightrightarrows (\bar{a}(t), \bar{x}(t))$ as $q \to \infty$ and (2) $\tilde{c}_m(t) \underset{L_2}{\longrightarrow} \bar{c}(t)$, $\tilde{z}_m(t) \underset{L_2}{\longrightarrow} \bar{z}(t)$ as $m \to \infty$.

Recall that $a_q(t)$ and $x_q(t)$ correspond to $c_q(t)$ and $z_q(t)$ as solutions to the respective differential equations (2) and (3).

So far, it is not clear whether $\tilde{c}_m(t)$ and $\tilde{z}_m(t)$ are admissible. It is immediately seen that they satisfy (4) and (5) but the corresponding $a(T)$ may fail to satisfy (6). However, we can show that the controls $\bar{c}(t)$ and $\bar{z}(t)$ are admissible.

To prove the last claim, note first that according to [7, Ch.7, §2.5, Prop.4] we can choose a subsequence of $\{\tilde{c}_m(t), \tilde{z}_m(t)\}$ that converges a.e. to $(\bar{c}(t), \bar{z}(t))$ and, after passing to the limit, we obtain that $\bar{c}(t)$ and $\bar{z}(t)$ satisfy (4) and (5).

It remains to show that $\bar{\bar{a}}(T) = e^{rT}\left[a_0 + \int_0^T [w(\bar{\bar{x}}(t)) - p\bar{c}(t) - \xi\bar{z}^2(t)]e^{-rt}dt\right] \ge 0$, where $\bar{\bar{x}}(t) = x_0 + \int_0^t \bar{z}(\tau)d\tau$.

Consider

$$\tilde{a}_m(T) = e^{rT}\left[a_0 + \int\limits_0^T [w(\tilde{x}_m(t)) - p\tilde{c}_m(t) - \xi\tilde{z}_m^2(t)]e^{-rt}dt\right], \tag{10}$$

with $\tilde{x}_m(t) = x_0 + \int_0^t \tilde{z}_m(\tau)d\tau = \frac{1}{m}\sum_{q=1}^m \left(x_0 + \int_0^t z_q(\tau)d\tau\right) = \frac{1}{m}\sum_{q=1}^m x_q(t)$. Adding and subtracting $\frac{1}{m}\sum_{q=1}^m w(x_q(t))$, and applying Jensen's inequality to the term $\tilde{z}_m^2(t)$, we obtain

$$\begin{aligned}\tilde{a}_m(T) \geq & e^{rT}\int\limits_0^T \left[w(\tilde{x}_m(t)) - \frac{1}{m}\sum_{q=1}^m w(x_q(t))\right]e^{-rt}dt + \\ & \frac{1}{m}\sum_{q=1}^m e^{rT}\left[a_0 + \int\limits_0^T [w(x_q(t)) - pc_q(t) - \xi z_q^2(t)]e^{-rt}dt\right] \geq \\ & e^{rT}\int\limits_0^T \left[w(\tilde{x}_m(t)) - \frac{1}{m}\sum_{q=1}^m w(x_q(t))\right]e^{-rt}dt\end{aligned} \tag{11}$$

By Lemma 1 both integrands inside the square brackets in the last line of (11) tend uniformly to $w(\bar{x}(t))$, so that the integral tends to zero. Thus, if $\lim_{m\to\infty}\tilde{a}_m(T)$ exists, we have $\lim_{m\to\infty}\tilde{a}_m(T) \geq 0$.

We proceed to check that $\lim_{m_j\to\infty}\tilde{a}_{m_j}(T) = \bar{\bar{a}}(T)$ for a suitable subsequence $\tilde{a}_{m_j}(T)$. We know that $\frac{1}{m}\sum_{q=1}^m x_q(t) = x_0 + \int_0^t \frac{1}{m}\sum_{q=1}^m z_q(\tau)d\tau$. Since $\frac{1}{m}\sum_{q=1}^m x_q(t) \rightrightarrows \bar{x}(t)$ and, additionally, it is easy to verify by applying Hölder's inequality that $\int_0^t \tilde{z}_m(\tau)d\tau \to \int_0^t \bar{z}(\tau)d\tau$ when $\tilde{z}_m(t) \underset{L_2}{\longrightarrow} \bar{z}(t)$, we obtain $\bar{x}(t) = x_0 + \int_0^t \bar{z}(\tau)d\tau = \bar{\bar{x}}(t)$.

As $\tilde{c}_m(t) \underset{L_2}{\longrightarrow} \bar{c}(t)$ and $\tilde{z}_m(t) \underset{L_2}{\longrightarrow} \bar{z}(t)$, there exist a subsequences $\tilde{c}_{m_j}(t)$ and $\tilde{z}_{m_j}(t)$ such that $\tilde{c}_{m_j}(t) \underset{a.e.}{\longrightarrow} \bar{c}(t)$ and $\tilde{z}_{m_j}(t) \underset{a.e.}{\longrightarrow} \bar{z}(t)$. To simplify notation, we refer to the new subsequences as $\tilde{c}_j(t)$ and $\tilde{z}_j(t)$. Since the function z^2 is bounded on $[-Z, Z]$, by Lebesgue's dominated convergence theorem $\int_0^T \xi\tilde{z}_j^2(t)e^{-rt}dt \to \int_0^T \xi\bar{z}^2(t)e^{-rt}dt$. It can also be verified that $\int_0^T \tilde{c}_j(t)e^{-rt}dt \to \int_0^T \bar{c}(t)e^{-rt}dt$. Lastly, we know that $\int_0^T w(\tilde{x}_j(t))e^{-rt}dt \to \int_0^T w(\bar{x}(t))e^{-rt}dt$ as $w(\tilde{x}_j(t)) \rightrightarrows w(\bar{x}(t))$. Consequently, the limit of (10) as $m_j \to \infty$ exists and is equal to $\bar{\bar{a}}(T)$, so that $\bar{\bar{a}}(T) \geq 0$. This shows that $\bar{c}(t)$ and $\bar{z}(t)$ are admissible.

By an application of Lebesgue's dominated convergence theorem to the respective terms in (1), we get $\lim_{j\to\infty} J(\tilde{c}_j(t), \tilde{z}_j(t)) = J(\bar{c}(t), \bar{z}(t))$.

Define $\rho_{m_j}(T) := e^{rT}\int_0^T \left[w(\tilde{x}_{m_j}(t)) - \frac{1}{m_j}\sum_{q=1}^{m_j} w(x_q(t))\right]e^{-rt}dt$. Obviously, $\tilde{\tilde{a}}_{m_j}(T) = \tilde{a}_{m_j}(T) - \rho_{m_j}(T)$ also tends to $\bar{\bar{a}}(T)$ and $\tilde{\tilde{a}}_{m_j}(T) \geq \frac{1}{m_j}\sum_{q=1}^{m_j} a_q(T)$, where $a_q(T)$ corresponds to $(c_q(t), z_q(t))$. Then, indexing by j instead of m_j to simplify notation, we get

$$J_0 \geq J(\bar{c}(t), \bar{z}(t)) = \lim_{j\to\infty}\left\{\int\limits_0^T \left[\frac{\tilde{c}_j(t)^{1-\theta}}{1-\theta} - \eta\tilde{z}_j^2(t)\right]e^{-\rho t}dt + e^{-\rho T}\frac{\tilde{\tilde{a}}_j^{1-\theta}(T)}{1-\theta}\right\} \geq$$

$$\lim_{j\to\infty}\left\{\frac{1}{j}\sum_{i=1}^{j}\left[\int_0^T\left[\frac{c_i^{1-\theta}(t)}{1-\theta}-\eta z_i^2(t)\right]e^{-\rho t}dt+e^{-\rho T}\frac{a_i^{1-\theta}(T)}{1-\theta}\right]\right\}=$$

$$\lim_{j\to\infty}\left\{\frac{1}{j}\sum_{i=1}^{j}J(c_i(t),z_i(t))\right\}=J_0,$$

where the inequality is a consequence of the fact that the functions $\sigma\mapsto\sigma^{1-\theta}$ and $z\mapsto(-z^2)$ are concave and we can apply Jensen's inequality. This shows that the admissible pair $(\bar{c}(t),\bar{z}(t))$ is optimal, as required. ■

4. Necessary conditions for optimality

In this section we turn to the derivation of a set of necessary conditions for optimality on the basis of Pontryagin's maximum principle. To apply the maximum principle, however, we need to ensure that the terminal utility from assets $e^{-\rho T}a(T)^{1-\theta}/(1-\theta)$ is well-behaved at least for the optimal value of terminal assets. To this end, we prove the following

Theorem 2. *For the optimal controls $(c(t),z(t))$ the terminal value of assets $a(T)$ is strictly positive for any $T>0$.*

Proof. Let us assume that there is a time $T_0>0$ for which $a(T_0)=0$.

Step 1. We first verify that it is impossible to have $c(t)\equiv 0$. Assuming that $c(t)\equiv 0$, together with $a(T_0)=0$, yields the objective functional

$$J(0,z(t))=-\eta\int_0^{T_0}z^2(t)e^{-\rho t}dt\le 0.$$

If one of the following two conditions is valid:

1. $a_0>0$ and $x_0\in[0,1]$;

2. $a_0=0$ and $x_0\in(0,1)$,

then we can choose the admissible pair $\bar{z}(t)\equiv 0$ (so that $x(t)\equiv x_0$) and $\bar{c}(t)\equiv c_0=const>0$, where c_0 is such that

$$a_0+\int_0^{T_0}[w(x_0)-pc_0]e^{-rt}dt=0.$$

The last condition is equivalent to

$$a_0+T_0w(x_0)\frac{e^{-rT_0}-1}{-r}=pc_0\frac{e^{-rT_0}-1}{-r}$$

and therefore $c_0 > 0$. Then

$$J(\bar{c}(t), \bar{z}(t)) = \int_0^{T_0} \frac{c_0^{1-\theta}}{1-\theta} e^{-\rho t} dt > 0,$$

contradicting the optimality of $(c(t), z(t))$.

The case $a_0 = 0$ and $x_0 = 0$ or 1 is pathological in the sense that the consumer has neither current income (w(0)=w(1)=0), nor initial wealth. Economically, it is implausible to expect that such a consumer will manage to obtain a loan. From a purely formal point of view, however, the consumer could get a loan and finance his relocation even in this case. Moreover, he will be able to attain positive consumption levels.

To illustrate the above claim, suppose that $x_0 = 0$, $a_0 = 0$ and the consumer spends all the income left after paying the relocation costs. Fix $\varepsilon_0 > 0$ in such a way that $w'(x) \geq w'(0)/2 > 0$ for $x \in [0, \varepsilon_0]$. Let the relocation strategy be given by the control $\bar{z}(t) = \varepsilon \sin \frac{\pi}{T} t,\ \varepsilon > 0$. Then consumption is given by $\bar{c}(t) = w(\bar{x}(t)) - \xi \bar{z}^2(t)$, where $\bar{x}(t)$ is

$$\bar{x}(t) = \varepsilon \int_0^t \sin\left(\frac{\pi}{T}\tau\right) d\tau = \frac{\varepsilon T}{\pi}\left(1 - \cos\frac{\pi t}{T}\right) = \frac{2\varepsilon T}{\pi} \sin^2 \frac{\pi t}{2T}.$$

Then, $\bar{x}(T) = \frac{2T\varepsilon}{\pi} < \varepsilon_0$ for ε sufficiently small. Notice that

$$w(\bar{x}(t)) = w(\bar{x}(t)) - w(0) = w'(x^*(t))\bar{x}(t) \geq \frac{w'(0)}{2}\bar{x}(t),$$

for some $x^*(t) \in (0, \bar{x}(t))$. Consequently, we obtain

$$w(\bar{x}(t)) - \xi \bar{z}^2(t) \geq \varepsilon \left[\frac{w'(0)}{2}\frac{2T}{\pi} \sin^2 \frac{\pi t}{2T} - \varepsilon\xi \sin^2 \frac{\pi t}{T}\right] = \varepsilon \sin^2 \frac{\pi t}{2T}\left[\frac{Tw'(0)}{\pi} - 4\varepsilon\xi \cos^2 \frac{\pi t}{2T}\right].$$

Consumption will be positive if

$$g(t) := \frac{Tw'(0)}{\pi} - 4\varepsilon\xi \cos^2 \frac{\pi t}{2T} > 0 \text{ for } t \in [0, T].$$

For ε small $g(0) = Tw'(0)/\pi - 4\xi\varepsilon > 0$. Also,

$$g'(t) = 4\varepsilon\xi \frac{\pi}{2T} 2\cos\frac{\pi t}{2T} \sin\frac{\pi t}{2T} = 2\varepsilon\xi\frac{\pi}{T}\sin\frac{\pi t}{T} \geq 0 \text{ for } t \in [0, T].$$

Thus, $g(t) \geq g(0) > 0$, as required.

Remark 2. It is easy to see that in the above example we can take $\bar{z}(t)$ to be any smooth function that is positive on $(0, T)$, zero for $t = 0, T$ and $\dot{\bar{z}}(0) > 0$.

Step 2. Since $c(t) \not\equiv 0$, there exists a set $A \subset [0, T]$, meas $A > 0$, such that

$$\operatorname*{essinf}_{t \in A} c(t) > \varepsilon_1 > 0.$$

Let us take the control pair $(\bar{c}(t), \bar{z}(t))$ with $\bar{c}(t) := c(t) - \varepsilon\chi_A(t)$ and $\bar{z}(t) := z(t)$, where $\chi_A(t)$ is the indicator function of the set A and $\varepsilon \in (0, \varepsilon_1)$. These controls are admissible if we have terminal assets $\bar{a}(T_0) > 0,\ \forall\varepsilon \in (0, \varepsilon_1)$. To verify the last claim, we take

$$\bar{a}(T_0) = e^{rT_0}\left[a_0 + \int_0^{T_0}[w(x(s)) - p(c(s) - \varepsilon\chi_A(s)) - \xi z^2(s)]e^{-rs}ds\right] = e^{rT_0}\int_A p\varepsilon e^{-rs}ds = \varepsilon C_1,$$

where $C_1 := pe^{rT_0}\int_A e^{-rs}ds > 0$.

An application of Taylor's formula yields

$$\frac{\bar{c}(t)^{1-\theta}}{1-\theta} = \frac{c(t)^{1-\theta}}{1-\theta} + (-\varepsilon\chi_A(t))c(t)^{-\theta} + (-\varepsilon\chi_A(t))^2\frac{-\theta}{2}c^*(t)^{-\theta-1},$$

where $c^*(t) = \alpha(t)\bar{c}(t) + (1-\alpha(t))c(t),\ \alpha(t) \in (0,1)$ or $c^*(t) = c(t) - \varepsilon\chi_A(t)\alpha(t)$. Note also that for $t \in A$ we have $0 < c(t) - \varepsilon_1 \cdot 1 \leq c^*(t) \leq c(t)$, so that $(c(t) - \varepsilon_1)^{-\theta-1} \geq c^*(t)^{-\theta-1} \geq c(t)^{-\theta-1}$.

Let us compare

$$J(c(t), z(t)) = \int_0^{T_0}\frac{c(t)^{1-\theta}}{1-\theta}e^{-\rho t}dt - \eta\int_0^{T_0} z^2(t)e^{-\rho t}dt$$

and

$$\begin{aligned} J(\bar{c}(t), \bar{z}(t)) = &\int_0^{T_0}\frac{\bar{c}(t)^{1-\theta}}{1-\theta}e^{-\rho t}dt - \eta\int_0^{T_0} z^2(t)e^{-\rho t}dt + \frac{\bar{a}(T_0)^{1-\theta}}{1-\theta}e^{-\rho T_0} \\ =&J(c(t), z(t)) + \left[-\varepsilon\int_A c(t)^{-\theta}e^{-\rho t}dt - \frac{\theta\varepsilon^2}{2}\int_A (c(t) - \varepsilon\alpha(t))^{-1-\theta}e^{-\rho t}dt\right] + \\ &+ \frac{(\varepsilon C_1)^{1-\theta}}{1-\theta}e^{-\rho T_0}. \end{aligned}$$

We will show that $J(\bar{c}(t), \bar{z}(t)) > J(c(t), z(t))$ for $\varepsilon \in (0, \varepsilon_1)$ sufficiently small. This will be true if we are able to establish that

$$\frac{(\varepsilon C_1)^{1-\theta}}{1-\theta}e^{-\rho T_0} > \varepsilon\int_A c(t)^{-\theta}e^{-\rho t}dt + \frac{\varepsilon^2\theta}{2}\int_A (c(t) - \varepsilon_1)^{-1-\theta}e^{-\rho t}dt,$$

where the last integral provides an upper bound on $\int_A (c(t) - \varepsilon\alpha(t))^{-1-\theta}e^{-\rho t}dt$. Denoting the respective positive constants in the above inequality by B_1, B_2 and B_3, we obtain

$$\varepsilon^{1-\theta}B_1 > \varepsilon B_2 + \varepsilon^2 B_3$$

or

$$B_1 > \varepsilon^{\theta}B_2 + \varepsilon^{1+\theta}B_3,$$

which is obviously true for $\varepsilon \in (0, \varepsilon_1)$ sufficiently small. This contradicts the optimality of $(c(t), z(t))$. Thus, $a(T_0) = 0$ cannot be true and hence $a(T_0) > 0$. ■

On the basis of Theorem 2 an optimal solution $\bar{c}(t), \bar{z}(t)$ to problem (1)-(3) (possibly non-unique) also solves the following problem, where the controls $(c(t), z(t)) \in \Delta_1 \subset \Delta$:

$$\max_{c(t),z(t)\in\Delta_1} J(c(t), z(t))$$

$$\dot{a}(t) = ra(t) + w(x(t)) - pc(t) - \xi z^2(t)$$

$$\dot{x}(t) = z(t)$$

$$a(0) = a_0 \geq 0,$$

$$x(0) = x_0 \in [0, 1],$$

$$a(T) \geq \delta > 0,$$

with δ being an appropriate constant, strictly smaller than the optimal value of terminal assets.

To avoid burdensome notation, from now on we do not append additional symbols to the state, costate and control variables in the model when referring to their optimal values. However, we use alternative symbols to denote alternative sets of variables to be compared with the optimal ones.

Taking into account that we do not impose any state constraints on the problem, Theorem 5.2.1 in [4] provides the set of necessary conditions. To derive the latter, we define the Hamiltonian for the problem

$$\begin{aligned} H &:= H(t, a, x, \varphi, \psi, p_1, p_2) = \\ &p_1(ra + w(x) - p\varphi - \xi\psi^2) + p_2\psi + \lambda_0 e^{-\rho t}\left(\frac{\varphi^{1-\theta}}{1-\theta} - \eta\psi^2\right), \end{aligned} \tag{12}$$

where $\lambda_0 \in \{0, 1\}$. Then

1) The costate variables $p_i(t),\ i = 1, 2$, satisfy a.e. on $(0, T)$

$$\dot{p}_1(t) = -rp_1(t), \tag{13}$$

$$\dot{p}_2(t) = -p_1(t)w'(x(t)). \tag{14}$$

2) The function $(\varphi, \psi) \mapsto H(t, a(t), x(t), \varphi, \psi, p_1(t), p_2(t))$ attains its maximum with respect to φ, ψ at the point $(c(t), z(t))$ for almost all $t \in [0, T]$, where φ, ψ satisfy the constraints on the function values arising from Δ_1, i.e. $\varphi \in [0, C]$ and $|\psi| \leq Z$:

$$\begin{aligned} H(t) &:= H(t, a(t), x(t), c(t), z(t), p_1(t), p_2(t)) = \\ &\max_{\varphi,\psi} H(t, a(t), x(t), \varphi, \psi, p_1(t), p_2(t)). \end{aligned} \tag{15}$$

3a) Since $a(t)$ and $x(t)$ are fixed at $t = 0$, the values of $p_i(0)$ are arbitrary, i.e.

$$p_1(0) = \lambda_1,\ \ \lambda_1 \in \mathbb{R}, \tag{16}$$

$$p_2(0) = \lambda_2,\ \ \lambda_2 \in \mathbb{R}. \tag{17}$$

3b) Since the terminal values $(a(T), x(T))$ of the state variables are at an interior point of the target set

$$\{(a, x) \in \mathbb{R}^2 | a \geq \delta > 0, x \in \mathbb{R}^1\},$$

the corresponding normal cone is trivial and the transversality condition at the right endpoint T has the form

$$p_1(T) = \lambda_0 e^{-\rho T} a(T)^{-\theta}, \tag{18}$$

$$p_2(T) = 0, \tag{19}$$

(cf. condition 4) in Theorem 5.2.1 and the functional form for $f(x(b))$ in §5.2 in [4]).

4) The variables $p_1(t), p_2(t), \lambda_0$ are not simultaneously equal to zero.

Below we specify the form of the necessary conditions in greater detail.

According to (13) and (16) we have

$$p_1(t) = \lambda_1 e^{-rt}. \tag{20}$$

Remark 3. As $c(t)$ is only an integrable function, it can be approximated in a.e. sense by a sequence of simple (even continuous) functions, which can be considered without loss of generality as taking values in $[0, C]$. Therefore, fixing a "representative" of $c(t)$, we can think that on $[0, T]\backslash E$, where E is an appropriate set of measure zero, $c(t)$ takes finite values and is the limit of a numerical sequence in $[0, C]$. For simplicity of formulation, however, we speak of *all* points $t \in [0, T]$ in what follows.

Proposition 1. *If there exists $t_0 \in [0, T]$ such that $c(t_0) \in (0, C)$, then $c(t) > 0$ for almost all $t \in [0, T]$.*

Proof. If there exists t_0 with the above properties, then (15) implies

$$\frac{\partial}{\partial \varphi} H(t_0, a(t_0), x(t_0), \varphi, z(t_0), p_1(t_0), p_2(t_0)) \left.\right|_{\varphi = c(t_0)} = 0,$$

i.e.

$$-p\lambda_1 e^{-rt_0} + \lambda_0 e^{-\rho t_0} c(t_0)^{-\theta} = 0. \tag{21}$$

If we assume that $\lambda_0 = 0$, then $\lambda_1 = 0$ and, because of (20), one obtains $p_1(t) \equiv 0$. This implies that $p_2(t) \equiv const = \lambda_2$. Now (19) shows that $\lambda_2 = 0$, which constitutes a contradiction with condition 4) from the cited theorem in [4]. Therefore, $\lambda_0 = 1$.

Assume that there exists $t_1 \in [0,T]$ for which $c(t_1) = 0$. Then, for all sufficiently small $\varphi > 0$ we have

$$\frac{H(t_1, a(t_1), x(t_1), \varphi, z(t_1), p_1(t_1), p_2(t_1)) - H(t_1, a(t_1), x(t_1), 0, z(t_1), p_1(t_1), p_2(t_1))}{\varphi - 0} \leq 0,$$

i.e.

$$-p_1(t_1)p + \lambda_0 e^{-\rho t_1} \frac{\varphi^{-\theta}}{1-\theta} \leq 0.$$

Since $\lambda_0 > 0$, for $\varphi \to 0+$ the last inequality leads to a contradiction. This proves the proposition. ■

Corollary 1. *If there exists $t_1 \in [0,T]$ for which $c(t_1) = 0$, then $\lambda_0 = 0$ and $c(t) = 0$ for almost all $t \in [0,T]$.*

Proof. The conclusion on λ_0 can be obtained in the same manner as in the proof of Proposition 1 by passing to the limit as $\varphi \to 0+$. If we assume the existence of a point $t_0 \in [0,T]$ for which $c(t_0) > 0$, we can proceed as in the proof of the proposition and get $p_1(t) \equiv p_2(t) \equiv 0$ and $\lambda_0 = 0$, which is impossible. ■

Proposition 2. *The optimal consumption cannot be identically zero.*

Proof. Assume that the controls $c(t) \equiv 0$ and $z(t)$ are optimal. Then

$$J(0, z(t)) = -\int_0^T \eta z^2(t) e^{-\rho t} dt + e^{-\rho T} \frac{a(T)^{1-\theta}}{1-\theta}.$$

Take the controls $\tilde{c}(t) = \varepsilon$ and $\tilde{z}(t) = z(t)$, where $\varepsilon > 0$ is sufficiently small. These controls are admissible, as the respective value of terminal assets is

$$\tilde{a}(T) = e^{rT} \left\{ a_0 + \int_0^T [w(x(t)) - p\varepsilon - \xi z^2(t)] e^{-rt} dt \right\} = a(T) - \varepsilon C_1,$$

where $C_1 := e^{rT} p \int_0^T e^{-rt} dt > 0$. It is evident that for ε sufficiently small we have $\tilde{a}(T) > \delta$, since $a(T) > \delta$. It remains to check that for ε close to zero we have

$$J(\varepsilon, z(t)) = \int_0^T \frac{\varepsilon^{1-\theta}}{1-\theta} e^{-\rho t} dt - \eta \int_0^T z^2(t) e^{-\rho t} dt + e^{-\rho T} \frac{(a(T) - \varepsilon C_1)^{1-\theta}}{1-\theta} >$$

$$J(0, z(t)) = -\int_0^T \eta z^2(t) e^{-\rho t} dt + e^{-\rho T} \frac{a(T)^{1-\theta}}{1-\theta},$$

which is equivalent to

$$\varepsilon^{1-\theta} C_2 > \frac{e^{-\rho T}}{1-\theta} \left[a(T)^{1-\theta} - (a(T) - \varepsilon C_1)^{1-\theta} \right], \; C_2 := const > 0.$$

The last expression is obviously true for all ε sufficiently small. ■

Remark 4. So far it is clear that the optimal consumption cannot be identically zero and that if there exists t_0 such that $c(t_0) \in (0, C)$, then $\lambda_0 = 1$. It remains to check whether we can have $c(t) = C$ for some t.

We first establish the following result.

Proposition 3. *It is impossible for the optimal $c(t)$ to satisfy*

$$c(t) \geq C_0 > 0, \tag{22}$$

where

$$C_0 > \frac{a_0 + T \sup_x |w(x)|}{p \frac{1-e^{-rT}}{r}}. \tag{23}$$

Proof. Notice that if condition (22) is true, then the inequality $a(T) \geq \delta$ is violated. Indeed, if (22) holds, then

$$\begin{aligned} a(T) =& e^{rT} \left[a_0 + \int_0^T \left[w(x(t)) - pc(t) - \xi z^2(t) \right] e^{-rt} dt \right] \leq \\ & e^{rT} \left[a_0 + \int_0^T \left[\sup_x |w(x)| - pC_0 \right] e^{-rt} dt \right], \end{aligned}$$

which is negative when (23) holds. ■

Proposition 4. *The number λ_1 is strictly positive.*

Proof. We know that for the optimal $c(t)$ it is impossible to have $c(t) \geq C_0$ or $c(t) \equiv 0$. Consequently, there exists $t_0 \in [0, T]$ for which $c(t_0) \in (0, C_0)$. Then

$$\frac{\partial}{\partial \varphi} H(t_0, a(t_0), x(t_0), \varphi, z(t_0), p_1(t_0), p_2(t_0)) \big|_{\varphi = c(t_0)} = 0,$$

and hence (21) holds. This in turn implies that $\lambda_0 = 1$, as well as

$$p \lambda_1 e^{-rt_0} = e^{-\rho t_0} c(t_0)^{-\theta}.$$

Therefore, we have $\lambda_1 > 0$ and

$$\lambda_1 = \frac{e^{(r-\rho)t_0} c(t_0)^{-\theta}}{p}. \tag{24}$$

■

Proposition 5. *There does not exist $t \in [0, T]$ for which $c(t) = C$.*

Proof. Assuming the contrary, by the maximum principle we obtain

$$\frac{H(t, a(t), x(t), \varphi, z(t), p_1(t), p_2(t)) - H(t, a(t), x(t), C, z(t), p_1(t), p_2(t))}{\varphi - C} \geq 0$$

for $\varphi \in (0, C)$ and so for $\varphi \to C - 0$ we get

$$-p\lambda_1 e^{-rt} + e^{-\rho t} C^{-\theta} \geq 0,$$

which implies

$$C^\theta \leq \frac{e^{(r-\rho)t}}{\lambda_1 p} = \frac{e^{(r-\rho)t} c(t_0)^\theta}{e^{(r-\rho)t_0}} < e^{(r-\rho)(t-t_0)} C_0^\theta \leq \mu C_0^\theta,$$

where $\mu := \max_{t,t_0 \in [0,T]} e^{(r-\rho)(t-t_0)} > 0$. In other words,

$$C \leq \mu^{\frac{1}{\theta}} C_0,$$

which is impossible. ■

The results obtained so far allow us to to find an expression for the optimal consumption $c(t)$.

Corollary 2. *For each $t \in [0, T]$ we have $c(t) \in (0, C)$. The optimal consumption rule has the form*

$$c(t) = \left[\frac{1}{p\lambda_1}\right]^{\frac{1}{\theta}} e^{\frac{r-\rho}{\theta} t} = \frac{1}{p^{\frac{1}{\theta}}} e^{\frac{\rho - r}{\theta}(T-t)} a(T). \tag{25}$$

Before deriving an expression for the optimal relocation control $z(t)$, we note that (14) and (19) imply

$$p_2(t) = \lambda_1 \int_t^T w'(x(\tau)) e^{-r\tau} d\tau = e^{(r-\rho)T} a(T)^{-\theta} \int_t^T w'(x(\tau)) e^{-r\tau} d\tau. \tag{26}$$

Proposition 6. *For each $t \in [0, T]$ we have the strict inequality*

$$|z(t)| < Z.$$

Proof. Assume, for example, that $z(t_1) = Z$ for some $t_1 \in [0, T]$. Then, after passing to the limit in the respective difference quotient, we obtain

$$-p_1(t_1)\xi 2Z + p_2(t_1) - 2\eta Z e^{-\rho t_1} \geq 0,$$

so that

$$p_2(t_1) \geq 2(\xi\lambda_1 e^{-rt_1} + \eta e^{-\rho t_1}) Z.$$

Similarly, the assumption that $z(t_2) = -Z$ for some $t_2 \in [0,T]$ leads to

$$-p_2(t_2) \geq 2(\xi\lambda_1 e^{-rt_2} + \eta e^{-\rho t_2})Z.$$

In both cases we have ($i = 1$ or 2)

$$Z \leq \frac{\pm p_2(t_i)}{2(\xi\lambda_1 e^{-rt_i} + \eta e^{-\rho t_i})} \leq \frac{|p_2(t_i)|}{2(\xi\lambda_1 e^{-rt_i} + \eta e^{-\rho t_i})} \leq \frac{\lambda_1 \left|\int_{t_i}^T w'(x(\tau))e^{-r\tau}d\tau\right|}{2(\xi\lambda_1 e^{-rt_i} + \eta e^{-\rho t_i})} \leq$$

$$\frac{\sup_x |w'(x)|\,|T - t_i|}{2(\xi e^{-rt_i} + \frac{\eta}{\lambda_1}e^{-\rho t_i})} < \frac{T\sup_x |w'(x)|}{2\xi e^{-rt_i}} \leq \frac{T\sup_x |w'(x)|}{2\xi}e^{rT},$$

which is impossible by the definition of Z. ■

Corollary 3. *For $t \in [0,T]$ we have for the optimal relocation speed $z(t) \in (-Z, Z)$ and then*

$$z(t) = \frac{p_2(t)}{2(\xi\lambda_1 e^{-rt} + \eta e^{-\rho t})}. \tag{27}$$

5. Existence of a solution of the system of necessary conditions

In order to facilitate the study of the differential equations arising from the set of necessary conditions in section 4, it would prove convenient to rewrite the differential system. Theorem 1 guarantees the existence of a solution to the problem (1)-(3) which in turn ensures the existence for each $T > 0$ of a solution to the following problem:

$$\left|\begin{array}{l} \dot{x}(t) = \frac{y(t)}{F(t)}, \\ \dot{y}(t) = -w'(x(t))\lambda_1 e^{-rt}, \\ x(0) = x_0, \\ y(T) = 0, \end{array}\right. \tag{28}$$

where $y(t) := p_2(t)$ and $F(t) := 2(\xi\lambda_1 e^{-rt} + \eta e^{-\rho t})$. It follows that there exists a solution to the problem

$$\left|\begin{array}{l} \frac{d}{dt}\left(F(t)\dot{x}(t)\right) + w'(x(t))\lambda_1 e^{-rt} = 0, \\ x(0) = x_0, \\ \dot{x}(T) = 0. \end{array}\right. \tag{29}$$

The latter fact can also be established without recourse to Theorem 1. Following the procedure described in §73 of [9], we construct the respective Green function and transform (29) in the form

$$x(t) = x_0 + \int_0^T K(t,\tau)\lambda_1 e^{-r\tau} w'(x(\tau))d\tau, \tag{30}$$

where

$$K(t,\tau) = \begin{cases} \int_0^t \frac{1}{F(s)}ds, & t \in [0,\tau] \\ \int_0^\tau \frac{1}{F(s)}ds, & t \in [\tau, T] \end{cases}$$

Since the function $w'(x)$ is bounded and continuous by assumption, a solution to (30) exists. This is a consequence of Leray-Schauder index theory (see §2.4 in [10]). Also, the solution to (29) may not be unique, as can be seen from simple examples of eigenfunction problems that possess nontrivial solutions.

A solution to (28) or (29) can be viewed as a particular member of the family of solutions $(x(t,\alpha), y(t,\alpha))$ to the Cauchy problem

$$\left| \begin{aligned} &\dot{x}(t) = \tfrac{y(t)}{F(t)}, \\ &\dot{y}(t) = -w'(x(t))\lambda_1 e^{-rt}, \\ &x(0) = x_0, \\ &y(0) = \alpha, \end{aligned} \right. \tag{31}$$

where α has been chosen appropriately, so that

$$y(T,\alpha) = 0. \tag{32}$$

The existence of a unique solution to (31) on the interval $[0,T]$ for initial data (x_0,α) and each $T > 0$ is ensured by Corollary 3.1, chapter 2, in [6].

Since (32) is equivalent to $\dot{x}(T,\alpha) = 0$, we can integrate the differential equation in (29) over $[0,T]$ to arrive at an equivalent form of (32):

$$\alpha = \lambda_1 \int_0^T w'(x(\tau,\alpha))e^{-r\tau}d\tau. \tag{33}$$

It is straightforward to verify the following

Proposition 7. *The function $x(t) \equiv x_0$ is a solution to* (29) *if and only if the point x_0 is a critical point for $w(x)$, i.e. $w'(x_0) = 0$.*

The analysis of the solutions of the system of necessary conditions, which is carried out in section 7, provides the dynamics of the behaviour of the economic agent, implied by this model, in the baseline case when the wage distribution has a single maximum point on the interval $[0,1]$. Prior to that, the next section studies the properties of the function $T \mapsto a(T)$ as $T \to \infty$.

6. Dynamics of terminal assets $a(T)$ for different time horizons

In this section we study the dependence of optimal terminal assets $a(T)$ on the length of the time horizon T. Although terminal assets is the natural object of study due to the fact that it is easily interpretable in economic terms, the discussion may equally well be framed in terms of the behaviour of λ_1, viewed as a function of the time horizon T and denoted $\lambda_1(T)$. This approach is feasible by virtue of the relationship

$$\lambda_1(T) = e^{(r-\rho)T} a(T)^{-\theta}. \tag{34}$$

Below we derive upper and lower bounds on $\lambda_1(T)$, which will be needed in the analysis of section 7. We assume for simplicity that $x_0 \in (0,1)$, i.e. $w(x_0) > 0$, as well as that $a_0 > 0$. Also, in this section we denote by C different constants that do not depend on T. Since we do not use the bound on the control $c(t)$ from (4) in this section, no confusion can arise from this convention.

6.1. An upper bound on $\lambda_1(T)$

The pair $(c(t) \equiv c_0 = const, z(t) \equiv 0)$ is admissible for c_0 appropriately chosen. Then $x(t) \equiv x_0$ and we set $c_0 := \frac{w(x_0)}{p}$. In this case terminal assets are

$$\tilde{a}(T) = e^{rT}\left[a_0 + \int_0^T [w(x(t)) - pc(t) - \xi z^2(t)] e^{-rT} dt\right] = a_0 e^{rT}.$$

Consequently,

$$J(c_0, 0) = c_1(1 - e^{-\rho T}) + c_2 e^{[r(1-\theta)-\rho]T},$$

where c_1 and c_2 are constants that depend on c_0 and a_0.

On the other hand, for the optimal controls $(c(t), z(t))$ we have

$$J(c(t), z(t)) = \int_0^T \left[a(T)\frac{1}{p^{1/\theta}} e^{\frac{\rho-r}{\theta}(T-t)}\right]^{1-\theta} \frac{e^{-\rho t}}{1-\theta} dt - \eta \int_0^T z^2(t) e^{-\rho t} dt + \frac{a(T)^{1-\theta}}{1-\theta} e^{-\rho T},$$

which takes different forms depending on whether $\rho - r(1-\theta)$ is different from zero.

Since $J(c(t), z(t)) \geq J(c_0, 0)$, for $\rho - r(1-\theta) \neq 0$ we obtain

$$\frac{a(T)^{1-\theta}}{1-\theta} e^{-\rho T}\left[1 + \frac{1}{p^{\frac{1-\theta}{\theta}}} \frac{e^{\frac{\rho - r(1-\theta)}{\theta}T} - 1}{\frac{\rho - r(1-\theta)}{\theta}}\right] \geq c_1 + c_2 e^{-(\rho - r(1-\theta))T} - c_1 e^{-\rho T}.$$

Thus,

$$a(T) \geq \left\{ \frac{\tilde{c}_1 e^{\rho T} + \tilde{c}_2 e^{r(1-\theta)T} - \tilde{c}_1}{1 + \frac{1}{p^{\frac{1-\theta}{\theta}}} \frac{e^{\frac{\rho - r(1-\theta)}{\theta}T} - 1}{\frac{\rho - r(1-\theta)}{\theta}}} \right\}^{\frac{1}{1-\theta}}, \tag{35}$$

where $\tilde{c}_1 = (1-\theta)c_1$, $\tilde{c}_2 = (1-\theta)c_2$. Using the last expression together with (34), we can derive upper bounds on $\lambda_1(T)$.

Proposition 8. *Under the assumptions of this section, we have ($\forall T > 0$):*

$$\lambda_1(T) \leq \begin{cases} C, & \text{if } \rho - r(1-\theta) > 0, \\ Ce^{-(\rho - r(1-\theta))T}, & \text{if } \rho - r(1-\theta) < 0, \\ C(1+T)^{\frac{\theta}{1-\theta}}, & \text{if } \rho - r(1-\theta) = 0. \end{cases} \tag{36}$$

and, accordingly,

$$a(T) \geq \begin{cases} Ce^{\frac{r-\rho}{\theta}T}, & \text{if } \rho - r(1-\theta) > 0, \\ Ce^{rT}, & \text{if } \rho - r(1-\theta) < 0, \\ C\frac{e^{\frac{\rho}{1-\theta}T}}{(1+T)^{\frac{1}{1-\theta}}}, & \text{if } \rho - r(1-\theta) = 0. \end{cases} \tag{37}$$

6.2. A lower bound on $\lambda_1(T)$

We first look at a particular case of the main problem, for which

$$w(x) \equiv W = const. \tag{38}$$

Then $\dot{p}_2 \equiv 0$ which, together with the transversality condition $p_2(T) = 0$, yields $p_2(t) \equiv 0$, i.e. $z(t) \equiv 0$ and $x(t) \equiv x_0$.

The optimal consumption rule is $c(t) = \frac{1}{p^{1/\theta}}\bar{a}(T)e^{\frac{\rho-r}{\theta}(T-t)}$, where $\bar{a}(T)$ is the optimal terminal value of assets for the problem with condition (38). We will calculate $\bar{a}(T)$ from

$$\bar{a}(T) = e^{rT}\left[a_0 + \int_0^T \left(W - p^{\frac{\theta-1}{\theta}}\bar{a}(T)e^{\frac{\rho-r}{\theta}(T-t)}\right)e^{-rt}dt\right].$$

Thus, we find

$$\bar{a}(T) \leq \begin{cases} Ce^{\frac{r-\rho}{\theta}T}, & \text{if } \rho - r(1-\theta) > 0, \\ Ce^{rT}, & \text{if } \rho - r(1-\theta) < 0, \\ C\frac{e^{rT}}{1+T}, & \text{if } \rho - r(1-\theta) = 0. \end{cases} \tag{39}$$

Proposition 9. *Let $w(x) \leq W$, $\forall x$, and let $a(T)$ and $\bar{a}(T)$ be the optimal terminal asset values for the problems with wage distributions $w(x)$ and W, respectively (all other parameters of the two problems being identical). Then*

$$a(T) \leq \bar{a}(T).$$

Proof. Since according to (25) optimal consumption for the two problems has the form $a(T)\Psi(t)$ and $\bar{a}(T)\Psi(t)$ with one and the same function $\Psi(t)$, we obtain

$$[\bar{a}(T) - a(T)]\left(1 + e^{rT}\int_0^T p\Psi(t)e^{-rt}dt\right) = e^{rT}\int_0^T [W - w(x(t))]e^{-rt}dt + e^{rT}\xi\int_0^T z^2(t)e^{-rt}dt,$$

where $x(t)$ and $z(t)$ refer to the variables in the problem with wage distribution $w(x)$. This completes the proof. ■

From Proposition 9 and equations (34) and (39), we obtain

Proposition 10. *Under the assumptions of this section, we have (*$\forall T > 0$*):*

$$a(T) \leq \begin{cases} Ce^{\frac{r-\rho}{\theta}T}, & \text{if } \rho - r(1-\theta) > 0, \\ Ce^{rT}, & \text{if } \rho - r(1-\theta) < 0, \\ C\frac{e^{rT}}{1+T}, & \text{if } \rho - r(1-\theta) = 0. \end{cases} \tag{40}$$

and, accordingly,

$$\lambda_1(T) \geq \begin{cases} C, & \text{if } \rho - r(1-\theta) > 0, \\ Ce^{-(\rho - r(1-\theta))T}, & \text{if } \rho - r(1-\theta) < 0, \\ C(1+T)^{\theta}, & \text{if } \rho - r(1-\theta) = 0. \end{cases} \tag{41}$$

Remark 5. The bounds derived above can be refined in some cases. For instance, the first inequality in (40) implies very different behaviour of $a(T)$ depending on whether $\rho \in (r(1-\theta), r)$, $\rho = r$ or $\rho > r$.

7. Optimal relocation strategies for single-peaked wage distributions

This section studies the optimal relocation behaviour of the consumer, as described by $x(t)$, in the important case of single-peaked wage distributions on the interval $[0, 1]$. We demonstrate first the validity of the following general claim (under the conditions stated at the end of section 2):

Proposition 11. *The optimal trajectory* $x(t)$ *remains in the interval* $[0, 1]$*, regardless of the particular form of the wage distribution* $w(x)$ *in* $[0, 1]$*.*

Proof. Notice that since $\dot{x}(t) = p_2(t)/F(t)$, with $F(t)$ defined as in section 5, in view of (26) we can write

$$\dot{x}(t) = \frac{\lambda_1}{F(t)} \int_t^T w'(x(\tau))e^{-r\tau}d\tau = G(t) \int_t^T w'(x(\tau))e^{-r\tau}d\tau,$$

where $G(t) := \lambda_1/F(t)$.

Assume first that at time t_1 the point $x(t)$ leaves the interval $[0,1]$ to the left (i.e. leaves the interval at $x = 0$) and remains to the left of zero until $t = T$, so that $x(t) < 0$ for $t \in (t_1, T]$. Then, for $t \in [t_1, T]$, $w'(x(t)) > 0$ and consequently $\dot{x}(t) > 0$. This would imply that for some $t_* \in (t_1, T)$, $x(T) - x(t_1) = (T - t_1)\dot{x}(t_*) > 0$, or $x(T) > x(t_1) = 0$, which contradicts the assumption that $x(t) < 0$ for $t \in (t_1, T]$. Thus, $x(t)$ cannot leave the interval $[0,1]$ to the left and remain outside it until the end of the planning horizon T. A similar argument shows that it is impossible for $x(t)$ to leave the interval $[0,1]$ to the right and stay there.

Let us now assume that $x(t)$ leaves the interval $[0,1]$ to the left of zero at time t_1 and returns back at time $t_2 > t_1$. Again, for $t \in (t_1, t_2)$ we have $x(t) < 0$ and $w'(x(t)) > 0$, which means that $\int_{t_1}^{t_2} w'(x(t))e^{-rt}dt > 0$. Since $x(t)$ leaves the interval $[0,1]$ to the left at t_1, it must be that $\dot{x}(t_1) \leq 0$. By the same logic, at time t_2 we should have $\dot{x}(t_2) \geq 0$. Then one obtains

$$\begin{aligned}
\dot{x}(t_1) =&G(t_1) \int_{t_1}^{T} w'(x(t))e^{-rt}dt = G(t_1) \int_{t_1}^{t_2} w'(x(t))e^{-rt}dt + \frac{G(t_1)}{G(t_2)} G(t_2) \int_{t_2}^{T} w'(x(t))e^{-rt}dt \\
=&G(t_1) \int_{t_1}^{t_2} w'(x(t))e^{-rt}dt + \frac{G(t_1)}{G(t_2)} \dot{x}(t_2) > 0,
\end{aligned}$$

which contradicts the condition $\dot{x}(t_1) \leq 0$. Hence it is impossible for $x(t)$ to temporarily leave the interval $[0,1]$ to the left. Naturally, this argument can be applied with obvious modifications to the hypothesis that $x(t)$ temporarily goes to the right of $x = 1$. Summarizing the above conclusions, we see that the optimal $x(t)$ remains in the interval $[0,1]$. ■

We turn next to the main object of study for this section: the case when the wage function has a single peak on the interval $[0,1]$. The example of the quadratic function $w(x) = x(1-x)$ in a neighbourhood of the interval $[0,1]$ may facilitate visualization.

Assume that in this case the initial location x_0 lies to the left of the wage peak, i.e. if $x_1 := \operatorname{argmax}_{x\in[0,1]} w(x)$, then $0 \leq x_0 < x_1$. For the remainder of this section we will assume that $w'(x) > 0,\ x \in [0, x_1]$ and $w'(x) < 0,\ x \in [x_1, 1]$. In this case, for T sufficiently small, $x(T)$ will remain in a small neighbourhood of x_0. However, this means that $w'(x(t)) > 0$ for any $t \in [0,T]$ and therefore $\dot{p}_2(t) < 0$, which implies $p_2(t) > 0$ since $p_2(T) = 0$. As $p_2(t) > 0$, we obtain $\dot{x}(t) > 0$. In words, for sufficiently small planning horizons the consumer unambiguously relocates toward the wage maximum.

If $x(T) \in (x_0, x_1)$, we have, using the notation in section 5, $\dot{y}(t) = -w'(x(t))p_1(t) < 0$. Since $y(T) = 0$, it follows that $y(t) > 0,\ t \in [0, T)$. Then $\dot{x}(t) = \frac{y(t)}{F(t)} > 0$, so that $x(t)$ is monotonically increasing.

We note that there does not exist a solution to the system of necessary conditions for which $x(T) = x_1$. For such a solution we would have $y(T) = 0$ and one could compare this solution of the stationary solution $\tilde{x}(t) \equiv x_1, \tilde{y}(t) \equiv 0$. Then, the uniqueness of the solution to a Cauchy problem (for identical data at $t = T$) shows that the two solutions coincide. This, however, is impossible, since for $t = 0$ the values of the two solutions are different ($x(0) = x_0 \neq \tilde{x}(0) = x_1$).

We would like to check whether it is possible for the terminal location $x(T)$ to lie to the right of the wage peak for $x_0 < x_1$. To this end, assume that $x(T) > x_1$ and let t_1 be the time when point x_1 is reached last, i.e. $x(t_1) = x_1$ and for $t \in (t_1, T)$ we have $x(t) > x_1$. (In other words, $t_1 = \sup\{t \in [0, T] | x(t) = x_1\}$.) Then, by the mean value theorem, $0 < x(T) - x(t_1) = (T - t_1)\frac{y(t_*)}{F(t_*)}$ for some $t_* \in (t_1, T)$. However, since $\dot{y}(t) > 0,\ t \in (t_1, T)$, and $y(T) = 0$ imply $y(t_*) < 0$, we obtain $\frac{y(t_*)}{F(t_*)} < 0$, which is a contradiction. Thus, $x(T)$ cannot lie to the right of x_1.

The systematic study of the relocation behaviour of the economic agent in this case can be reduced to the analysis of the way in which the solutions to the Cauchy problem (31) behave for different values of the parameter α. Those solutions that satisfy (32) are also solutions to the system of necessary conditions (28), i.e. extremals. Through this approach we can also obtain information on the number of solutions to the problem at hand. Of course, if only one extremal exists, then it is the solution we seek.

We remind the reader that the number $\lambda_1 = \lambda_1(T)$ is fixed, insofar as T is fixed.

Case I: $\alpha \leq 0$**.** It is obvious that for small t we have $y(t) < 0$ since $\dot{y}(t) < 0$. For such t we have

$$x(t) = x_0 + \int_0^t \frac{y(\tau)}{F(\tau)} d\tau < x_0,$$

i.e. the agent shifts toward $x = 0$. It means that $\dot{y}(t)$ remains negative, so that $y(t) = \alpha + \int_0^t \dot{y}(\tau) d\tau$ also remains negative and $x(t)$ keeps moving to the left. Thus, it is impossible for (32) to become true, i.e. there are no extremals among the solutions of (31) for $\alpha \leq 0$.

Case II: $\alpha > 0$**.** In this case we have $y(t) > 0$ in a neighbourhood of $t = 0$ and so $x(t)$ moves to the right in the direction of the point x_1. However, $\dot{y}(t) = -w'(x(t))p_1(t) < 0$, so that $y(t)$ decreases. If it turns out that $y(T) = 0$ and $x(T) < x_1$, then the respective solution is an extremal. The existence of such an extremal is guaranteed by Theorem 1. The latter claim can be established through an alternative approach, which allows us to ascertain the number of extremals.

We introduce the notation $M(\alpha)$ for the right-hand side of (33). Since $x(t) < x_1$ for $t \in [0, T]$, we get

$$M(\alpha) \leq \lambda_1 \sup_x |w'(x)| \frac{1 - e^{-rT}}{r} =: M_0,$$

i.e. in view of (33) the relevant values of α lie in the interval $(0, M_0)$.

It is easy to see that the function

$$g(\alpha) := \alpha - M(\alpha)$$

is continuous on the interval $[0, M_0 + 1]$ and satisfies the inequalities

$$g(0) < 0 < g(M_0 + 1).$$

Consequently, there exists $\alpha > 0$ for which $g(\alpha) = 0$, i.e. which satisfies (33).

Proposition 12. *For the case of a single-peaked wage distribution $w(x)$ with $w''(x) \leq 0$ in $[0, 1]$, there exists a unique extremal for the system* (28).

Proof. Assume that at least two different extremals exist. They solve the system (31) for different positive values $\alpha_1 \neq \alpha_2$, for which $\alpha_i - M(\alpha_i) = 0, \ i = 1, 2$. Then

$$(\alpha_2 - \alpha_1) \left(1 - \frac{d}{d\alpha} M(\alpha^*)\right) = 0, \ \alpha^* = \varkappa \alpha_1 + (1 - \varkappa)\alpha_2, \ \varkappa \in (0, 1). \tag{42}$$

The derivative

$$\frac{d}{d\alpha} \left(\lambda_1 \int_0^T w'(x(t, \alpha)) e^{-rt} dt \right) \Bigg|_{\alpha = \alpha^*}$$

has the form

$$\lambda_1 \int_0^T w''(x(t, \alpha^*)) \frac{\partial x(t, \alpha^*)}{\partial \alpha} e^{-rt} dt,$$

with $x_\alpha(t, \alpha) := \frac{\partial x(t,\alpha)}{\partial \alpha}$ and $y_\alpha(t, \alpha) := \frac{\partial y(t,\alpha)}{\partial \alpha}$ satisfying the equations of variation [6, Ch.V, Theorem 3.1]:

$$\left| \begin{array}{l} \dot{x}_\alpha(t, \alpha) = \frac{y_\alpha(t,\alpha)}{F(t)}, \\ \dot{y}_\alpha(t, \alpha) = -w''(x(t, \alpha)) x_\alpha(t, \alpha) \lambda_1 e^{-rt}, \\ x_\alpha(0, \alpha) = 0, \\ y_\alpha(0, \alpha) = 1. \end{array} \right.$$

Consequently, $x_\alpha(t, \alpha^*)$ solves the linear equation

$$\frac{d}{dt} \left(F(t) \dot{x}_\alpha(t, \alpha^*) \right) + w''(x(t, \alpha^*)) \lambda_1 e^{-rt} x_\alpha(t, \alpha^*) = 0$$

for initial data $x_\alpha(0,\alpha^*) = 0$ and $\dot{x}_\alpha(0,\alpha^*) = 1$. Multiplying by $x_\alpha(t,\alpha^*)$ and integrating over $(0,t)$, we obtain

$$F(t)\dot{x}_\alpha(t,\alpha^*)x_\alpha(t,\alpha^*) = \int_0^t F(\tau)\dot{x}_\alpha^2(\tau,\alpha^*)d\tau + \int_0^T (-w''(x(\tau,\alpha^*)))\lambda_1 e^{-r\tau}x_\alpha^2(\tau,\alpha^*)d\tau \geq 0.$$

Taking into account that $F(t)\dot{x}_\alpha(t,\alpha^*)x_\alpha(t,\alpha^*) = \frac{d}{dt}(x_\alpha^2(t,\alpha^*))F(t)/2$, we establish that the function $x_\alpha^2(t,\alpha^*)$ is increasing. In view of the initial conditions, in a small interval $(0,\varepsilon)$ we have $x_\alpha(t,\alpha^*) > 0$. This inequality holds for all $t \in (0,T)$, for otherwise there would exist $\bar{t} \in (\varepsilon,T)$ for which $x_\alpha(\bar{t},\alpha^*) = 0$. The latter would lead to the contradiction $0 < x_\alpha^2(\varepsilon/2,\alpha^*) \leq x_\alpha^2(\bar{t},\alpha^*) = 0$. Taking into account that $w''(x) \leq 0$, we obtain from (42) that $\alpha_1 = \alpha_2$, i.e. the two extremals coincide. ■

We augment the above results by investigating the dependence of the final location $X(T) := x(T;T,x_0)$ of the agent on the length of the time horizon T. Since $X(T) < x_1$, $\forall T > 0$, we have $l := \sup_{T>0} X(T) \leq x_1$.

Proposition 13. *Under the assumptions of Proposition 12, we have the following classification. If $\rho \geq r$ or $\rho \in (0, r(1-\theta)]$, then $l = x_1$. If $\rho \in (r(1-\theta), r)$, it is possible to have $l < x_1$ for appropriate values of the parameters of the problem.*

Proof. Assume that $l < x_1$. For $\rho \geq r$, we have $e^{(r-\rho)t} \leq 1$ and so

$$X(T) = x_0 + \int_0^T \frac{\int_\tau^T \lambda_1(T)w'(x(s))e^{-rs}ds}{2(\xi\lambda_1(T)e^{-r\tau} + \eta e^{-\rho\tau})}d\tau \geq x_0 + \frac{\lambda_1(T)w'(l)}{2(\xi\lambda_1(T)+\eta)}\int_0^T e^{r\tau}\left(\int_\tau^T e^{-rs}ds\right)d\tau,$$

where the integral evaluates to $\frac{1}{r}\left[T - \frac{1}{r} + \frac{e^{-rT}}{r}\right]$.

According to the results from section 6, the expression $\frac{\lambda_1(T)}{2(\xi\lambda_1(T)+\eta)}$ does not tend to zero as $T \to \infty$, so $\lim_{T\to\infty} X(T) = \infty$, which contradicts the fact that $X(T)$ is bounded.

For $\rho < r$, we study three cases according to the behaviour of $\lambda_1(T)$:

i) $\rho \in (r(1-\theta), r)$. In this case $\lim_{T\to\infty} \lambda_1(T) = const$,

ii) $\rho \in (0, r(1-\theta))$. In this case $\lim_{T\to\infty} \lambda_1(T) = \infty$.

iii) $\rho = r(1-\theta)$. In this case $C_1(1+T)^\theta \leq \lambda_1(T)$.

For case i) we have

$$X(T) \leq x_0 + \frac{\lambda_1(T)w'(x_0)}{2\eta}\int_0^T e^{\rho\tau}\left(\int_\tau^T e^{-rs}ds\right)d\tau \leq x_0 + \frac{\lambda_1(T)w'(x_0)}{2\rho(r-\rho)\eta},$$

after taking into account that the integral evaluates to $\frac{1}{r}\left[\frac{1}{r-\rho} - \frac{r}{(r-\rho)\rho}e^{-(r-\rho)T} + \frac{e^{-rT}}{\rho}\right]$. It is clear that, for instance, for large values of η this upper bound on $X(T)$ can be strictly smaller than x_1.

In case ii), defining $A := r(1-\theta) - \rho > 0$, we have from section 6

$$C_1 e^{AT} \le \lambda_1(T) \le C_2 e^{AT} \text{ with } C_i > 0,\ i = 1, 2.$$

Consequently, assuming that $l < x_1$, we have

$$X(T) \ge x_0 + \frac{C_1 w'(l)}{2r} \int_0^T e^{AT} \frac{e^{-r\tau} - e^{-rT}}{\zeta e^{AT} e^{-r\tau} + \eta e^{-\rho\tau}} d\tau, \tag{43}$$

where $\zeta := \xi C_2 > 0$. Denote the integral in (43) by I. We have

$$I = \int_0^T \frac{1 - e^{r\tau}e^{-rT}}{\zeta + \eta\frac{e^{(A+r\theta)\tau}}{e^{AT}}} d\tau \ge \int_0^{\frac{A}{A+r\theta}T} \frac{1 - e^{r\tau}e^{-rT}}{\zeta + \eta\frac{e^{(A+r\theta)\tau}}{e^{AT}}} d\tau.$$

Since $e^{-AT}e^{(A+r\theta)\tau} \le 1$ for $\tau \in [0, AT/(A+r\theta)]$, we have from the last expression

$$I \ge \int_0^{\frac{A}{A+r\theta}T} \frac{1 - e^{r\tau}e^{-rT}}{\zeta + \eta} d\tau = \frac{A}{(A+r\theta)(\zeta+\eta)} T - \frac{e^{-rT}}{\zeta+\eta} \cdot \frac{e^{\frac{A}{A+r\theta}rT} - 1}{r}.$$

The last expression tends to infinity as $T \to \infty$, implying that $X(T)$ is unbounded. This contradiction shows that $l = x_1$.

In case iii) the condition from section 6 is equivalent to

$$\frac{1}{\lambda_1(T)} \le \frac{1}{C_1(1+T)^\theta}.$$

Assume that $l < x_1$. Then

$$\begin{aligned} X(T) \ge & x_0 + \frac{w'(l)\lambda_1(T)}{2r} \int_0^T \frac{1 - e^{-rT}e^{r\tau}}{\xi\lambda_1(T) + \eta e^{r\theta\tau}} d\tau \ge x_0 + \frac{w'(l)}{2r} \int_0^T \frac{1 - e^{-rT}e^{r\tau}}{\xi + \frac{\eta e^{r\theta\tau}}{C_1(1+T)^\theta}} d\tau = \\ & x_0 + \frac{w'(l)}{2r\eta} C_1(1+T)^\theta \int_0^T \frac{1 - e^{-rT}e^{r\tau}}{\xi\frac{C_1(1+T)^\theta}{\eta} + e^{r\theta\tau}} d\tau. \end{aligned}$$

Set $B = B(T) := \xi\frac{C_1(1+T)^\theta}{\eta}$ and introduce the change of variables $\mu = e^{r\theta\tau}$ in the last expression to obtain

$$X(T) \ge x_0 + const(1+T)^\theta \int_1^{e^{r\theta T}} \frac{1 - e^{-rT}\mu^{\frac{1}{\theta}}}{r\theta\mu(\mu + B)} d\tau.$$

This requires us to study the behaviour of two expressions.

First, we have

$$(1+T)^{\theta}\int\limits_{1}^{e^{r\theta T}}\frac{d\mu}{\mu(\mu+B)}=(1+T)^{\theta}\frac{1}{B}\left[\ln\left(\frac{1}{1+\frac{const(1+T)^{\theta}}{e^{r\theta T}}}\right)+\ln\left(1+const(1+T)^{\theta}\right)\right].$$

When $T\to\infty$, the first logarithm tends to zero and the second one tends to infinity, i.e. the whole expression tends to infinity.

Second, note that

$$0\le(1+T)^{\theta}e^{-rT}\int\limits_{1}^{e^{r\theta T}}\frac{\mu^{\frac{1}{\theta}}}{\mu(\mu+B)}d\mu\le\frac{(1+T)^{\theta}}{e^{rT}}\int\limits_{1}^{e^{r\theta T}}\frac{\mu^{\frac{1}{\theta}}}{\mu^{2}}d\mu=\frac{\theta}{1-\theta}\left[\frac{(1+T)^{\theta}}{e^{r\theta T}}-\frac{(1+T)^{\theta}}{e^{rT}}\right].$$

The last expression tends to zero as $T\to\infty$.

Combining the above results, we obtain $X(T)\to\infty$, which contradicts the fact that $X(T)$ is bounded. Thus, in this case $l=x_1$. ■

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References

[1] R. Baldwin, The Core-Periphery Model with Forward-Looking Expectations, *Regional Science and Urban Economics*, 31, 21-49 (2001).

[2] R. Boucekkine, C. Camacho and B. Zou, Bridging the gap between growth theory and economic geography: The spatial Ramsey model, mimeo (2006).

[3] P. Brito, The dynamics of growth and distribution in a spatially heterogeneous world, *Working papers, Department of Economics, ISEG, WP13/2004/DE/UECE* (2004).

[4] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley, New York, 1983.

[5] J. Diestel, *Geometry of Banach Spaces: Selected Topics*, Springer-Verlag, Berlin, 1975.

[6] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, New York, 1964.

[7] A.N. Kolmogorov, S.V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Nauka, Moscow (in Russian), 1976.

[8] E.B. Lee, L. Markus, *Foundations of Optimal Control Theory*, John Wiley, New York, 1968.

[9] W.V. Lovitt, *Linear Integral Equations*, McGraw-Hill Book Co., New York, 1924.

[10] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Inst., NY Univ., 1974.

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The Mixing Properties on Maps of the Warsaw Circle*

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Abstract: Let W be a Warsaw circle and $f : W \to W$ be a continuous map. In this paper some dynamical properties of f are studied. Firstly, it is shown that f is topological transitive if and only if f is chaotic in the sense of Devaney. Secondly, It is shown that f is topological transitive if and only if f is mixing and that f is topological transitive implies f has horseshoe.

MSC 2000: 58F10, 54H20.

Keywords: Warsaw Circle; Topological transitivity; Topological Mixing;

1 Introduction

Let N denote the set of all positive integers. Being a dynamical system we mean a pair (X, f), where X is a compact metric space with the metric d and $f : X \to X$ is a continuous map. We assume that $f^n = f \circ f^{n-1}, n = 1, 2, \cdots,$ and f^0 is the identity mapping. The sequence $f^n(x), n = 0, 1, 2, \cdots$, is called the ***trajectory*** of a point $x \in X$ and is denoted by $O(x, f)$. A point $x \in X$ is said to be a ***periodic point*** if $f^n(x) = x$ for some $n \in N$.The least such n is called the period of x. x is said to be a ***fixed point*** if $f(x) = x$. Let $\mathrm{Fix}(f), P(f)$ denote respectively the set of all fixed, periodic points.

We define the ***limit set*** of a point $x \in X$ to be the set $\omega(x) = \omega(x, f) = \bigcap_{n\in N} \overline{\{f^k(x) : k \geq n\}}$. Write $\omega(f) = \bigcup_{x\in X} \omega(x, f)$. A point $x \in X$ is said to be a ***recurrent*** if $x \in \omega(x, f)$ and ***non-wandering*** if every open set containing x contains at least two points of some trajectory. Let $R(f), \Omega(f)$ denote respectively the set of all recurrent and non-wandering points. It is well known that $\mathrm{Fix}(f) \subset P(f) \subset R(f) \subset \omega(f) \subset \Omega(f)$.

f is said to be ***topologically transitive*** if for every pair of nonempty open sets U and V in X, there is a positive integer k such that $f^k(U) \cap V \neq \emptyset$. f

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is said to be ***totally transitive*** if f^n is transitive for every $n \in N$. f is said to be ***topologically mixing*** if for every pair of nonempty open sets U and V in X, there exits N_0 such that $f^n(U) \cap V \neq \emptyset$ for every $n \geq N_0$. It is obvious that f is topologically mixing implies f is totally transitive, which implies f is transitive.

f is said to be ***chaotic in the sense of Devaney*** if it satisfies the following three conditions:

(1) f is topological transitive;

(2) the periodic points of f are dense in X and

(3) f is sensitive dependence on initial conditions(i.e. there is a positive number $\delta > 0$ such that for every point $x \in X$ and every neighborhood U of x there exits a point $y \in U$ and $n \in N$ such that $d(f^n(x), f^n(y)) \geq \delta$).

It is well known that the conditions (1) and (2) imply condition (3). Also, for functions on intervals in $\mathcal{R}$, it was shown by Vallekoop and Berglund that transitive implies chaos(see [2] and [3]).

Proposition A[1]. A continuous map $f : X \to X$ of a compact metric space into itself is transitive if and only if there exists a point $x \in X$ such that $\omega(x, f) = X$.

Warsaw circle W is the union of the following five sets of the plane R^2:

$$\begin{array}{ll} W_0 = \{(x,y) \in R^2 | x = 0, -1 \leq y \leq 1 & \} \\ W_1 = \{(x,y) \in R^2 | x = 0, -3 \leq y \leq -1 & \} \\ W_2 = \{(x,y) \in R^2 | 0 \leq x \leq 1, y = -3 & \} \\ W_3 = \{(x,y) \in R^2 | x = 1, -3 \leq y \leq 0 & \} \\ W_4 = \{(x,y) \in R^2 | 0 < x \leq 1, y = \sin(\frac{2\pi}{x}) & \} \end{array}$$

Write $z = (0,1) \in W_0$. For each $x \in W - \{z\}$, we denote $W_x(z)$ the arc-connected component of $W - \{x\}$ which contains z. Denote $x <_w y$ and $y >_w x$ if $x \in W_y(z)$. Denote $(x,y) = \{a \in W : x <_w a <_w y\}$, $[x,y] = (x,y) \cup \{x\} \cup \{y\}$, $[x,y) = (x,y) \cup \{x\}$, and $(x,y] = (x,y) \cup \{y\}$ for each $x, y \in W$. The sets $[x,y]$, $[x,y)$, $(x,y]$ and (x,y) are said to be the intervals in W. Warsaw circle is simple connected but not locally connected, and it can cut the plane. Warsaw circle often appears as an example of circle-like an non arc-like in the theory of continuum [4]. So, it is worthy of discussing the dynamical properties of continuous map on Warsaw circle. Xiong. etal[5] proved the following theorem B:

Theorem B. Let $f : W \to W$ be a continuous map. Then $\overline{P(f)} = \overline{R(f)}$.

In [6], Gu discussed some equivalent conditions of the continuous map on Warsaw circle W being equicontinuous.

In this paper, we will prove the following main Theorem:

Main Theorem. Let $f : W \to W$ be a continuous map on Warsaw , then

1. f is topological transitive if and only if f is chaotic in the sense of Devaney;
2. f is topological transitive if and only if f is mixing;
3. f has horseshoe if f is topological transitive ;

2 Main results and proof

In this section, we denote $a = (0, -1) \in W_0$.

Lemma 2.1 Let $f : W \to W$ be topological transitivity, then $\overline{P(f)} = W$.
Proof: Let $T = \{x : \overline{O(x, f)} = W\}$. Firstly, we claim that $\overline{T} = W$.
Let $\{U_n\}_{n\geq 1}$ be a countable base of W. Then $T = \cap_{n\geq 1} \cup_{m\geq 0} f^{-m}(U_n)$. Since f is topological transitivity, it is easy to see that $\overline{\cup_{m\geq 0} f^{-m}(U_n)} = W$ for every $n > 0$. Hence $\overline{T} = W$. The Claim is proved.
Secondly, by Theorem B and $T \subset R(f)$, we have $\overline{P(f)} = \overline{R(f)} = W$.

According to Lemma 2.1, It is easy to get the following theorem:
Theorem 2.2 Let $f : W \to W$ be a continuous map on Warsaw circle W, then the following conclusions are equivalent:
(1) f is topological transitivity;
(2) f is chaotic in the sense of Devaney.

Lemma 2.3 Let $f : W \to W$ be a continuous maps on Warsaw circle W. then $f([z, b]) \neq W$ for each $b \in W$.
Proof Suppose that there exists some point b such that $f([z, b]) = W$. Let $\{y_n\} \subset W_4$ such that $\lim\limits_{n\to\infty} y_n = z$. Choose $x_n \in [z, b]$ such that $f(x_n) = y_n$. Without loss of generality, we assume that $\lim\limits_{n\to\infty} x_n = c$ for some point c. Therefore, $f(c) = z$. If $c <_w x_n$ then $f([c, x_n]) \supseteq [z, y_n]$. Else, If $c >_w x_n$ then $f([x_n, c]) \supseteq [z, y_n]$. It follows that there exist points s_n with $\lim\limits_{n\to\infty} s_n = c$ such that $\lim\limits_{n\to\infty} f(s_n) = d$ for some $b \neq z$, which contradicts with the continuity of f.

Lemma 2.4 Let $f : W \to W$ be transitive on Warsaw circle W, then $F(f) \cap \{x : x >_w b\} \neq \emptyset$ for each $b >_w a$.
Proof Suppose that there exists some $b >_w a$ such that $F(f) \cap \{x : x >_w b\} = \emptyset$.
Case 1 $x <_w f(x)$ for each $x >_w b$. Let $U_1 = \{x : x >_w b\}$ and $U_2 = (a, b)$. It is obvious that U_1, U_2 are two nonempty open set in W and $f^n(U_1) \cap U_2 = \emptyset$ for each $n \in N$, which contradicts with the transitivity of f.
Case 2 $x >_w f(x)$ for each $x >_w b$. There exists some point $c >_w b$ such that $f([z, b]) \subset [z, c)$ since $[z, b]$ is compact. $U_1 = \{x : x >_w c\}$ and $U_2 = (a, c)$. It is obvious that U_1, U_2 are two nonempty open set in W and $f^n(U_2) \cap U_1 = \emptyset$,which contradicts with the transitivity of f.

Theorem 2.5 Let $f : W \to W$ be transitive on Warsaw circle W and $x \in W$ be such that $\omega(x, f) = W$. Then $\omega(x, f^s) = W$ for every positive integer s.
Proof Fix an arbitrary integer s and set $B_i = \omega(f^i(x), f^s)$ for $0 \leq i < s$. Since $\bigcup\limits_{i=0}^{s-1} B_i = W$, at least one B_i has non-empty interior. More over,since the orbit of x cannot contain a periodic, f cannot collapse an interval to a point.

Since $f(B_i) = B_{i+1}$ for $0 \le i < s-1$ and $f(B_{s-1}) = B_0$, it follows that each B_i has non-empty interior.

We claim that $B_i = B_j$ if the interiors of B_i and B_j intersect. To see this, suppose that $int(B_i) \cap int(B_j) \ne \emptyset$. Then for some positive integer n, $f^{sn+i}(x) \in int(B_i) \cap int(B_j)$. It follows that $B_i \subset B_j$, since B_j is f^s-invariant and $B_i = \omega(f^{sn+i}(x), f^s)$. Since i and j can be changed in this argument we must actually have $B_i = B_j$.

Let $\mathcal{A}$ denote the collection of all subset of W which are components of $intB_i$ for some $i \in \{0, 1, \cdots, s-1\}$. Thus $\mathcal{A}$ is a collection of disjoint open intervals whose union is dense in W. If $A_1 \in \mathcal{A}$ then $f(\overline{A_1}) \subset \overline{A_2}$ for some $\overline{A_2} \in \mathcal{A}$, again because f cannot collapse an interval to a point. Moreover, since the orbit of x is dense in W,for every $A_2 \in \mathcal{A}$ there is a positive integer k such that $f^k(\overline{A_1}) \subset \overline{A_2}$. It follows that $\mathcal{A}$ is finite, say $\mathcal{A} = \{A_1, A_2, \cdots, A_h\}$, and that the sets $C_i = \overline{A_i}(i = 1, 2, \cdots, h)$ are cyclically permuted by f.

If $h = 1$ then $B_i = W$ for $i = 0, 1, \cdots, s-1$ and , in particular, $\omega(x, f^s) = W$. If $h > 1$, then there exists $i < h$ such that $\{x : x >_w b\} \subset A_i$ for some b. Therefore, by Lemma 2.3, there exists some fixed point y such that $y \in A_i$, then $f(C_i) = C_i$. Which contradicts the fact that $C_i = \overline{A_i}(i = 1, 2, \cdots, h)$ are cyclically permuted by f.

Corollary 2.6 Let $f : W \to W$ be continuous map such that f is transitive. Then f^s is transitive for every positive integer s and f is totally transitive.

Lemma 2.7 Let $f : W \to W$ ba continuous map, then f is mixing if and only if for all points $c >_w b >_w a$ and all non degenerate intervals $J \subset W - W_0$ there exists an integer N such that $\forall\ n \ge N$, $f^n(J) \supset [b, c]$.

Proof Suppose first that f is mixing and put $U_1 = (a, b)$ and $U_2 = \{x : x >_w c\}$. If $J \subset W - W_0$ is a non degenerate open interval in W, there exists N_1 such that $\forall\ n \ge N_1$, $f^n(J) \cap U_1 \ne \emptyset$ because f is mixing. In the same way, there exists N_2 such that $\forall\ n \ge N_2$, $f^n(J) \cap U_2 \ne \emptyset$. Therefore, for all $n \ge \max\{N_1, N_2\}$, $f^n(J)$ meets both U_1 and U_2, which implies that $f^n(J) \supset [b, c]$ by connectedness. If $J \subset W - W_0$ is a non interval in W, one considers $\text{Int}(J)$ which is not empty.

Suppose that for all points $c > b > a$ and all non degenerate intervals $J \subset W - W_0$ there exists an integer N such that $\forall\ n \ge N$, $f^n(J) \supset [b, c]$. Let U, V be two nonempty open sets in W. Let J, K be two non degenerate intervals such that $J \subset U \cap (W - W_0)$ and $K \subset V \cap (W - W_0)$ and that there exist points $c >_w b >_w a$ such that $K \subset [b, c]$. By assumption, there exists N such that $\forall$ $n \ge N$, $f^n(J) \supset [b, c] \supset K$, which implies $f^n(U) \cap V \ne \emptyset$.

Theorem 2.8 Let $f : W \to W$ be a continuous map on W. Then f is transitive if and only if it is mixing.

Proof The sufficiency is obviously.

Now we suppose that f is transitive. Let $J \subset W - W_0$ be a non degenerate interval and b, c be points with $b \in (a, c)$. According to Theorem 2.1, there exists a periodic point $x \in J$. By Theorem 2.1 again, there exist periodic points $x_1 \in (a, b)$ and $x_2 \in \{x : x >_w c\}$. Let k be the common multiple of the periods of x, x_1, x_2. Put $g = f^k$ and $K_i = \bigcup_{n=0}^{\infty} g^n(f^i(J))$ for $i = 0, 1, \cdots, k-1$. For each $i \leq k-1$, the interval K_i is arc connected since for all n, $g^n(f^i(J))$ contains $f^i(x)$. More over, g is transitive by Corollary 2.6; This implies that K_0 is dense in W. It follows that K_i is dense in W for each $i \leq k-1$. Therefore, there exist positive integers N_i such that $g^{N_i}(f^i(J)) \supset \{x_1, x_2\}$ for each $i \leq k-1$. Let $N = \max\{N_0, N_1, \cdots, N_{k-1}\}$. Then $g^n(f^i(J)) \supset \{x_1, x_2\}$ for each $i \leq k-1$ and $n \geq N$ since x_1, x_2 are fixed points of g. Which means that $f^n(J) \supset \{x_1, x_2\}$ for every $n \geq Nk$. It follows that $f^n(J) \supset [b, c]$ for every $n \geq Nk$. Then, by Lemma 2.7, f is mixing.

Theorem 2.9 Let $f : W \to W$ be a continuous map on W. Then f has a horseshoe if it is transitive.

Proof According to Lemma 2.3, $F(f) \cap \{x : x >_w b\} \neq \emptyset$ for each $b >_w a$. Let x_1, x_2 are two fixed points of f in W_4. Without loss of generality, we assume that $x_1 <_w x_2$ and $(x_1, x_2) \cap Fix(f) = \emptyset$ since $Fix(f)$ is a closed set of empty interior by transitivity. Therefore, $f(x) >_w x$ for every $x_1 <_w x <_w x_2$ or $f(x) <_w x$ for every $x_1 <_w x <_w x_2$.

Case 1 $f(x) >_w x$ for every $x_1 <_w x <_w x_2$;

Let $U = \{x : x >_w x_1\}$. If $f(x) >_w x_1$ for all $x \in U$, then U is invariant, which is impossible by transitivity. We deduce that there exists $x >_w x_1$ such that $f(x) \leq_w x_1$. Since $f(x_2) = x_2 >_w x_1$, there exists $b >_w x_1$ such that $f(b) = x_1$ by connectedness; we choose b such that $f^{-1}(x_1) \cap (x_1, b) = \emptyset$.

If for all x in (x_1, b) one has $f(x) \neq b$, then $x_1 <_w f(x) <_w b$ for all $x \in (x_1, b)$. Thus, the interval (x_1, b) is invariant, which is impossible by transitivity. we deduce that there exists $y \in (x_1, b)$ such that $f(y) = b$. Let $J_1 = [x_1, y]$ and $J_2 = [y, b]$, then $f(J_1) \cap f(J_2) = J_1 \cup J_2$ and f has a horseshoe.

Case 2 $f(x) <_w x$ for every $x_1 <_w x <_w x_2$;

Let $U = \{x : x <_w x_2\}$. If $f(x) <_w x_2$ for all $x \in U$, then U is invariant, which is impossible by transitivity. We deduce that there exists $x <_w x_2$ such that $f(x) \geq_w x_2$. Since $f(x_1) = x_1 <_w x_2$, there exists $b <_w x_2$ such that $f(b) = x_2$ by connectedness; we choose b such that $f^{-1}(x_2) \cap (b, x_2) = \emptyset$.

If for all x in (b, x_2) one has $f(x) \neq b$, then $b <_w f(x) <_w x_2$ for all $x \in (b, x_2)$. Thus, the interval (b, x_2) is invariant, which is impossible by transitivity. we deduce that there exists $y \in (b, x_2)$ such that $f(y) = b$. Let $J_1 = [y, x_2]$ and $J_2 = [b, y]$, then $f(J_1) \cap f(J_2) = J_1 \cup J_2$ and f has a horseshoe.

Now by Theorems 2.2, 2,8 and 2.9, we can deduce the following Main Theorem:

Main Theorem Let $f : W \to W$ be a continuous map on Warsaw, then

1. f is topological transitive if and only if f is chaotic in the sense of Devaney;
2. f is topological transitive if and only if f is mixing;
3. f has horseshoe if f is topological transitive;

3 Conclusion

Let W be a Warsaw circle and $f : W \to W$ be a continuous map. In this paper some dynamical properties of f are studied. Firstly, it is shown that f is topological transitive if and only if f is chaotic in the sense of Devaney. Secondly, we show that f is topological transitive if and only if f is mixing and that f is topological transitive implies f has horseshoe. As the Corollary of Theorem 2.8, we have f is transitive if and only if $\overline{f}$ is transitive by Theorem 3.2 of [7](Where $\overline{f}$ defined as the induced map on the hyperspace, see[6] and [7]); As the Corollary of Theorem 2.9, we have that f is transitive implies the topological entropy $h(f) \geq \log 2$. In more general context and directly with the above ideas, an interesting question is the following: ***Is there any continuous map*** $f : W \to W$ ***with*** f ***is transitive and*** $h(f) = \log 2$***?***

References

[1] L. S. Block and W. Coppel. *Dynamics in One Dimension.* Lecture Notes in Mathematics. no 1513. Springer-Verlag, 1992.

[2] J. Nanks et,al *On the Deavney's defintions os chaos.* Amer. Math. Monthly,1992, 99(4). 332-334;

[3] M. Vellekoop, R. Berglund. *On intervals, transitive=chaos* Amer. Math. Monthly,1994, 101(4). 353-355;

[4] S. B. Nadler, *Continuum Theory,* New York:Marcel Dekker, Inc, 1992.

[5] J. Xiong, X. Ye, Z. Zhang and J. Huang, *Some dynamical properties for continuous map on Warsaw circle.* Acta Math.Sinica,1996,3:294-299

[6] R. Gu , *Equiconitinuity of maps on Warsaw Circle.* Journal of Mathematical Study, 2002, 35 (3):249-256

[7] J. Mai, *Pointwise-recurrent graph maps.* Ergod. Th. Dynam. 2005,25(2), 629-637.

[8] G. Zhang F. Zeng and X. Liu, *Devaney's Chaotic on Induced Maps of Hyperspace.* Chaos,Solitios and Fractals 2006(27)471-475.

[9] R. Gu and W. Guo, *On Mixing property in set-valued discrete system.*Chaos,Solitons and Fractals 2006, (28)747-754.

[10] T. Sun, M. Xie and J. Zhao, *Equivalent condition of a tree map with zero topological entropy.* Bull. Austral. Math. Soc.2006(73) 321-327.

[11] J. Mai, T. Sun, *The ω-limitset of a graph map.* Topology andits Applications. 2007(154) 2306C2311

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On the Generalized Test Likelihood Ratio for Multivariate Normal Distribution Applied to Classification

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Abstract

In this paper we shall generalize the likelihood ratio criterion [1] in order to obtain another criterion which can be used in pattern recognition.

AMS Subject Classification: 62-xx, 62Hxx, 62H30

Keywords: likelihood ratio criterion, null hypothesis, alternative hypothesis, maximum likelihood estimates, decision rule

1 Introduction

Assume that we have two independent multivariate samplings $X_1^{(1)}, \ldots, X_{N_1}^{(1)}$ and $X_1^{(2)}, \ldots, X_{N_2}^{(2)}$ which are assumed to be normally distributed $\mathcal{N}(\mu^{(1)}, \Sigma^{(1)})$, $\mathcal{N}(\mu^{(2)}, \Sigma^{(2)})$ respectively, with $\mu^{(1)}, \mu^{(2)}, \Sigma^{(1)}, \Sigma^{(2)}$ unknown parameters, $\mu^{(1)}$, $\mu^{(2)}$ being $d \times 1$ vectors and $\Sigma^{(1)}, \Sigma^{(2)}$ being $d \times d$ positive definite matrices.

The $N = N_1 + N_2$ multivariate sampling values could be random characteristics of N objects which must be classified into two classes.

By some means, the objects are separated into two classes corresponding to the two normal populations $\mathcal{N}(\mu^{(1)}, \Sigma^{(1)})$, $\mathcal{N}(\mu^{(2)}, \Sigma^{(2)})$ as above.

The maximum likelihood estimates of parameters under respective hypotheses are:

$$\mu^{(1)} \cong \overline{X}^{(1)} = \frac{1}{N_1}\sum_{i=1}^{N_1} X_i{}^{(1)} \tag{1}$$

$$\mu^{(2)} \cong \overline{X}^{(2)} = \frac{1}{N_2}\sum_{i=1}^{N_2} X_i{}^{(2)} \tag{2}$$

$$\Sigma^{(1)} \cong \hat{\Sigma}^{(1)} = \frac{1}{N_1 - 1}\left[\sum_{i=1}^{N_1}(X_i{}^{(1)} - \overline{X}^{(1)})(X_i{}^{(1)} - \overline{X}^{(1)})^t\right] \tag{3}$$

$$\Sigma^{(2)} \cong \hat{\Sigma}^{(2)} = \frac{1}{N_2 - 1}\left[\sum_{i=1}^{N_2}(X_i{}^{(2)} - \overline{X}^{(2)})(X_i{}^{(2)} - \overline{X}^{(2)})^t\right] \tag{4}$$

Assume that we have a new object with the random characteristic X (a $d \times 1$ vector) which must be included into one of the two classes.

Therefore, the problem consist in testing the composite null hypothesis

$$\mathrm{H} : X, X_1^{(1)}, \ldots, X_{N_1}^{(1)} \ \textit{are drawn from}\ \mathcal{N}(\mu^{(1)}, \Sigma^{(1)}) \ \textit{and}$$

$$X_1^{(2)}, \ldots, X_{N_2}^{(2)} \ \textit{are drawn from}\ \mathcal{N}(\mu^{(2)}, \Sigma^{(2)}),$$

against the composite alternative hypothesis

$$\mathcal{NH} : X_1^{(1)}, \ldots, X_{N_1}^{(1)} \ \textit{are drawn from}\ \mathcal{N}(\mu^{(1)}, \Sigma^{(1)}) \ \textit{and}$$

$$X, X_1^{(2)}, \ldots, X_{N_2}^{(2)} \ \textit{are drawn from}\ \mathcal{N}(\mu^{(2)}, \Sigma^{(2)}),$$

with $\mu^{(1)},\ \mu^{(2)}$ and $\Sigma^{(1)},\ \Sigma^{(2)}$ unspecified.

Under the null hypothesis, the maximum likelihood estimates of $\mu^{(1)},\ \mu^{(2)}$ and $\Sigma^{(1)},\ \Sigma^{(2)}$ are respectively

$$\hat{\mu}_1^{(1)} = \frac{N_1\overline{X}^{(1)} + X}{N_1 + 1}, \tag{5}$$

$$\hat{\mu}_1^{(2)} = \overline{X}^{(2)}, \tag{6}$$

$$\hat{\Sigma}_1^{(1)} = \frac{1}{N_1}\left[\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \hat{\mu}_1^{(1)}\right)\left(X_\alpha^{(1)} - \hat{\mu}_1^{(1)}\right)^t + \left(X - \hat{\mu}_1^{(1)}\right)\left(X - \hat{\mu}_1^{(1)}\right)^t\right] \tag{7}$$

$$\hat{\Sigma}_1^{(2)} = \frac{1}{N_2 - 1}\left[\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)} - \hat{\mu}_1^{(2)}\right)\left(X_\alpha^{(2)} - \hat{\mu}_1^{(2)}\right)^t\right] \tag{8}$$

We can write

$$\hat{\Sigma}_1^{(1)} = \frac{1}{N_1}\left[\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \overline{X}^{(1)}\right)\left(X_\alpha^{(1)} - \overline{X}^{(1)}\right)^t + \frac{N_1^2}{(N_1+1)^2}\left(X - \overline{X}^{(1)}\right)\left(X - \overline{X}^{(1)}\right)^t\right] \tag{9}$$

or, using (3) it results

$$\hat{\Sigma}_1^{(1)} = \frac{1}{N_1}\left[(N_1 - 1)\cdot\Sigma^{(1)} + \frac{N_1^2}{(N_1+1)^2}\left(X - \overline{X}^{(1)}\right)\left(X - \overline{X}^{(1)}\right)^t\right] \tag{10}$$

Using (4) and (6) we shall have

$$\hat{\Sigma}_1^{(2)} = \frac{1}{N_2 - 1}\cdot(N_2 - 1)\cdot\Sigma^{(2)} = \Sigma^{(2)}. \tag{11}$$

Under the assumptions of the alternative hypothesis, the maximum likelihood estimates of the parameters $\mu^{(1)},\ \mu^{(2)}$ and $\Sigma^{(1)},\ \Sigma^{(2)}$ are respectively

$$\hat{\mu}_2^{(1)} = \overline{X}^{(1)}, \tag{12}$$

$$\hat{\mu}_2^{(2)} = \frac{N_2\overline{X}^{(2)} + X}{N_2 + 1}, \tag{13}$$

$$\hat{\Sigma}_2^{(1)} = \frac{1}{N_1 - 1}\left[\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \hat{\mu}_2^{(1)}\right)\left(X_\alpha^{(1)} - \hat{\mu}_2^{(1)}\right)^t\right], \tag{14}$$

namely

$$\hat{\Sigma}_2^{(1)} = \frac{1}{N_1 - 1}\cdot(N_1 - 1)\cdot\Sigma^{(1)} = \Sigma^{(1)} \tag{15}$$

and

$$\hat{\Sigma}_2^{(2)} = \frac{1}{N_2}\left[\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)} - \hat{\mu}_2^{(2)}\right)\left(X_\alpha^{(2)} - \hat{\mu}_2^{(2)}\right)^t + \left(X - \hat{\mu}_2^{(2)}\right)\left(X - \hat{\mu}_2^{(2)}\right)^t\right], \tag{16}$$

namely

$$\hat{\Sigma}_2^{(2)} = \frac{1}{N_2}\left[(N_2 - 1)\cdot\Sigma^{(2)} + \frac{N_2^2}{(N_2+1)^2}\left(X - \overline{X}^{(2)}\right)\left(X - \overline{X}^{(2)}\right)^t\right]. \tag{17}$$

2 The generalized likelihood ratio criterion

Theorem 1 (I.Iatan) *The decision rule used for including a new object with the random characteristic X (a $d \times 1$ vector) into one of the two classes is*

$$\begin{cases} if \ \ln \frac{|\hat{\Sigma}_2^{(1)}|^{N_1/2}}{|\hat{\Sigma}_1^{(1)}|^{(N_1+1)/2}} + \ln \frac{|\hat{\Sigma}_2^{(2)}|^{(N_2+1)/2}}{|\hat{\Sigma}_1^{(2)}|^{N_2/2}} > K \ then \ X \in \omega_1, \\ \\ else \ if \ \ln \frac{|\hat{\Sigma}_2^{(1)}|^{N_1/2}}{|\hat{\Sigma}_1^{(1)}|^{(N_1+1)/2}} + \ln \frac{|\hat{\Sigma}_2^{(2)}|^{(N_2+1)/2}}{|\hat{\Sigma}_1^{(2)}|^{N_2/2}} < K \ then \ X \in \omega_2 \end{cases} \tag{18}$$

or

$$\begin{cases} if \ \frac{|\hat{\Sigma}_2^{(1)}|^{N_1/2}}{|\hat{\Sigma}_1^{(1)}|^{(N_1+1)/2}} \cdot \frac{|\hat{\Sigma}_2^{(2)}|^{(N_2+1)/2}}{|\hat{\Sigma}_1^{(2)}|^{N_2/2}} > \mathrm{e}^K \ then \ X \in \omega_1, \\ \\ else \ if \ \frac{|\hat{\Sigma}_2^{(1)}|^{N_1/2}}{|\hat{\Sigma}_1^{(1)}|^{(N_1+1)/2}} \cdot \frac{|\hat{\Sigma}_2^{(2)}|^{(N_2+1)/2}}{|\hat{\Sigma}_1^{(2)}|^{N_2/2}} < \mathrm{e}^K \ then \ X \in \omega_2, \end{cases}$$

where

$$K = -\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \hat{\mu}_2^{(1)}\right)(\hat{\Sigma}_2^{(1)})^{-1}\left(X_\alpha^{(1)} - \hat{\mu}_2^{(1)}\right)^t - \frac{1}{2}\left(X - \hat{\mu}_2^{(2)}\right)(\hat{\Sigma}_2^{(2)})^{-1}\left(X - \hat{\mu}_2^{(2)}\right)^t -$$

$$-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)} - \hat{\mu}_2^{(2)}\right)(\hat{\Sigma}_2^{(2)})^{-1}\left(X_\alpha^{(2)} - \hat{\mu}_2^{(2)}\right)^t + \frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \hat{\mu}_1^{(1)}\right)(\hat{\Sigma}_1^{(1)})^{-1}\left(X_\alpha^{(1)} - \hat{\mu}_1^{(1)}\right)^t +$$

$$+\frac{1}{2}\left(X - \hat{\mu}_1^{(1)}\right)(\hat{\Sigma}_1^{(1)})^{-1}\left(X - \hat{\mu}_1^{(1)}\right)^t + \frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)} - \hat{\mu}_1^{(2)}\right)(\hat{\Sigma}_1^{(2)})^{-1}\left(X_\alpha^{(2)} - \hat{\mu}_1^{(2)}\right)^t,$$

$$\frac{|\hat{\Sigma}_2^{(1)}|}{|\hat{\Sigma}_1^{(1)}|} = \frac{1}{\frac{N_1-1}{N_1}\left[1 + \frac{N_1^2}{(N_1+1)^2} \cdot (X - \overline{X}^{(1)})^t (N_1-1)^{-1} {\Sigma^{(1)}}^{-1} (X - \overline{X}^{(1)})\right]}$$

and

$$\frac{|\hat{\Sigma}_2^{(2)}|}{|\hat{\Sigma}_1^{(2)}|} = \frac{N_2 - 1}{N_2}\left[1 + \frac{N_2^2}{(N_2+1)^2}(X - \overline{X}^{(2)})^t (N_2-1)^{-1} {\Sigma^{(2)}}^{-1} (X - \overline{X}^{(2)})\right].$$

Proof:

Under the first hypothesis, the maximum value of the likelihood function is

$$\mathrm{L}_1(X, X_1^{(1)}, \ldots, X_{N_1}^{(1)}, X_1^{(2)}, \ldots, X_{N_2}^{(2)};\ \hat{\mu}_1^{(1)}, \hat{\mu}_1^{(2)}, \hat{\Sigma}_1^{(1)}, \hat{\Sigma}_1^{(2)}) =$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}(N_1+N_2+1)} \cdot |\hat{\Sigma}_1^{(1)}|^{\frac{N_1+1}{2}} \cdot |\hat{\Sigma}_1^{(2)}|^{\frac{N_2}{2}}} \cdot$$

$$\cdot \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \hat{\mu}_1^{(1)}\right)(\hat{\Sigma}_1^{(1)})^{-1}\left(X_\alpha^{(1)} - \hat{\mu}_1^{(1)}\right)^t - \frac{1}{2}\left(X - \hat{\mu}_1^{(1)}\right)(\hat{\Sigma}_1^{(1)})^{-1}\left(X - \hat{\mu}_1^{(1)}\right)^t\right] \cdot$$

$$\cdot \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)} - \hat{\mu}_1^{(2)}\right)(\hat{\Sigma}_1^{(2)})^{-1}\left(X_\alpha^{(2)} - \hat{\mu}_1^{(2)}\right)^t\right],$$

while, under the second hypothesis, the maximum value of the likelihood function is

$$\mathrm{L}_2(X, X_1^{(1)}, \ldots, X_{N_1}^{(1)}, X_1^{(2)}, \ldots, X_{N_2}^{(2)};\ \hat{\mu}_2^{(1)}, \hat{\mu}_2^{(2)}, \hat{\Sigma}_2^{(1)}, \hat{\Sigma}_2^{(2)}) =$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}(N_1+N_2+1)} \cdot |\hat{\Sigma}_2^{(1)}|^{\frac{N_1}{2}} \cdot |\hat{\Sigma}_2^{(2)}|^{\frac{N_2+1}{2}}} \cdot$$

$$\cdot \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \hat{\mu}_2^{(1)}\right)^t(\hat{\Sigma}_2^{(1)})^{-1}\left(X_\alpha^{(1)} - \hat{\mu}_2^{(1)}\right)\right] \cdot$$

$$\cdot \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)} - \hat{\mu}_2^{(2)}\right)^t(\hat{\Sigma}_2^{(2)})^{-1}\left(X_\alpha^{(2)} - \hat{\mu}_2^{(2)}\right) - \frac{1}{2}\left(X - \hat{\mu}_2^{(2)}\right)^t(\hat{\Sigma}_2^{(2)})^{-1}\left(X - \hat{\mu}_2^{(2)}\right)\right],$$

over the d- dimensional space.

We shall obtain

$$\ln \mathrm{L}_1 = -\frac{d}{2}(N_1+N_2+1)\ln(2\pi) - \frac{N_1+1}{2}\ln|\hat{\Sigma}_1^{(1)}| - \frac{N_2}{2}\ln|\hat{\Sigma}_1^{(2)}| -$$

$$-\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \hat{\mu}_1^{(1)}\right)(\hat{\Sigma}_1^{(1)})^{-1}\left(X_\alpha^{(1)} - \hat{\mu}_1^{(1)}\right)^t - \frac{1}{2}\left(X - \hat{\mu}_1^{(1)}\right)(\hat{\Sigma}_1^{(1)})^{-1}\left(X - \hat{\mu}_1^{(1)}\right)^t -$$

$$-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)} - \hat{\mu}_1^{(2)}\right)(\hat{\Sigma}_1^{(2)})^{-1}\left(X_\alpha^{(2)} - \hat{\mu}_1^{(2)}\right)^t$$

and

$$\ln \mathrm{L}_2 = -\frac{d}{2}(N_1+N_2+1)\ln(2\pi) - \frac{N_1}{2}\ln|\hat{\Sigma}_2^{(1)}| - \frac{N_2+1}{2}\ln|\hat{\Sigma}_2^{(2)}| -$$

$$-\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)}-\hat{\mu}_2^{(1)}\right)(\hat{\Sigma}_2^{(1)})^{-1}\left(X_\alpha^{(1)}-\hat{\mu}_2^{(1)}\right)^t - \frac{1}{2}\left(X-\hat{\mu}_2^{(2)}\right)(\hat{\Sigma}_2^{(2)})^{-1}\left(X-\hat{\mu}_2^{(2)}\right)^t -$$

$$-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)}-\hat{\mu}_2^{(2)}\right)(\hat{\Sigma}_2^{(2)})^{-1}\left(X_\alpha^{(2)}-\hat{\mu}_2^{(2)}\right)^t$$

The decision rule is

$$\begin{cases} if\ \ \ln \mathrm{L}_1 - \ln \mathrm{L}_2 > 0\ then\ X \in \omega_1, \\ else\ \ if\ \ \ln \mathrm{L}_1 - \ln \mathrm{L}_2 < 0\ then\ X \in \omega_2. \end{cases} \tag{19}$$

deriving from:

$$\begin{cases} if\ \ \ln \frac{\mathrm{L}_1}{\mathrm{L}_2} > 0\ then\ X \in \omega_1, \\ else\ \ if\ \ \ln \frac{\mathrm{L}_1}{\mathrm{L}_2} < 0\ then\ X \in \omega_2. \end{cases} \tag{20}$$

Using the notation

$$K = -\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)}-\hat{\mu}_2^{(1)}\right)(\hat{\Sigma}_2^{(1)})^{-1}\left(X_\alpha^{(1)}-\hat{\mu}_2^{(1)}\right)^t - \frac{1}{2}\left(X-\hat{\mu}_2^{(2)}\right)(\hat{\Sigma}_2^{(2)})^{-1}\left(X-\hat{\mu}_2^{(2)}\right)^t -$$

$$-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)}-\hat{\mu}_2^{(2)}\right)(\hat{\Sigma}_2^{(2)})^{-1}\left(X_\alpha^{(2)}-\hat{\mu}_2^{(2)}\right)^t + \frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)}-\hat{\mu}_1^{(1)}\right)(\hat{\Sigma}_1^{(1)})^{-1}\left(X_\alpha^{(1)}-\hat{\mu}_1^{(1)}\right)^t +$$

$$+\frac{1}{2}\left(X-\hat{\mu}_1^{(1)}\right)(\hat{\Sigma}_1^{(1)})^{-1}\left(X-\hat{\mu}_1^{(1)}\right)^t + \frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)}-\hat{\mu}_1^{(2)}\right)(\hat{\Sigma}_1^{(2)})^{-1}\left(X_\alpha^{(2)}-\hat{\mu}_1^{(2)}\right)^t,$$

the decision rule from (19) becomes

$$\begin{cases} if\ \ -\frac{N_1+1}{2}\ln|\hat{\Sigma}_1^{(1)}| - \frac{N_2}{2}\ln|\hat{\Sigma}_1^{(2)}| + \frac{N_1}{2}\ln|\hat{\Sigma}_2^{(1)}| + \frac{N_2+1}{2}\ln|\hat{\Sigma}_2^{(2)}| > K\ then\ X \in \omega_1, \\ else\ if\ \ -\frac{N_1+1}{2}\ln|\hat{\Sigma}_1^{(1)}| - \frac{N_2}{2}\ln|\hat{\Sigma}_1^{(2)}| + \frac{N_1}{2}\ln|\hat{\Sigma}_2^{(1)}| + \frac{N_2+1}{2}\ln|\hat{\Sigma}_2^{(2)}| < K\ then\ X \in \omega_2. \end{cases} \tag{21}$$

The decision rule may be written also in the form

$$\begin{cases} if\ \ \ln \frac{|\hat{\Sigma}_2^{(1)}|^{N_1/2}}{|\hat{\Sigma}_1^{(1)}|^{(N_1+1)/2}} + \ln \frac{|\hat{\Sigma}_2^{(2)}|^{(N_2+1)/2}}{|\hat{\Sigma}_1^{(2)}|^{N_2/2}} > K\ then\ X \in \omega_1, \\ \\ else\ if\ \ \ln \frac{|\hat{\Sigma}_2^{(1)}|^{N_1/2}}{|\hat{\Sigma}_1^{(1)}|^{(N_1+1)/2}} + \ln \frac{|\hat{\Sigma}_2^{(2)}|^{(N_2+1)/2}}{|\hat{\Sigma}_1^{(2)}|^{N_2/2}} < K\ then\ X \in \omega_2 \end{cases} \tag{22}$$

or

$$\begin{cases} if\ \ \frac{|\hat{\Sigma}_2^{(1)}|^{N_1/2}}{|\hat{\Sigma}_1^{(1)}|^{(N_1+1)/2}} \cdot \frac{|\hat{\Sigma}_2^{(2)}|^{(N_2+1)/2}}{|\hat{\Sigma}_1^{(2)}|^{N_2/2}} > \mathrm{e}^K\ then\ X \in \omega_1, \\ \\ else\ if\ \ \frac{|\hat{\Sigma}_2^{(1)}|^{N_1/2}}{|\hat{\Sigma}_1^{(1)}|^{(N_1+1)/2}} \cdot \frac{|\hat{\Sigma}_2^{(2)}|^{(N_2+1)/2}}{|\hat{\Sigma}_1^{(2)}|^{N_2/2}} < \mathrm{e}^K\ then\ X \in \omega_2, \end{cases} \tag{23}$$

respectively,

$$\begin{cases} if\ \ \ \frac{|\hat{\Sigma}_2^{(2)}|}{|\hat{\Sigma}_1^{(1)}|} \cdot \left(\frac{|\hat{\Sigma}_2^{(1)}|}{|\hat{\Sigma}_1^{(1)}|}\right)^{N_1/2} \cdot \left(\frac{|\hat{\Sigma}_2^{(2)}|}{|\hat{\Sigma}_1^{(2)}|}\right)^{N_2/2} > \mathrm{e}^K\ then\ X \in \omega_1, \\ \\ else\ if\ \ \ \frac{|\hat{\Sigma}_2^{(2)}|}{|\hat{\Sigma}_1^{(1)}|} \cdot \left(\frac{|\hat{\Sigma}_2^{(1)}|}{|\hat{\Sigma}_1^{(1)}|}\right)^{N_1/2} \cdot \left(\frac{|\hat{\Sigma}_2^{(2)}|}{|\hat{\Sigma}_1^{(2)}|}\right)^{N_2/2} < \mathrm{e}^K\ then\ X \in \omega_2, \end{cases} \tag{24}$$

From (10) and (15) we obtain

$$\frac{|\hat{\Sigma}_2^{(1)}|}{|\hat{\Sigma}_1^{(1)}|} = \frac{|\Sigma^{(1)}|}{\left|\frac{1}{N_1}\left[(N_1-1)\cdot\Sigma^{(1)} + \frac{N_1^2}{(N_1+1)^2}(X-\overline{X}^{(1)})(X-\overline{X}^{(1)})^t\right]\right|}$$

or using [3] it results

$$= \frac{|\Sigma^{(1)}|}{\frac{N_1-1}{N_1}|\Sigma^{(1)}|\left[1+\frac{N_1}{N_1+1}\cdot(X-\overline{X}^{(1)})^t\left(\frac{N_1-1}{N_1}\right)^{-1}{\Sigma^{(1)}}^{-1}\frac{N_1}{N_1+1}\cdot(X-\overline{X}^{(1)})\right]},$$

namely

$$\frac{|\hat{\Sigma}_2^{(1)}|}{|\hat{\Sigma}_1^{(1)}|} = \frac{1}{\frac{N_1-1}{N_1}\left[1+\frac{N_1^2}{(N_1+1)^2}\cdot(X-\overline{X}^{(1)})^t(N_1-1)^{-1}{\Sigma^{(1)}}^{-1}(X-\overline{X}^{(1)})\right]}. \tag{25}$$

Analogous, using (11) and (17) we shall obtain

$$\frac{|\hat{\Sigma}_2^{(2)}|}{|\hat{\Sigma}_1^{(2)}|} = \frac{\left|\frac{1}{N_2}\left[(N_2-1)\cdot\Sigma^{(2)} + \frac{N_2^2}{(N_2+1)^2}(X-\overline{X}^{(2)})(X-\overline{X}^{(2)})^t\right]\right|}{|\Sigma^{(2)}|} =$$

$$= \frac{N_2-1}{N_2}\left[1+\frac{N_2^2}{(N_2+1)^2}(X-\overline{X}^{(2)})^t(N_2-1)^{-1}\Sigma^{(2)^{-1}}(X-\overline{X}^{(2)})\right]. \quad (26)$$

3 Conclusions

Our goal is to generalize the likelihood ratio criterion [1] in order to obtain another criterion which can be used in pattern recognition.

The region of classification into the first class consists of those patterns for which the ratio from (23) is grater than e^K.

References

[1] Anderson, T.,W., 1962. *An Introduction to Multivariate Statistical Analysis*. John Wiley and Sons Inc., New York, London.

[2] Duda, D.,O., Hart P., E., Stork, D., G., 2001. *Pattern Classification*. John Wiley, second edition.

[3] Mihoc, Gh., Craiu, V., 1977. *Treatise of Mathematical Statistics. Testing Statistical Hypotheses*. Vol. II, Ed. Academiei, Bucharest.

[4] Stockburger, D., W, 1996. *Multivariate Statistics: Concepts, Models, and Applications*. Southwest Missouri State University.

[5] Văduva, I., 1970. *Analysis of Variance*. Technical Publishing House, Bucharest (Romanian).

[6] Webb, A., 2002. *Statistical Pattern Recognition*. John Wiley and Sons, N.York, second edition.

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A Common Fixed Point Theorem Through Commutativity

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Abstract

In this paper, we generalized Jungck's common fixed point theorem to intuitionistic fuzzy metric spaces. Jungck [4] proved his theorem more general than contraction principle and Grabiec [3] proved the contraction principle in the setting of fuzzy metric spaces.

Keywords: Complete intuitionistic fuzzy metric space; Contraction principle; Commutativity; Fixed point.

AMS Subject Classifications: 47H10, 54H25

1 Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [10]. sing the idea of fuzzy sets, Kramosil and Michalek [6] defined the notion of intuitionistic fuzzy metric spaces, and George and Veeramani [2] modified the concept of fuzzy metric spaces in order to have the Hausdorff topology on fuzzy metric spaces. They showed also that every metric induces a fuzzy metric.

Atanassov [1] introduced and stuided the the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Using the idea of intuitionistic fuzzy sets, Park [8] defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space due to George and Veeramani [2], introduce the notion of Cauchy sequences in an intuitionistic fuzzy metric space and find a necessary and suffient condition for an intuitionistic fuzzy metric spaces to be complete.

The purpose of this paper is to generalize Jungck's common fixed point theorem to intuitionistic fuzzy metric spaces. Jungck [4] proved his theorem more general than contraction principle, Grabiec [3] proved the contraction principle in the setting of fuzzy metric spaces and Kutukcu et al. [7] proved the contraction principle in the setting of intuitionistic fuzzy metric spaces.

2 Intuitionistic fuzzy metric spaces

Definition 2.1. *A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:*

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2.2. *A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\Diamond$ is satisfying the following conditions:*
(a) $\Diamond$ is commutative and associative;
(b) $\Diamond$ is continuous;
(c) $a \Diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Several examples and detals for the concepts of triangular norms (t-norms) and triangular conorms (t-conorms) were proposed by many authors (see [7,8]).

Definition 2.3 ([8]). *A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,*
(IFM-1) $M(x,y,t) + N(x,y,t) \leq 1$;
(IFM-2) $M(x,y,t) > 0$;
(IFM-3) $M(x,y,t) = 1$ if and only if $x = y$;
(IFM-4) $M(x,y,t) = M(y,x,t)$;
(IFM-5) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(IFM-6) $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous;
(IFM-7) $N(x,y,t) > 0$;
(IFM-8) $N(x,y,t) = 0$ if and only if $x = y$;
(IFM-9) $N(x,y,t) = N(y,x,t)$;
(IFM-10) $N(x,y,t) \Diamond N(y,z,s) \geq N(x,z,t+s)$;
(IFM-11) $N(x,y,.) : (0,\infty) \to (0,1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.1. *Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated, i.e. $x \Diamond y = 1 - ((1-x) * (1-y))$ for any $x, y \in [0,1]$.*

Remark 2.2 ([8]). *In intuitionistic fuzzy metric space X, $M(x,y,.)$ is non-decreasing and $N(x,y,.)$ is non-increasing for all $x, y \in X$.*

Definition 2.4 ([8]). *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then*
(a) a sequence $\{x_n\}$ in X is convergent to x in X if for each $t > 0$, $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} N(x_n, x, t) = 0$,
(b) a sequence $\{x_n\}$ in X is called Cauchy if for each $t > 0$ and $p > 0$, $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ and $\lim_{n\to\infty} N(x_n, x_{n+p}, t) = 0$.
(c) An intuitionistic fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Throughout this paper, $(X, M, N, *, \Diamond)$ will denote the intuitionistic fuzzy metric space with the following conditions: for all $x, y \in X$ and $t > 0$,
(IFM-7$'$) $N(x,y,t) < 1$;
(IFM-11$'$) $N(x,y,.) : (0,\infty) \to [0,1)$ is continuous;
(IFM-12) $\lim_{t\to\infty} M(x,y,t) = 1$;
(IFM-13) $\lim_{t\to\infty} N(x,y,t) = 0$.

3 Main Results

THEOREM 3.1. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and let $f, g : X \to X$ be maps that satisfy the following conditions:*

(a) $g(X) \subseteq f(X)$;

(b) f is continuous;

(c) $M(g(x), g(y), \alpha t) \geq M(f(x), f(y), t)$ and $N(g(x), g(y), \alpha t) \leq N(f(x), f(y), t)$ for $0 < \alpha < 1$ and all x, y in X.

Then f and g have a unique common fixed point provided f and g commute.

Proof. Let $x_0 \in X$. By (a), we can find x_1 such that $f(x_1) = g(x_0)$. By induction, we can define a sequence x_n in X such that $f(x_n) = g(x_{n-1})$. By induction again, we have

$$\begin{aligned} M(f(x_n), f(x_{n+1}), t) &= M(g(x_{n-1}), g(x_n), t) \\ &\geq M(f(x_{n-1}), f(x_n), t/\alpha) \\ &\geq ... \geq M(f(x_0), f(x_1), t/\alpha^n) \end{aligned}$$

and

$$\begin{aligned} N(f(x_n), f(x_{n+1}), t) &= N(g(x_{n-1}), g(x_n), t) \\ &\leq N(f(x_{n-1}), f(x_n), t/\alpha) \\ &\leq ... \leq N(f(x_0), f(x_1), t/\alpha^n). \end{aligned}$$

So, for any positive integer p,

$$\begin{aligned} M(f(x_n), f(x_{n+p}), t) &\geq M(f(x_n), f(x_{n+1}), t/p) \\ &\quad *\overset{(p)}{\cdots} * M(f(x_{n+p-1}), f(x_{n+p}), t/p) \\ &\geq M(f(x_0), f(x_1), t/p\alpha^n) \\ &\quad *\overset{(p)}{\cdots} * M(f(x_0), f(x_1), t/p\alpha^{n+p-1}) \end{aligned}$$

and

$$\begin{aligned} N(f(x_n), f(x_{n+p}), t) &\leq N(f(x_n), f(x_{n+1}), t/p) \\ &\quad \diamondsuit\overset{(p)}{\cdots}\diamondsuit N(f(x_{n+p-1}), f(x_{n+p}), t/p) \\ &\leq N(f(x_0), f(x_1), t/p\alpha^n) \\ &\quad \diamondsuit\overset{(p)}{\cdots}\diamondsuit N(f(x_0), f(x_1), t/p\alpha^{n+p-1}). \end{aligned}$$

By (IFM-12) and (IFM-13), we get

$$lim_{n\to\infty} M(f(x_0), f(x_1), t/p\alpha^n) = 1 \text{ and } lim_{n\to\infty} N(f(x_0), f(x_1), t/p\alpha^n) = 0.$$

Thus, $lim_{n\to\infty} M(f(x_n), f(x_{n+p}), t) \geq 1 * ... * 1 \geq 1$ and $lim_{n\to\infty} N(f(x_n), f(x_{n+p}), t) \leq 0\diamondsuit...\diamondsuit 0 \leq 0$. Therefore $\{f(x_n)\}$ is a Cauchy sequence. By the completeness of X, $\{f(x_n)\}$ converges to y. So $g(x_{n-1})[= f(x_n)]$ tends to y. It can be seen from (c) that the continuity of f implies that of g. So $g(f(x_n)) \to g(y)$. However, by the commutativity of f and g, $g(f(x_n)) = f(g(x_n))$. So $f(g(x_n))$ converges to $f(y)$. Since the limits are unique, $f(y) = g(y)$. By the commutativity $f(f(y)) = f(g(y))$ and

$$\begin{aligned} M(g(y), g(g(y)), t) &\geq M(f(y), f(g(y)), t/\alpha) \geq M(g(y), g(g(y)), t/\alpha) \\ &\geq ... \geq M(g(y), g(g(y)), t/\alpha^n) \end{aligned}$$

and

$$\begin{aligned} N(g(y),g(g(y)),t) &\leq N(f(y),f(g(y)),t/\alpha) \leq N(g(y),g(g(y)),t/\alpha) \\ &\leq ... \leq N(g(y),g(g(y)),t/\alpha^n). \end{aligned}$$

By (IFM-3), (IFM-8), (IFM-12) and (IFM-13), we get $g(y) = g(g(y))$, therefore $g(y) = g(g(y)) = f(g(y))$. So $g(y)$ is a common fixed point of f and g.

If y and z are two common fixed points of f and g, then

$$\begin{aligned} 1 &\geq M(y,z,t) = M(g(y),g(z),t) \geq M(f(y),f(z),t/\alpha) = M(y,z,t/\alpha) \\ &\geq ... \geq M(y,z,t/\alpha^n) \to 1 \end{aligned}$$

and

$$\begin{aligned} 0 &\leq N(y,z,t) = N(g(y),g(z),t) \leq N(f(y),f(z),t/\alpha) = N(y,z,t/\alpha) \\ &\leq ... \leq N(y,z,t/\alpha^n) \to 0. \end{aligned}$$

So, $y = z$.

Remark 3.1. *Jungck [4,5] has shown by means of an example that his theorem is more general than the contraction principle and Subrahmanyam [9] has shown his theorem is more general than Grabiec's result in [3], so our intuitionistic fuzzy version is an extension of their results.*

Following, we give an example that illustrates Theorem 3.1;

Example. Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. For each $x, y \in X$ and $t \in (0,\infty)$, define

$$M(x,y,t) = \frac{t}{t + |x-y|} \text{ and } N(x,y,t) = \frac{|x-y|}{t+|x-y|}$$

Clearly $(X, M, N, *, \Diamond)$ is a complete intuitionistic fuzzy metric on X, where $a * b = ab$ and $a \Diamond b = \min\{1, a+b\}$ for all $a, b \in [0,1]$. Define $g(x) = x/6$ and $f(x) = x/2$ on X. It is evident that $g(X) \subseteq f(X)$. Also, for $\alpha = 1/3$,

$$\begin{aligned} M(g(x),g(y),t/3) &= \frac{t/3}{t/3 + |x-y|/6} = \frac{2t}{2t+|x-y|} \\ &\geq M(f(x),f(y),t) \\ &= \frac{t}{t+|x-y|/2} = \frac{2t}{2t+|x-y|} \end{aligned}$$

and

$$\begin{aligned} N(g(x),g(y),t/3) &= \frac{|x-y|/6}{t/3 + |x-y|/6} = \frac{|x-y|}{2t+|x-y|} \\ &\leq N(f(x),f(y),t) \\ &= \frac{|x-y|/2}{t+|x-y|/2} = \frac{|x-y|}{2t+|x-y|}. \end{aligned}$$

Thus, all conditions of Theorem 3.1 are satisfied, and f and g have the common fixed point 0.

Problem. It is natural to ask whether Theorem 3.1 would remain true if we can weaken the commutativity of f and g to compatibility.

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986) 87-96.
[2] A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems 64 (1994) 395-399.
[3] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems 27 (1988) 385-389.
[4] G. Jungck, *Commutating maps and fixed points*, Amer. Math. Monthly 83 (1976) 261-263.
[5] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci 9 (1986) 771-774.
[6] O. Kramosil, J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetika 11 (1975) 326-334.
[7] S. Kutukcu, C. Yildiz, D. Turkoglu, *Fixed points of contractive mappings in intuitionistic fuzzy metric spaces*, J. Comput. Anal. Appl., in press.
[8] J.H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons & Fractals 22 (2004) 1039-1046.
[9] P.V. Subrahmanyam, *A common fixed point theorem in fuzzy metric spaces*, Inform. Sci. 83 (1995) 109-112.
[10] L.A. Zadeh, *Fuzzy sets*, Inform. Control 8 (1965) 338-353.

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A NEW APPROACH TO q-EULER NUMBERS AND POLYNOMIALS

L.C. Jang and T. Kim

Abstract. In the recent, several mathematician have studied the second kind ordinary Euler numbers which were defined by

$$\left(\frac{2}{e^x+e^{-x}}\right) = sechx = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

The purpose of this paper is to construct q-expansion of the second kind Euler numbers. From these numbers, we will derive some interesting properties and identities.

§1. Introduction

Let p be a fixed prime, and let $\mathbb{C}_p$ denote the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. For a fixed odd positive integer d with $(p,d)=1$, let

$$\begin{aligned} X = X_d &= \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \\ X_1 &= \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a+dp\mathbb{Z}_p), \\ a+dp^N\mathbb{Z}_p &= \{x \in X | x \equiv a \pmod{dp^n}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. Let q be variously considered as an indeterminate a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we always assume $|q|<1$. If $q \in \mathbb{C}_p$, we always assume $|q-1|_p < 1$. Throughout this paper we use the notation

$$[x]_q = [x:q] = \frac{1-q^x}{1-q} = 1+q+q^2+\cdots+q^{x-1}, \quad \text{cf. [5,6,7]}.$$

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y}$$

have a limit $l = f'(a)$ as $(x, y) \to (a, a)$. cf. [6].

The p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{0\le x<p^N} f(x) q^x, \quad \text{cf. } [6,7].$$

Let us define $I_{-1}(f)$ in the sense of fermionic as follows:

$$\lim_{q\to -1} I_q(f) = I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).$$

In [6,7], it was known that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0),$$

where f_1 is translation with $f_1(x) = f(x+1)$.

The first kind ordinary Euler numbers were defined by

$$F(t) = \frac{2}{e^t+1} = e^{E^* t} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}, \quad |t| < \pi,$$

where we use the technical method notation by replacing E^{*n} by $E_n^* (m \ge 0)$, symbolically, cf. [3,5,9,10,11]. In [5], author have constructed q-extension of E_n^* as follows:

$$\left(\frac{2}{e^t+1}\right)_q = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}.$$

Recently, several mathematicians are studying Euler numbers and polynomials, cf.[1-10].

In [10], Simsek have studied the second kind Euler numbers and polynomials. He also studied congruences for higher-order the second kind Euler numbers.

The purpose of this paper is to construct q-extension of the second kind Euler numbers and polynomials as follows:

$$\left(\frac{2}{e^x+e^{-x}}\right)_q = (sechx)_q = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m+1]_q t} = \sum_{m=0}^{\infty} E_{m,q} \frac{t^m}{m!}.$$

By using these q-Euler numbers, we give another constructions of q-Genocchi numbers, which are different q-extension of the first kind Genocchi numbers.

Finally, we will investigate the relations between q-Euler numbers and q-extension of Genocchi numbers.

§2. q- Euler numbers and polynomials

For an integer k, the Euler numbers E_n were defined by

$$\left(\frac{2}{e^x + e^{-x}}\right) = sechx = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}. \tag{1}$$

Note that

$$\left(\frac{2}{e^{2x}+1}e^x\right) = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}.$$

From (1), we derive

$$(E+1)^n + (E-1)^n = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0, \end{cases} \tag{2}$$

where we have used the symbolic notation E_n for E^n.

By (2), we easily see that

$$E_0 = 1, \quad E_1 = 0, \quad E_2 = -1, \quad E_3 = 0, \quad E_4 = 5, \cdots$$

$E_{2k+1} = 0$ for $k \in \mathbb{N}$. In particular,

$$E_{2n} = -\sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k}.$$

In [6], the p-adic integral was defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=1}^{p^N-1} f(x) q^x. \tag{3}$$

Let

$$\lim_{q\to -1} I_q(f) = I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).$$

Then we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0),$$

where $f_1(x)$ is a translation with $f_1(x) = f(x+1)$. If we take $f(x) = e^{(2x+1)t}$, then we have

$$\int_{\mathbb{Z}_p} e^{t(2x+1)} d\mu_{-1}(x) = \frac{2}{e^t + e^{-t}} = secht = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{4}$$

From (4), we derive

$$\int_{\mathbb{Z}_p} (2x+1)^n d\mu_{-1}(x) = E_n. \tag{5}$$

It was well known that the first kind Euler numbers are defined by

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n^*\frac{t^n}{n!},\qquad \text{cf. [6]}. \tag{6}$$

From (5), we derive the following:

$$\int_{\mathbb{Z}_p}x^n d\mu_{-1}(x)=E_n^*,\quad E_0^*=1,\qquad E_1^*=-\frac{1}{2},\qquad E_2^*=0,\cdots,$$
$$E_{2k}^*=0 \text{ for } k\in\mathbb{N}. \tag{7}$$

By (1), (3), (4), (5) and (6), we easily see that

$$E_n=\sum_{l=0}^{n}\binom{n}{l}2^l E_l^*. \tag{8}$$

In (3), we note that

$$I_{-q}(f)=\int_{\mathbb{Z}_p}f(x)d\mu_{-q}(x)=\frac{[2]_q}{2}\lim_{N\to\infty}\sum_{x=0}^{p^N-1}f(x)(-1)^x q^x \qquad \text{cf. [5]}. \tag{9}$$

By using (9), we define the q-extension of n-th Euler numbers which were defined in (1) as follows:

$$E_{n,q}=\int_{\mathbb{Z}_p}[2x+1]_q^n d\mu_{-q}(x). \tag{10}$$

From (10), we derive

$$E_{n,q}=[2]_q\left(\frac{1}{1-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^l\frac{1}{1+q^{2l+1}}. \tag{11}$$

Let $F_q(t)$ be the generating function of $E_{n,q}$ by

$$F_q(t)=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!}. \tag{12}$$

It follows from (12) that

$$\begin{aligned}F_q(t)=&[2]_q\sum_{n=0}^{\infty}\left(\frac{1}{1-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^l}{1+q^{2l+1}}\frac{t^n}{n!}\\ &=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[2m+1]_q t}\\ &=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{([m]_q+q^m[m]_q)t}.\end{aligned} \tag{13}$$

Note that

$$\lim_{q\to 1} F_q(t) = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

From (1), we can also consider the n-th q-Euler polynomials as follows:

$$\left(\frac{2}{e^t + e^{-t}}\right) e^{xt} = (secht)e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{14}$$

By (1) and (14), we easily see that

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l}. \tag{15}$$

It is easy to see that

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{(2y+1+x)t} d\mu_{-1}(y) &= \frac{2}{e^t + e^{-t}} e^{xt} = (secht)e^{xt} \\ &= \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \end{aligned} \tag{16}$$

Thus, we have

$$\int_{\mathbb{Z}_p} (2y + 1 + x)^n d\mu_{-1}(y) = E_n(x). \tag{17}$$

We now give distribution relation for $E_n(x)$ as follows:

$$E_n(x) = d^n \sum_{a=0}^{d-1} (-1)^a E_n\left(\frac{2a + x + 1 - d}{d}\right), \tag{18}$$

where d is a positive odd integer. We now consider q-extension of Euler polynomial which were defined in (14) as follows:

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [2x + 1 + x]_q^n d\mu_{-q}(x), \quad n \geq 0. \tag{19}$$

From (19), we derive

$$E_{n,q}(x) = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{(x+1)l} \frac{1}{1 + q^{2l+1}}. \tag{20}$$

Let $F_q(t,x)$ be the generating function of $E_{n,q}(x)$ as follows:

$$F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{21}$$

By (20) and (21), we easily see that

$$\begin{aligned} F_q(t,x) &= [2]_q \sum_{n=0}^{\infty} \left((\frac{1}{1-q})^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{(x+1)l} \frac{1}{1+q^{2l+1}} \right) \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[2n+1+x]_q t}. \end{aligned} \tag{22}$$

Note that

$$\lim_{q\to 1} F_q(t,x) = \frac{2}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{23}$$

By (21) and (23), we note that

$$\lim_{q\to 1} E_{n,q}(x) = E_n(x).$$

Now, we define the second kind Genocchi polynomials as follows:

$$G(t) = \frac{2t}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{24}$$

In the case $x = 0$, $G_n = G_n(0)$ will be called Genocchi numbers.

Let n be odd. Then we obtain the multiplication theorem for the second kind Genocchi polynomials as follows:

$$n^{m-1} \sum_{j=0}^{n-1} (-1)^j G_m \left(\frac{2j + x + 1 - n}{n} \right) = G_m(x). \tag{25}$$

This is equivalent to

$$n^{1-m} G_m(n(x+1)) = \sum_{j=0}^{n-1} (-1)^j G_m(x + \frac{2j+1}{n}). \tag{26}$$

We now consider the q-extension of the second kind Genocchi polynomials which were defined in (24) as follows:

$$G_q(x,t) = tF_q(x,t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^n e^{[2n+1+x]_q t}$$
$$= \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \tag{27}$$

From (27), we derive

$$\begin{aligned} G_q(x,t) &= [2]_q t \sum_{n=0}^{\infty} (-1)^n q^n e^{[2n+1+x]_q t} \\ &= [2]_q t \sum_{m=0}^{\infty} \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{m} \binom{m}{l} \frac{(-1)^l q^{l(x+1)}}{1+q^{2l+1}} \frac{t^m}{m!} \\ &= [2]_q \sum_{m=0}^{\infty} \left(\frac{1}{1-q}\right)^{m-1} m \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{(-1)^l q^{l(x+1)}}{1+q^{2l+1}} \frac{t^m}{m!}. \end{aligned} \tag{28}$$

By (27) and (28), we easily see that

$$G_{m,q}(x) = [2]_q m \left(\frac{1}{1-q}\right)^{m-1} \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{(-1)^l}{1+q^{2l+1}} q^{l(x+1)}. \tag{29}$$

Note that $G_{0,q} = 0$.
From (22) and (28), we derive

$$\begin{aligned} F_q(t,x) &= [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[2n+1+x]_q t} = \frac{e^{[x]_q t}}{t} [2]_q t \sum_{n=0}^{\infty} (-1)^n q^n e^{q^x [2n+1]_q t} \\ &= \frac{1}{t} e^{[x]_q t} \sum_{m=0}^{\infty} G_{m,q} q^{mx} \frac{t^m}{m!} = \frac{e^{[x]_q t}}{t} \sum_{m=1}^{\infty} G_{m,q} q^{mx} \frac{t^m}{m!} \\ &= e^{[x]_q t} \sum_{m=0}^{\infty} \frac{G_{m+1,q}}{m+1} q^{(m+1)x} \frac{t^m}{m!} \\ &= \left(\sum_{l=0}^{\infty} [x]_q^l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} q^{(m+1)x} \frac{G_{m+1,q}}{m+1} \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} q^{(m+1)x} \frac{G_{m+1,q}}{m+1} [x]_q^{n-m} \frac{n!}{m!(n-m)!} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} q^{(m+1)x} \frac{G_{m+1,q}}{m+1} [x]_q^{n-m} \right) \frac{t^n}{n!}. \end{aligned} \tag{30}$$

Thus, we note that

$$E_{n,q}(x)=\sum_{m=0}^{n}\binom{n}{m}q^{(m+1)x}\frac{G_{m+1,q}}{m+1}[x]_q^{n-m}. \tag{31}$$

It follows from (22) that

$$E_{k,q}(x)=\left(\frac{d}{dt}\right)^k F_q(t,x)\mid_{t=0}=[2]_q\sum_{n=0}^{\infty}(-1)^n q^n[2n+1+x]_q^k.$$

Thus, we obtain the following:
For $k\in\mathbb{N}$, we have

$$E_{k,q}(x)=[2]_q\sum_{n=0}^{\infty}(-1)^n q^n[2n+1+x]_q^k. \tag{32}$$

For $s\in\mathbb{C}$, we define Hurwitz's type q-zeta function as follows:

$$\zeta_{q,E}(s,x)=[2]_q\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{[2n+1+x]_q^k}.$$

Note that

$$\zeta_{q,E}(-k,x)=E_{k,q}(x),\quad \text{for } k\geq 0.$$

In [10], the second Euler number $E_n^{(k)}$ of order k (the index k may be negative) is defined by

$$(sechx)^k=\sum_{n=0}^{\infty}E_n^{(k)}\frac{x^n}{n!}, \tag{33}$$

or equivalently

$$\left(\frac{2}{e^x+e^{-x}}\right)^k=\sum_{n=0}^{\infty}E_{2n}^{(k)}\frac{x^{2n}}{(2n)!}. \tag{34}$$

The numbers $E_n^{(1)}=E_n$ are the second kind ordinary Euler numbers.
From (4) and (34), we derive

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}e^{t(2(x_1+x_2+\cdots+x_k)+k)}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\ &=\left(\frac{2}{e^t+e^{-t}}\right)^k=\sum_{n=0}^{\infty}E_n^k\frac{t^n}{n!}. \end{aligned}\tag{35}$$

By using Taylor expansion in (35), we obtain

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}(2(x_1+\cdots+x_k)+k)^n d\mu_{-1}(x_1)\cdots\mu_{-1}(x_k)=E_n^{(k)}.$$

Thus, we note that

$$E_n^{(k)}=\sum_{\substack{n=a_1+\cdots+a_k\\ a_1,\cdots,a_k\geq 0}}\binom{n}{a_1,\cdots,a_k}E_{a_1}E_{a_2}\cdots E_{a_k}, \tag{36}$$

where $\binom{n}{a_1,\cdots,a_k}=\dfrac{n!}{a_1!a_2!\cdots a_k!}$.

We now consider q-extension of the second Euler numbers of higher order as follows:

$$E_{n,q}^{(k)}=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}[2x_1+\cdots+2x_k+k]_q^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k). \tag{37}$$

From (37), we derive

$$E_{n,q}^{(k)}=[2]_q^k\left(\frac{1}{1-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{kl}\left(\frac{1}{q^{2l+1}+1}\right)^k. \tag{38}$$

Note that

$$\lim_{q\to 1}E_{n,q}^{(k)}=E_n^{(k)}.$$

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References

[1] M. Cenkci, M. Can, *Some results on q-analogue of the Lerch zeta function*, Adv. Stud. Contemp. Math. **12** (2006), 213–223.

[2] M. Cenkci, M. Can and V. Kurt, *p-adic interpolation functions and Kummer-type congruences for q-twisted Euler numbers*, Advan. Stud. Contemp. Math. **9** (2004), 203–216.

[3] F. T. Howard, *Applications of a recurrence for the Bernoulli numbers*, J. Number Theory **52** (1995), 157–172.

[4] T. Kim, *A note on q-Bernoulli numbers and polynomials*, J. Nonlinear Math. Phys. **13(3)** (2006), 315-322.

[5] T. Kim, *Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function*, Russ. J. Math. Phys. **12** (2005), 186-196.

[6] T. Kim, *A note on p-adic invariant integral in the rings of p-adic integers*, Advan. Stud. Contemp. Math. **13** (2006), 95–99.
[7] T. Kim, *A note on q-Volkenborn integration*, Proc. Jangjeon Math. Soc. **8** (2005), 13–17.
[8] T. Kim, *On the q-extension of Euler and Genocchi numbers*, J. Math. Anal. Appl. **326** (2007), 1458-1465.
[9] H. Ozden, Y. Simsek, S.-H. Rim, I. N. Cangul, *A note on p-adic q-Euler measures*, Advan. Stud. Contemp. Math. **14** (2007), 233-239.
[10] Y. Simsek, *q-analogue of twisted l-series and q-twisted Euler numbers*, J. Number Theory **110** (2005), 267–278.
[11] K. Shiratani, S. Yamamoto, *On a p-adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci. Kyushu Univ. Ser. A **39** (1985), 113–125.
[12] J. Sandor, E. Egri, *Arithmetic functions in algebra, geometry and analysis*, Advan. Stud. Contemp. Math. **14** (2007), 161-213.

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q-analogue of the p-adic twisted l-function

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Abstract : The purpose of this paper is to give a q-analogue of the p-adic twisted l-function, which is an answer to a part of the problem in a previous publication(see J. Math. Anal. Appl. (in press)).

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ are respectively denoted as the ring of p-adic rational integers, the field of p-adic rational numbers, the complex numbers field and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$.

When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|1-q|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_q \leqslant 1$.

We use the notations as

$$\begin{aligned} [x]_q &= \frac{1-q^x}{1-q} = 1 + q + q^2 + \cdots + q^{x-1}, \\ [x]_{-q} &= \frac{1-(-q)^x}{1+q} = 1 - q + q^2 - q^3 + \cdots + (-1)^x q^{x-1}. \end{aligned}$$

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, Kim originally defined the p-adic invariant q-integral on $\mathbb{Z}_p$ as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ cf. [6], [7], [10]},$$

where N is natural number. Let

$$I_1(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),$$

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where N is a natural number (see [2], [3], [4], [5], [9]).

Let d be a fixed integer. For any positive integer N, we set

$$\begin{aligned} \mathbb{X} &= \mathbb{X}_d = \varprojlim_N \left(\mathbb{Z}/dp^N\mathbb{Z}\right), \\ \mathbb{X}^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N &= \left\{x \in \mathbb{X} : x \equiv a \left(\mathrm{mod} dp^N\right)\right\}, \end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \leqslant a < dp^N$.

Let us define $I_{-q}(f)$ as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{P^N-1} f(x)(-q)^x, \text{ see [9]} .$$

This integral, $I_{-q}(f)$, can be considered as the q-deformed p-adic invariant integral on $\mathbb{Z}_p$ in the sense of fermionic, cf. [3], [4], [5], [6], [8].

In [6], multiple q-Euler polynomials of higher order were defined by

$$E_{n,q}^{(h,k)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} [x + x_1 + \cdots + x_k]_q^n \, q^{\sum_{i=1}^k x_i(h-i)} d\mu_{-q}(x) \cdots d\mu_{-q}(x),$$

where $h \in \mathbb{Z}$, $k \in \mathbb{N}$. The q-Euler polynomials of higher order at $x = 0$ are called q-Euler numbers of higher order.

In [1], Carlitz originally constructed q-Bernoulli numbers and polynomials. These numbers and polynomials are studied by many authors (see [7], [8], [10], [11]). In particular, twisted (h, q)-Bernoulli numbers and polynomials were also studied by several authors (see [9], [10], [11]).

In [8], Kim-Rim introduced an interesting twisted q-Euler numbers and polynomials associated with basic twisted q-l-functions and suggested the following question:

"Find a q-analogue of the p-adic twisted l-function which interpolates generalized twisted q-Euler numbers attached to χ, $E_{n,w,\chi,q}$."

In this paper, we give some interesting properties related to twisted (h, q)-Euler numbers and polynomials. The purpose of this paper is to construct p-adic twisted q-l-function which is a part of answer for the question in [8].

2. p-adic Invariant Integral on $\mathbb{Z}_p$ Associated with Twisted (h, q)-Euler Numbers and Polynomials

Let $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$. From the invariant integral on $\mathbb{Z}_p$ in the sense of fermionic, we define

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x), \text{ see [9]},$$

where $f \in UD(\mathbb{Z}_p)$, cf. [3]. Note that $I_{-1}(f_1) + I_{-1}(f) = 2f(0)$, where $f_1(x) = f(x + 1)$. Let $C_{p^n} = \left\{\xi : \xi^{p^n} = 1\right\}$ be the cyclic group of order p^n and let $T_p = \lim_{n\to\infty} C_{p^n} = C_{p^\infty}$. Then T_p is

p-adic locally constant space. For $\xi \in T_p$, we denote by $\phi_\xi : \mathbb{Z}_p \to \mathbb{C}_p$ defined by $\phi_\xi(x) = \xi^x$ be the locally constant function. If we take $f(x) = \phi_\xi(x) e^{tx}$, then we have

$$\int_{\mathbb{Z}_p} e^{tx} \phi_\xi(x) \, d\mu_{-1}(x) = \frac{2}{\xi e^t + 1}, \text{ (see [3])}. \tag{1}$$

In complex case the twisted Euler numbers were defined by Kim-Rim [8]

$$\frac{2}{\xi e^t + 1} = \sum_{n=0}^{\infty} E_{n,\xi} \frac{t^n}{n!},$$

where $|\log\xi + t| < \pi$. Thus we have

$$\int_{\mathbb{Z}_p} x^n \phi_\xi(x) \, d\mu_{-1}(x) = E_{n,\xi}, \ (n \geqslant 0).$$

By using iterative method of p-adic invariant integral on $\mathbb{Z}_p$ in the sense of fermionic, we see that

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \tag{2}$$

where $f_n(x) = f(x+n)$. If n is odd positive integer, then we have

$$I_{-1}(f_n) + I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^l f(l). \tag{3}$$

Let χ be the Dirichlet's character with conductor d (=odd)$\in \mathbb{N}$, and let $f(x) = \chi(x) \phi_\xi(x) e^{tx} \in UD(\mathbb{Z}_p)$. From (3), we can derive the following:

$$\int_{\mathbb{X}} e^{tx} \phi_\xi(x) \chi(x) \, d\mu_{-1}(x) = 2 \frac{\sum_{a=0}^{d-1} \chi(a) \phi_\xi(x) e^{at}}{\xi^d e^{dt} + 1}. \tag{4}$$

Now we define twisted generalized Euler numbers attached to χ as follows:

$$\frac{2 \sum_{a=0}^{d-1} \chi(a) \phi_\xi(x) e^{at}}{\xi^d e^{dt} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,\xi} \frac{t^n}{n!}.$$

From (1) and (4), we note that

$$\begin{aligned} \int_{\mathbb{Z}_p} x^n \phi_\xi(x) \, d\mu_{-1}(x) &= E_{n,\xi}, \\ \int_{\mathbb{X}} x^n \phi_\xi(x) \chi(x) \, d\mu_{-1}(x) &= E_{n,\chi,\xi} \end{aligned} \tag{5}$$

In [6], (h,q)-Euler numbers were defined by

$$E_{n,q}^{(h,1)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)y} [x+y]_q^n \, d\mu_{-q}(y),$$

where $h \in \mathbb{Z}$. $E_{n,q}^{(h,1)}(0) = E_{n,q}^{(h,1)}$ will be called (h,q)-Euler number. In the special case $h = 1$, we note that $\lim_{h \to 1} E_{n,q}^{(h,1)} = E_{n,q}^{(1,1)}$ becomes q-Euler numbers, which were originally defined by Kim [6]. That is $\lim_{h \to 1} E_{n,q}^{(h,1)} = E_{n,q}^{(1,1)} = E_{n,q}$.

In the viewpoint of (5), we consider twisted (h,q)-Euler numbers using p-adic invariant q-integral on $\mathbb{Z}_p$ in the sense of fermionic as follows:

$$E_{n,\xi,q}^{(h,1)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)y} \phi_\xi(y) [x+y]_q^n \, d\mu_{-q}(y), \tag{6}$$

which are called twisted (h,q)-Euler polynomials. In the special case $x = 0$, we use notation $E_{n,\xi,q}^{(h,1)}(0) = E_{n,\xi,q}^{(h,1)}$ which are called (h,q)-twisted Euler numbers. Note that

$$E_{n,\xi,q}^{(h,1)}(x) = \frac{[2]_q}{(1-q)^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{xj} \frac{1}{1+\xi q^{h+j}}, \tag{7}$$

where

$$\binom{n}{j} = \frac{n(n-1)\cdots(n-j+1)}{j!}.$$

From (7), we note that

$$E_{n,\xi,q}^{(h,1)}(x) = [2]_q \sum_{k=0}^{\infty} (-1)^k \xi^k q^{hk} [x+k]_q^n, \tag{8}$$

where $h \in \mathbb{Z}$, $n \in \mathbb{N}$. Equation (8) is equivalent to

$$E_{n,\xi,q}^{(h,1)}(x) = \frac{[2]_q}{[2]_{q^d}} [d]_q^n \sum_{a=0}^{d-1} (-1)^a \xi^a q^{ha} E_{n,\xi^d,q^d}^{(h,1)}\left(\frac{x+a}{d}\right)$$

(distribution for $E_{n,\xi,q}^{(h,1)}(x)$), where n, $d\,(=\text{odd}) \in \mathbb{N}$.

Let χ be the Dirichlet character with conductor $f\,(=\text{odd}) \in \mathbb{N}$. Then we define the generalized twisted (h,q)-Euler numbers attached to χ as follows: For $n \geqslant 0$,

$$E_{n,\xi,\chi,q}^{(h,1)} = \int_{\mathbb{X}} \chi(x) q^{(h-1)x} \xi^x [x]_q^n \, d\mu_{-q}(x), \tag{9}$$

where $h \in \mathbb{Z}$. Note that $E_{n,1,\chi,q}^{(1,1)} = E_{n,\chi,q}$, see [3]. From (9), we also derive

$$\begin{aligned} E_{n,\xi,\chi,q}^{(h,1)} &= [f]_q^n \frac{[2]_q}{[2]_{q^f}} \sum_{a=0}^{f-1} \chi(a) (-1)^a \xi^a q^{ha} E_{n,\xi^f,q^f}^{(h,1)}\left(\frac{a}{f}\right) \\ &= [2]_q \sum_{k=1}^{\infty} \chi(k) (-1)^k \xi^k q^{hk} [k]_q^n, \end{aligned} \tag{10}$$

where n, $d\,(=\text{odd}) \in \mathbb{N}$.

3. Twisted (h,q)-Euler Zeta Function in $\mathbb{C}$

For $q \in \mathbb{C}$ with $|q| < 1$, $s \in \mathbb{C}$, we define

$$\zeta_{E,q,\xi}^{(h,1)}(s) = [2]_q \sum_{k=1}^{\infty} \frac{(-1)^k \xi^k q^{hk}}{[k]_q^s}.$$

Then we see that $\zeta_{E,q,\xi}^{(h,1)}(s)$ is analytic continuation in whole complex plane. We easily see that

$$\zeta_{E,q,1}^{(h,1)}(s)=\zeta_{E,q}^{(h,1)}(s)\,,\text{ see [6].}$$

We now also consider Hurwitz's type twisted (h,q)-Euler zeta function as follows: For $s\in\mathbb{C}$, define

$$\zeta_{E,q,\xi}^{(h,1)}(s,x)=[2]_q\sum_{k=0}^{\infty}\frac{(-1)^a\,\xi^k q^{hk}}{[x+k]_q^s},\tag{11}$$

for $s\in\mathbb{C}$, $h\in\mathbb{Z}$. By (8) and (11), we see that

$$\zeta_{E,q,\xi}^{(h,1)}(-n,x)=E_{n,\xi,q}^{(h,1)}(x)\,,$$

where $n\in\mathbb{N}$, $h\in\mathbb{Z}$.

Let χ be the Dirichlet's character with conductor $f\,(=\text{odd})\in\mathbb{N}$. Then we define twisted (h,q)-l-function which interpolates twisted generalized (h,q)-Euler numbers attached to χ as follows: For $s\in\mathbb{C}$, $h\in\mathbb{Z}$, we define

$$l_{q,\xi}^{(h,1)}(s,\chi)=[2]_q\sum_{k=1}^{\infty}\frac{\chi(k)\,(-1)^k\,q^{hk}\xi^k}{[k]_q^s}.\tag{12}$$

For any positive integer n, we have

$$l_{q,\xi}^{(h,1)}(-n,\chi)=E_{n,\xi,\chi,q}^{(h,1)},\ n\in\mathbb{N}.\tag{13}$$

From (13), we derive

$$\begin{aligned}l_{q,\xi}^{(h,1)}(s,\chi)&=[2]_q\sum_{k=1}^{\infty}\frac{\chi(k)\,(-1)^k\,q^{hk}\xi^k}{[k]_q^s}\\&=[f]_q^{-s}\,\frac{[2]_q}{[2]_{q^f}}\sum_{a=1}^{f}\chi(a)\,(-1)^a\,\xi^a q^{ha}\zeta_{E,\xi^f,q^f}^{(h,1)}\left(s,\frac{a}{f}\right).\end{aligned}$$

Let s be a complex variable and let a and $F\,(=\text{odd})$ be integer with $0<a<F$. We consider the following twisted (h,q)-harmonic sums (or partial (h,q)-zeta function):

$$\begin{aligned}H_{E,q,\xi}^{(h,1)}(s,a|F)&=\sum_{\substack{m\equiv 0(\text{mod}F)\\ m>0}}\frac{(-1)^m\,q^{hm}\xi^m}{[m]_q^s}\\&=\sum_{n=0}^{\infty}\frac{(-1)^{a+nF}\,q^{h(a+nF)}\xi^{a+nF}}{[a+nF]_q^s}\\&=(-1)^a\,q^{ha}\xi^a\sum_{n=0}^{\infty}\frac{(-1)^n\,\left(q^F\right)^{hn}\left(\xi^F\right)^n}{[F]_q^s\left[\frac{a}{F}+n\right]_{q^F}^s}\\&=[F]_q^{-s}\,\frac{(-1)^a\,q^{ha}\xi^a}{[2]_{q^F}}\zeta_{E,\xi^F,q^F}^{(h,1)}\left(s,\frac{a}{F}\right).\end{aligned}$$

Thus we have

$$H_{E,q,\xi}^{(h,1)}(s,a|F)=[F]_q^{-s}\,\frac{(-1)^a\,q^{ha}\xi^a}{[2]_{q^F}}\zeta_{E,\xi^F,q^F}^{(h,1)}\left(s,\frac{a}{F}\right).\tag{14}$$

By (12), (13) and (14), we see that

$$l_{q,\xi}^{(h,1)}(s,\chi)=[2]_q\sum_{a=1}^{F}\chi(a)\,H_{E,q,\xi}^{(h,1)}(s,a|F). \tag{15}$$

From (14), we note that

$$H_{E,q,\xi}^{(h,1)}(-n,a|F)=\frac{[F]_q^n}{[2]_{q^F}}(-1)^a q^{ha}\xi^a E_{n,\xi^F,q^F}^{(h,1)}\left(\frac{a}{F}\right), \tag{16}$$

where n is a positive integer. By (10), (15) and (16), we see that

$$l_{q,\xi}^{(h,1)}(-n,\chi)=E_{n,\xi,\chi,q}^{(h,1)}.$$

The Euler (h,q)-twisted harmonic sum $H_{E,q,\xi}^{(h,1)}(s,a|F)$ will be called partial twisted (h,q)-zeta function which interpolates twisted (h,q)-Euler polynomials at negative integers. The values $l_{q,\xi}^{(h,1)}(s,\chi)$ are algebraic, hence regarded as lying in the extension of $\mathbb{Q}_p$.

4. p-adic Twisted q-l-Function

Let $\omega(x)$ be the Teichmüller character and let $\langle x\rangle_q=\langle x\rangle=\frac{[x]_q}{\omega(x)}$. When $F\,(=\text{odd})$ is multiple of p, and $(a,p)=1$, we define p-adic partial (h,q)-zeta function as follows: For $h\in\mathbb{Z}$, $q\in\mathbb{C}_p$ with $|1-q|_p<p^{-\frac{1}{p-1}}$, we define

$$H_{E,p,q,\xi}^{(h,1)}(s,a|F)=\frac{(-1)^a\xi^a q^{ha}}{[2]_{q^F}}\langle a\rangle^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}\left(\frac{[F]_q}{[a]_q}\right)^j q^{aj}E_{j,\xi^F,q^F}^{(h,1)},$$

where $s\in\mathbb{Z}_p$. Thus we note that

$$\begin{aligned}
H_{E,p,q,\xi}^{(h,1)}(-n,a|F) &= \frac{(-1)^a\xi^a q^{ha}}{[2]_{q^F}}\langle a\rangle^{n}\sum_{j=0}^{n}\binom{n}{j}\left(\frac{[F]_q}{[a]_q}\right)^j q^{aj}E_{j,\xi^F,q^F}^{(h,1)}\\
&= \frac{(-1)^a\xi^a q^{ha}}{[2]_{q^F}}[F]_q^n\,\omega^{-n}(a)\sum_{j=0}^{n}\binom{n}{j}\left(\frac{[a]_q}{[F]_q}\right)^{n-j}\left(q^F\right)^{\frac{a}{F}j}E_{j,\xi^F,q^F}^{(h,1)}\\
&= [F]_q^n\frac{(-1)^a\xi^a q^{ha}}{[2]_{q^F}}\omega^{-n}(a)\,E_{n,\xi^F,q^F}^{(h,1)}\left(\frac{a}{F}\right)\\
&= \omega^{-n}(a)\,H_{E,q,\xi}^{(h,1)}(-n,a|F).
\end{aligned}$$

Therefore, we obtain the following formula:

$$H_{E,p,q,\xi}^{(h,1)}(-n,a|F)=\omega^{-n}(a)\,H_{E,q,\xi}^{(h,1)}(-n,a|F). \tag{17}$$

Now we consider p-adic interpolating function for twisted generalized (h,q)-Euler numbers attached to χ as follows:

$$l_{p,q,\xi}^{(h,1)}(s,\chi)=[2]_q\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\,H_{E,p,q,\xi}^{(h,1)}(s,a|F), \tag{18}$$

for $s \in \mathbb{Z}_p$. Let n be a natural number. Then we have

$$\begin{aligned} l_{p,q,\xi}^{(h,1)}(-n,\chi) &= [2]_q \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) H_{E,p,q,\xi}^{(h,1)}(-n, a|F) \\ &= [2]_q \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) [F]_q^n \frac{(-1)^a \xi^a q^{ha}}{[2]_{q^F}} \omega^{-n}(a) E_{n,\xi^F,q^F}^{(h,1)}\left(\frac{a}{F}\right) \\ &= [F]_q^n \frac{[2]_q}{[2]_{q^F}} \sum_{\substack{a=1 \\ (a,p)=1}}^{F} (-1)^a \chi\omega^{-n}(a) \xi^a q^{ha} E_{n,\xi^F,q^F}^{(h,1)}\left(\frac{a}{F}\right) \\ &= E_{n,\xi,\chi\omega^{-n},q}^{(h,1)} - \chi\omega^{-n}(p) [p]_q^n \frac{[2]_q}{[2]_{q^p}} E_{n,\xi^p,\chi\omega^{-n},q^p}^{(h,1)}. \end{aligned}$$

Therefore we obtain the following theorem:

Theorem : *For $s \in \mathbb{Z}_p$, we define p-adic twisted q-l-function as follows:*

$$\begin{aligned} l_{p,q,\xi}^{(h,1)}(s,\chi) &= [2]_q \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) H_{E,p,q,\xi}^{(h,1)}(s, a|F) \\ &= [2]_q \sum_{\substack{k=1 \\ (k,p)=1}}^{\infty} \frac{(-1)^k \chi(k) q^{hk} \xi^k}{[k]_q^s}. \end{aligned}$$

Then we have

$$l_{p,q,\xi}^{(h,1)}(-n,\chi) = E_{n,\xi,\chi\omega^{-n},q}^{(h,1)} - \chi\omega^{-n}(p) [p]_q^n \frac{[2]_q}{[2]_{q^p}} E_{n,\xi^p,\chi\omega^{-n},q^p}^{(h,1)}.$$

Remark : *From the above theorem, we note that*

$$l_{p,q,\xi}^{(h,1)}(s,\chi) = \int_{\mathbb{X}^*} \chi(x) \langle x \rangle^{-s} q^{(h-1)x} \xi^x d\mu_{-q}(x), \text{ see [11-15]}.$$

References

[1] L. Carlitz, q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc. 76 (1954), 332-370.

[2] M. Cenkci, M. Can, Some results on q-analogue of the Lerch zeta function, Advan. Stud. Contemp. Math. 12 (2006), 213-223.

[3] T. Kim, On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$, J. Math. Anal. Appl. (2006), 779-792 .

[4] T. Kim, On p-adic q-l-functions and sums of powers, J. Math. Anal. Appl. (2006), doi:10.1016/j.jmaa.2006.07.071.

[5] T. Kim, On the q-extension of Euler and Genocchi numbers, J. Math. Anal. Appl. 326 (2007), 1458-1469.

[6] T. Kim, q-Euler numbers and polynomials associated with p-adic q-integrals , J. Nonlinear Math. Phys. 4(2007), 15-27.

[7] T. Kim, S. H. Rim, q-Volkenborn integation, Russian J. Math. Phys. 9 (2002), 288-299.

[8] T. Kim, S. H. Rim, On the twisted q-Euler numbers and polynomials associated with basic q-l-functions, J. Math. Anal. Appl.(in press: see arXiv.math.NT/0611807).

[9] T. Kim, A note on p-adic q-integrals on $\mathbb{Z}_p$ associated with q-Euler numbers, arXiv:0706.4341v1

[10] Y. Simsek, Twisted (h, q)-Bernoulli numbers and polynomials related to twisted (h, q)-zeta function and L-function, J. Math. Anal. Appl. 324 (2006), 790-804.

[11] Y. Simsek, On p-adic twisted q-L-functions related to generalized tiwsted Bernoulli numbers, Russian J. Math. Phys. 13 (3) (2006), 340-348.

[12] T. Kim, A note on p-adic invariant integral in the rings of p-adic integers, Advan. Stud. Contemp. Math. 13 (1) (2006), 95-99.

[13] Y. Simsek, Theorems on twisted L-function and twisted Bernoulli numbers, Advan. Stud. Contemp. Math. 11 (2) (2005), 205-218.

[14] Y. Simsek, D. Kim, S.-H. Rim, On the two-variable Dirichlet q-L-series, Advan. Stud. Contemp. Math. 10 (2) (2005), 131-142.

[15] M. Cenkci, M. Can, V. Kurt, p-adic interpolation functions and Kummer-type congruences for q-twisted and q-generalized twisted Euler numbers, Advan. Stud. Contemp. Math. 9 (2) (2004), 203-216.

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Geometric and Approximation Properties of Some Complex Sikkema and Spline Operators in the Unit Disk*

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Dedicated to Professor Paul Butzer on the occasion of his 80th birthday anniversary

Abstract. The first purpose of this paper is to obtain Jackson-type estimates in approximation by a complex Sikkema-type operator and by a complex spline operator in the unit disk. Then, in addition, it is proved that these operators preserve the univalence, starlikeness and convexity of analytic functions.

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1 Introduction

In a series of papers, Sikkema [9]–[12] and Totik [13], studied the approximation properties of integral operators (for $\rho \to +\infty$),

$$U_\rho(f)(x) = \frac{1}{I_\rho}\int_{-\infty}^{+\infty} f(x-t)\beta^\rho(t)\,dt, \quad I_\rho = \int_{-\infty}^{+\infty} \beta^\rho(t)\,dt,$$

where $f, \beta\colon \mathbb{R} \to \mathbb{R}$ satisfy some appropriate properties.

For $r \geq 1$, let $D_r = \{z \in \mathbb{C};\ |z| < r\}$ and $A(\overline{D_r}) = \{f\colon \overline{D_r} \to \mathbb{C};\ f$ is continuous on $\overline{D_r}$, analytic on D_r, $f(0) = 0$, $f'(0) = 1\}$. For $r = 1$ simply we denote D_r by D and $A(\overline{D_r})$ by $A(\overline{D})$.

In Section 2 of the paper we study approximation and geometric properties of the complexified version of the above operators, given by

$$L_\rho(f)(z) = \frac{1}{I_\rho}\int_{-\infty}^{+\infty} f(ze^{-it})\beta^\rho(t)\,dt, \quad f \in A(\overline{D_r}), \rho \geq 1.$$

In Section 3 we study approximation and geometric properties for a complexified version of spline operator introduced in the real case by [1].

The results of this paper continue the ideas of approximation by complex convolution operators in the very recent papers [2]–[6].

2 Complex Sikkema Operators

For $f \in A(\overline{D_r})$, $r \geq 1$, let us consider the complex operators

$$L_\rho(f)(z) = \frac{1}{I_\rho}\int_{-\infty}^{+\infty} f(ze^{-it})\beta^\rho(t)\,dt, \rho \geq 1,$$

with $I_\rho = \int_{-\infty}^{+\infty} \beta^\rho(t)\,dt$, where $\beta\colon \mathbb{R} \to \mathbb{R}$ satisfies the following five properties:

a) $\beta(t) \geq 0$, $\forall t \in \mathbb{R}$, $\beta(0) = 1$;

b) $\forall \delta > 0$, $\sup\{\beta(t);\ |t| \geq \delta\} < 1$;

c) $\beta(t)$ is continuous at 0;

d) $t^2\beta(t)$ is Lebesgue integrable over $\mathbb{R}$;

e) β is even on $\mathbb{R}$, i.e. $\beta(-t) = \beta(t)$, $\forall t \in \mathbb{R}$.

First we present:

Theorem 2.1. *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in D and continuous in $\overline{D}$, then $L_\rho(f)$ is analytic in D and continuous in $\overline{D}$, for all $\rho \geq 1$. Also, we can write*

$$L_\rho(f)(z) = \sum_{k=0}^{\infty} A_k(\rho) z^k, \quad z \in D,$$

where

$$A_k(\rho) = \frac{a_k}{I_\rho} \cdot \int_{-\infty}^{+\infty} \cos(kt)\beta^\rho(t)\,dt, \quad k = 0, 1, 2, \ldots,$$

and

$$|A_k(\rho)| \le |a_k|, \quad k = 0, 1, 2, \ldots .$$

Proof. Let $z_0, z_n \in \overline{D}$ be such that $\lim_{n\to\infty} z_n = z_0$. We have

$$\begin{aligned}
|L_\rho(f)(z_n) - L_\rho(f)(z_0)| &\le \frac{1}{I_\rho} \int_{-\infty}^{+\infty} |f(z_n e^{-it}) - f(z_0 e^{-it})|\beta^\rho(t)\,dt \\
&\le \frac{1}{I_\rho} \int_{-\infty}^{+\infty} \omega_1\big(f; |e^{-it}| \cdot |z_n - z_0|\big)_{\overline{D}} \beta^\rho(t)\,dt \\
&= \omega_1\big(f; |z_n - z_0|\big)_{\overline{D}},
\end{aligned}$$

which proves the continuity of $L_\rho(f)$ in $\overline{D}$.

Now, let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in D$, be analytic in D. For fixed $z \in D$, we get $f(ze^{-it}) = \sum_{k=0}^{\infty} a_k e^{-ikt} z^k$, and since $|a_k e^{-ikt}| = |a_k|$, for all $t \in \mathbb{R}$ and the series $\sum_{k=0}^{\infty} a_k z^k$ is convergent, it follows that the series $\sum_{k=0}^{\infty} a_k e^{-ikt} z^k$ is uniformly convergent with respect to $t \in \mathbb{R}$. This immediately implies that the series can be integrated term by term, that is

$$\begin{aligned}
L_\rho(f)(z) &= \sum_{k=0}^{\infty} a_k z^k \cdot \frac{1}{I_\rho} \int_{-\infty}^{+\infty} [\cos(kt) - i\sin(kt)]\beta^\rho(t)\,dt \\
&= \sum_{k=0}^{\infty} a_k \left(\frac{1}{I_\rho} \int_{-\infty}^{+\infty} \cos(kt)\beta^\rho(t)\,dt \right) z^k.
\end{aligned}$$

Then, it is immediate that

$$|A_k| \le |a_k| \cdot \frac{1}{I_\rho} \int_{-\infty}^{+\infty} |\cos(kt)|\beta^\rho(t)\,dt \le |a_k|, \quad k = 0, 1, \ldots . \qquad \blacksquare$$

Remark. In the rest of the section those particular choices of $\beta(t)$ for which $\int_{-\infty}^{+\infty} (\cos t)\beta^\rho(t)\,dt \ne 0$ will be important.

Concerning the approximation properties, we have

Theorem 2.2. *If $f \in A(\overline{D})$ then*

$$|L_\rho(f)(z) - f(z)| \le 2\big(1 + B_\rho(\delta)\big)\omega_1(f;\delta)_{\partial D}, \quad \forall z \in \overline{D},\ \delta > 0,\ \rho \ge 1,$$

where

$$B_\rho(\delta) = \frac{1}{I_\rho} \int_{-\infty}^{+\infty} |t|\delta^{-1}\beta^\rho(t)\,dt.$$

Proof. By Theorem 2.1, $L_\rho(f)$ is analytic in D and continuous on $\overline{D}$, $\forall \rho \geq 1$, $f \in A(\overline{D})$, so from the Maximum Modulus Principle, for the estimate $|L_\rho(f)(z) - f(z)|$ it suffices to take $|z| = 1$.

For $f \in A(\overline{D})$ and $|z| = 1$, we can write

$$f(z) = U(\cos u, \sin u) + iV(\cos u, \sin u), \quad \forall z = e^{iu} \in \partial D.$$

Denoting $F(u) = U(\cos u, \sin u)$, $G(u) = V(\cos u, \sin u)$, $u \in \mathbb{R}$, by [9, p. 356, Theorem 2] we get

$$\begin{aligned}
&|U_\rho(F)(u) - F(u)| \leq \big(1 + B_\rho(\delta)\big)\omega_1(F;\delta)_{\mathbb{R}}, \\
&|U_\rho(G)(u) - G(u)| \leq \big(1 + B_\rho(\delta)\big)\omega_1(G;\delta)_{\mathbb{R}}, \quad \forall u \in \mathbb{R},\ \delta > 0,\ \rho \geq 1.
\end{aligned}$$

But, for $|z| = 1$, $z = e^{iu}$, we have $L_\rho(f)(z) = U_\rho(F)(u) + iU_\rho(G)(u)$ and for

$$\omega_1(f;\delta)_{\partial D} = \sup\{|f(e^{iu}) - f(e^{iv})|; u, v \in \mathbb{R}, |u - v| \leq \delta\},$$

it is easy to check the inequalities

$$\omega_1(F;\delta)_{\mathbb{R}} \leq \omega_1(f;\delta)_{\partial D}, \quad \omega_1(G;\delta)_{\mathbb{R}} \leq \omega_1(f;\delta)_{\partial D}, \quad \delta > 0,$$

since

$$|F(u) - F(v)| \leq |f(e^{iu}) - f(e^{iv})|, \quad |G(u) - G(v)| \leq |f(e^{iu}) - f(e^{iv})|.$$

In conclusion, for $|z| = 1$ we get

$$|L_\rho(f)(z) - f(z)| \leq 2\big(1 + B_\rho(\delta)\big)\omega_1(f;\delta)_{\partial D}, \quad \forall \delta > 0,\ \rho \geq 1,$$

which proves the theorem. ■

Remark. In [9]–[12], for several particular choices of $\beta(t)$ and δ, various estimates in approximation of real functions by U_ρ were obtained. For the same choices of $\beta(t)$ and δ, we get similar estimates for approximation by L_ρ.

Now, concerning the geometric properties, first we present

Theorem 2.3. *Let $f \in A(\overline{D})$ and $L_\rho(f)(z) = \sum_{k=0}^{\infty} A_k(\rho) z^k$. Suppose that $\beta(t)$ is chosen such that $A_1(\rho) \neq 0$, for all $\rho \geq 1$. Then, for all $\rho \geq 1$ we have*

$$\frac{1}{A_1(\rho)} L_\rho\big(S_{3,A_1(\rho)}\big) \subset S_3, \quad \frac{1}{A_1(\rho)} L_\rho(S_M) \subset S_{M/|A_1(\rho)|},$$

where $M > 1$ and

$$\begin{aligned}
S_3 &= \big\{f \in A(\overline{D}); |f''(z)| \leq 1,\ \forall z \in D\big\}, \\
S_{3,A_1(\rho)} &= \big\{f \in A(\overline{D}); |f''(z)| \leq |A_1(\rho)|,\ \forall z \in D\big\}, \\
S_B &= \big\{f \in A(\overline{D}); |f'(z)| < B,\ \forall z \in D\big\}.
\end{aligned}$$

Proof. Let $f \in A(\overline{D})$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in D$. It follows $a_0 = 0$, $a_1 = 1$, which by Theorem 2.1 immediately implies

$$\frac{1}{A_1(\rho)} \cdot L_\rho(f)(0) = 0, \quad \frac{1}{A_1(\rho)} L'_\rho(f)(0) = \frac{A_1(\rho)}{A_1(\rho)} = 1,$$

i.e.

$$\frac{1}{A_1(\rho)} L_\rho(f) \in A(\overline{D}).$$

Then, by

$$\frac{1}{A_1(\rho)} L'_\rho(f)(z) = \frac{1}{I_\rho} \int_{-\infty}^{+\infty} e^{-it} f'(ze^{-it}) \beta^\rho(t)\, dt, \quad z \in D,$$

and

$$\frac{1}{A_1(\rho)} L''_\rho(f)(z) = \frac{1}{I_\rho} \int_{-\infty}^{+\infty} e^{-2it} f''(ze^{-it}) \beta^\rho(t)\, dt, \quad z \in D,$$

we get:

$f \in S_{3,A_1(\rho)}$ implies $|f''(z)| \le |A_1(\rho)|$, $\forall z \in D$, i.e.

$$\frac{1}{|A_1(\rho)|} \cdot |L''_\rho(f)(z)| \le \frac{1}{|A_1(\rho)|} \frac{1}{I_\rho} \int_{-\infty}^{+\infty} |f''(ze^{-it}) e^{-2it}| \beta^\rho(t)\, dt \le 1, \ \forall z \in D$$

and

$f \in S_M$ implies $|f'(z)| < M$, $\forall z \in D$, i.e.

$$\frac{1}{|A_1(\rho)|} \cdot |L'_\rho(f)(z)| \le \frac{1}{|A_1(\rho)|} \frac{1}{I_\rho} \int_{-\infty}^{+\infty} |f'(ze^{-it}) e^{-it}| \beta^\rho(t)\, dt < \frac{M}{|A_1(\rho)|}, \ \forall z \in D,$$

which proves the theorem. ∎

Remarks. 1) It is known (see e.g. [8]) that $f \in S_3$ implies that f is starlike (and univalent) in D and that $f \in S_M$ implies that f is univalent in $\{z \in \mathbb{C}; |z| < \frac{1}{M}\} \subset D$ (see e.g. [7, p. 111, Exercise 5.4.1]).

Since by Theorem 2.1 we have $|A_1(\rho)| \le |a_1| = 1$, it follows that $S_{3,A_1(\rho)} \subset S_3$ and $\frac{M}{|A_1(\rho)|} \ge M > 1$, i.e. if $f \in S_{3,A_1(\rho)}$ then $L_\rho(f)$ remains starlike (and univalent) in D and if $f \in S_M$ then $L_\rho(f)$ is univalent in

$$\left\{z \in \mathbb{C}; |z| < \frac{|A_1(\rho)|}{M}\right\} \subset \left\{z \in \mathbb{C}; |z| < \frac{1}{M}\right\}.$$

2) Denote $A = \inf\{|A_1(\rho)|; \rho \ge 1\}$. If $A > 0$, then by Theorem 2.3 we get the following invariant geometric properties: if $f \in S_{3,A}$ then $L_\rho(f) \in S_3$ for all $\rho \ge 1$ and if $f \in S_M$, $M > 1$, then $L_\rho(f)$ is univalent in

$$\left\{z \in \mathbb{C}; |z| < \frac{A}{M}\right\}, \quad \forall \rho \ge 1.$$

Therefore, would remain to calculate A and to check that $A > 0$ for various choices of $\beta(t)$, problems which are left to the reader as open questions.

The second result concerning geometric properties of $L_\rho(f)$ is the following.

Theorem 2.4. *Let $f \in A(\overline{D_r})$, $r > 1$, and suppose that $\beta(t)$ is such that for any bounded $g : \mathbb{R} \to \mathbb{R}$, we have $lim_{\rho\to\infty} U_\rho(g) = g$, uniformly in any compact interval of $\mathbb{R}$.*

(i) If f is starlike in $\overline{D}$ (that is, Re $\left(\frac{zf'(z)}{f(z)}\right) > 0$, for all $z \in \overline{D}$), then there exists $\rho_0 > 0$ (depending on f), such that for all $\rho \geq \rho_0$, $L_\rho(f)(z)$ are starlike in $\overline{D}$.

If f is starlike only in D, then for any disk of radius $0 < \lambda < 1$ denoted by D_λ , there exists ρ_0 (depending on f and D_λ), such that for all $\rho \geq \rho_0$, $L_\rho(f)(z)$ are starlike in $\overline{D_\lambda}$ (that is, Re $\left(\frac{zL'_\rho(f)(z)}{L_\rho(f)(z)}\right) > 0$, for all $z \in \overline{D_\lambda}$).

(ii) If f is convex in $\overline{D}$ (that is, Re $\left(\frac{zf''(z)}{f'(z)}\right) + 1 > 0$, for all $z \in \overline{D}$), then there exists ρ_0 (depending on f), such that for all $\rho \geq \rho_0$, $L_\rho(f)(z)$ are convex in $\overline{D}$.

If f is convex only in D, then for any disk of radius $0 < \lambda < 1$ denoted by D_λ, there exists ρ_0 (depending on f and D_λ), such that for all $\rho \geq \rho_0$, $L_\rho(f)(z)$ are convex in $\overline{D_\lambda}$ (that is, Re $\left(\frac{zL''_\rho(f)(z)}{L'_\rho(f)(z)}\right) + 1 > 0$, for all $z \in \overline{D_\lambda}$).

Proof. First let us make some general useful considerations. By hypothesis, it follows that for $\rho \to \infty$, we have $L_\rho(f)(z) \to f(z)$, uniformly in any compact disk included in D_r, that is in $\overline{D}$ too. Indeed, this is immediate from the relationship $L_\rho(f)(z) = U_\rho(F_\lambda)(u) + iU_\rho(G_\lambda)(u)$, where $F_\lambda(u) = U(\lambda cos(u), \lambda sin(u)), G_\lambda(u) = V(\lambda cos(u), \lambda sin(u)), f(z) = U(x, y) + iV(x, y)$, $|z| = \lambda \in [0, r), z = \lambda e^{iu} = x + iy, u \in [0, 2\pi]$.

By the well-known Weierstrass' result, this implies that $L'_\rho(f)(z) \to f'(z)$ and $L''_\rho(f)(z) \to f''(z)$, uniformly in any compact disk in D_r and therefore in $\overline{D}$ too, when $\rho \to \infty$.

Then, denoting by $A_1(\rho)$ the coefficient of z in the Taylor series in Theorem 2.1 representing the analytic function $L_\rho(f)(z)$, since $A_1(\rho) = L'_\rho(0)$ and $lim_{\rho\to\infty} L'_\rho(0) = f'(0) = 1$, it follows that $lim_{\rho\to\infty} A_1(\rho) = 1$ and for all $\rho \geq \rho_0$ we have $A_1(\rho) > 0$.

Let us denote $P_\rho(f)(z) = \frac{L_\rho(f)(z)}{A_1(\rho)}$, for all $\rho \geq \rho_0$.

By $f(0) = f'(0) - 1 = 0$ we get $P_\rho(f)(0) = \frac{f(0)}{A_1(\rho)} = 0$ and $P'_\rho(f)(0) = \frac{L'_\rho(f)(0)}{A_1(\rho)} = 1$. Also, we obviously have $P_\rho(f)(z) \to f(z)$, $P'_\rho(f)(z) \to f'(z)$ and $P''_\rho(f)(z) \to f''(z)$, uniformly in $\overline{D}$.

(i) By hypothesis we get $|f(z)| > 0$ for all $z \in \overline{D}$ with $z \neq 0$, which from the univalence of f in D, implies that we can write $f(z) = zg(z)$, with $g(z) \neq 0$, for all $z \in \overline{D}$, where g is analytic in D and continuous in $\overline{D}$.

Write $P_\rho(f)(z)$ in the form $P_\rho(f)(z) = zQ_\rho(f)(z)$.

Let $|z| = 1$. We have

$$|f(z) - P_\rho(f)(z)| = |z| \cdot |g(z) - Q_\rho(f)(z)| = |g(z) - Q_\rho(f)(z)|,$$

which by the uniform convergence in $\overline{D}$ of $P_\rho(f)$ to f and by the Maximum Modulus Principle, implies the uniform convergence in $\overline{D}$ of $Q_\rho(f)(z)$ to $g(z)$, as $\rho \to \infty$.

Since g is continuous in $\overline{D}$ and $|g(z)| > 0$ for all $z \in \overline{D}$, there exist ρ_0 and $a > 0$ depending on g, such that $|Q_\rho(f)(z)| > a > 0$, for all $z \in \overline{D}$ and all $\rho \geq \rho_0$.

Also, for all $|z| = 1$, we have

$$|f'(z) - P'_\rho(f)(z)| = |z[g'(z) - Q'_\rho(f)(z)] + [g(z) - Q_\rho(f)(z)]| \geq$$

$$\left|\quad |z| \cdot |g'(z) - Q'_\rho(f)(z)| - |g(z) - Q_\rho(f)(z)| \quad\right| =$$

$$\left|\quad |g'(z) - Q'_\rho(f)(z)| - |g(z) - Q_\rho(f)(z)| \quad\right|,$$

which from the Maximum Modulus Principle, the uniform convergence of $P'_\rho(f)$ to f' and of $Q_\rho(f)$ to g, evidently implies the uniform convergence of $Q'_\rho(f)$ to g', as $\rho \to \infty$.

Then, for $|z| = 1$, we get

$$\frac{zP'_\rho(f)(z)}{P_\rho(f)} = \frac{z[zQ'_\rho(f)(z) + Q_\rho(f)(z)]}{zQ_\rho(f)(z)} =$$

$$= \frac{zQ'_\rho(f)(z) + Q_\rho(f)(z)}{Q_\rho(f)(z)} \to \frac{zg'(z) + g(z)}{g(z)} = \frac{f'(z)}{g(z)} = \frac{zf'(z)}{f(z)},$$

which again from the Maximum Modulus Principle, implies

$$\frac{zP'_\rho(f)(z)}{P_\rho(f)} \to \frac{zf'(z)}{f(z)}, \text{ uniformly in } \overline{D}.$$

Since $Re\left(\frac{zf'(z)}{f(z)}\right)$ is continuous in $\overline{D}$, there exists $\alpha \in (0,1)$, such that

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha, \text{ for all } z \in \overline{D}.$$

Therefore

$$Re\left[\frac{zP'_\rho(f)(z)}{P_\rho(f)(z)}\right] \to Re\left[\frac{zf'(z)}{f(z)}\right] \geq \alpha > 0,$$

uniformly in $\overline{D}$, i.e. for any $0 < \beta < \alpha$, there is ρ_0 such that for all $\rho \geq \rho_0$ we have

$$Re\left[\frac{zP'_\rho(f)(z)}{P_\rho(f)(z)}\right] > \beta > 0, \text{ for all } z \in \overline{D}.$$

Since $P_\rho(f)(z)$ differs from $L_\rho(f)(z)$ only by a constant, this proves the first part in (i).

For the second part, the proof is identical with the first part, with the only difference that instead of $\overline{D}$, we reason for $\overline{D_\lambda}$.

(ii) For the first part, by hypothesis there is $\alpha \in (0,1)$, such that

$$Re\left[\frac{zf''(z)}{f'(z)}\right]+1 \geq \alpha > 0,$$

uniformly in $\overline{D}$. It is not difficult to show that this is equivalent with the fact that for any $\beta \in (0,\alpha)$, the function $zf'(z)$ is starlike of order β in $\overline{D_1}$ (see e.g. [7], p. 77), which implies $f'(z) \neq 0$, for all $z \in \overline{D}$, i.e. $|f'(z)| > 0$, for all $z \in \overline{D}$. Also, by the same type of reasonings as those from the above point (i), we get

$$Re\left[\frac{zP''_\rho(f)(z)}{P'_\rho(f)(z)}\right]+1 \to Re\left[\frac{zf''(z)}{f'(z)}\right]+1 \geq \alpha > 0,$$

uniformly in $\overline{D}$. As a conclusion, for any $0 < \beta < \alpha$, there is $\rho_0 > 0$ depending on f, such that for all $\rho \geq \rho_0$ we have

$$Re\left[\frac{zP''_\rho(f)(z)}{P'_\rho(f)(z)}\right]+1 > \beta > 0, \text{ for all } z \in \overline{D}.$$

The proof of second part in (ii) is similar, which proves the theorem. ■

3 Complex Spline Operator

In the paper [1], for a real-valued function f, of real variable, the following B-splines were constructed

$$P_{n,(\delta_1,\ldots,\delta_n)}(f)(x) = \int_{x+x_c}^{x+x_c+s} f(u)M(u-x;x_c,x_c+\delta_1,\ldots,x_c+s)\,du, \quad x \in \mathbb{R},\ n \in \mathbb{N},$$

where $s = \delta_1 + \delta_2 + \cdots + \delta_n$,

$$x_c = -\frac{\sum_{k=1}^{n}(n+1-k)\delta_k}{(n+1)}, \quad \delta_i \geq 0,\ \forall i = 1,\ldots,n,\ s > 0,$$

$$M(t;x_0,\ldots,x_n) = n[x_0,\ldots,x_n](\bullet - t)_+^{n-1},$$

$(v-t)_+^{n-1} = (v-t)^{n-1}$ if $v \geq t$, $(v-t)_+^{n-1} = 0$ if $v < t$, and $[x_0,\ldots,x_n]F$ denotes the divided difference of F on the knots $x_0,\ldots,x_n$ (if $x_0 = x_n$ then by definition $[x_0,\ldots,x_n]F = \frac{F^{(n)}(x_0)}{n!}$).

By [1, Proposition 3.1], it follows that $f \in C^r$ implies $P_{n,(\delta_1,\ldots,\delta_n)}(f) \in C^r$, where C^r denotes the class of r-th differentiable functions with continuous $f^{(r)}$.

Changing the variable $u - x = v$, we can write

$$P_{n,(\delta_1,\ldots,\delta_n)}(f)(x) = \int_{x_c}^{x_c+s} f(v+x)M(v;x_c,x_c+\delta_1,\ldots,x_c+s)\,dv.$$

In this section we study the complexified version given by

$$P_{n,(\delta_1,\ldots,\delta_n)}(f)(z) = \int_{x_c}^{x_c+s} f(ze^{iu})M(u;x_c,x_c+\delta_1,\ldots,x_c+s)\,du,$$

for $f \in A(\overline{D})$.

The first result is the following.

Theorem 3.1. *If $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ is analytic in D and continuous in $\overline{D}$, then $P_{n,(\delta_1,\ldots,\delta_n)}$ is analytic in D and continuous in $\overline{D}$. Also, we can write*

$$P_{n,(\delta_1,\ldots,\delta_n)}(f)(z) = \sum_{k=0}^{\infty} B_{k,n} z^k, \quad z \in D,$$

where

$$B_{k,n} = a_k \cdot \int_{x_c}^{x_c+s} [\cos(ku) + i\sin(ku)]M(u;x_c,x_c+\delta_1,\ldots,x_c+s)\,du,$$
$$|B_{k,n}| \le |a_k|, \quad k = 0,1,\ldots .$$

Proof. The reasonings for the formulas of $B_{k,n}$ are similar to those for $A_k(\rho)$ in the proof of Theorem 2.1. Then, by

$$\int_{x_c}^{x_c+s} M(u;x_c,x_c+\delta_1,\ldots,x_c+s)\,du = 1,$$

we easily get

$$|B_{k,n}| \le |a_k| \int_{x_c}^{x_c+s} |\cos(ku) + i\sin(ku)| \cdot M(u;x_c,x_c+\delta_1,\ldots,x_c+s)\,du = |a_k|,$$

$k = 0,1,\ldots .$ ■

Concerning the approximation properties, we have

Theorem 3.2. *If $f \in A(\overline{D})$ then*

$$\left|P_{n,\left(\frac{s}{2},0,\ldots,0,\frac{s}{2}\right)}(f)(z) - f(z)\right| \le 3\omega_2\left(f;\frac{s}{\sqrt{2(n+1)(n+2)}}\right)_{\partial D}, \quad z \in \overline{D},$$

and

$$\left|P_{n,\left(\frac{s}{n},\frac{s}{n},\ldots,\frac{s}{n}\right)}(f)(z) - f(z)\right| \le 3\omega_2\left(f;\frac{s}{\sqrt{12n}}\right)_{\partial D}, \quad z \in \overline{D},$$

where

$$\omega_2(f;\delta)_{\partial D} = \sup\left\{|f(e^{i(x+u)}) - 2f(e^{iu}) + f(e^{i(x-u)})|; x \in \mathbb{R}, |u| \le \delta\right\}.$$

Proof. By Theorem 3.1, if $f \in A(\overline{D})$ then $P_{n,(\delta_1,\ldots,\delta_n)}(f)$ is analytic in D and continuous in $\overline{D}$, so from the Maximum Modulus Principle, for the estimate $|P_{n,(\delta_1,\ldots,\delta_n)}(f)(z) - f(z)|$ it suffices to take $|z| = 1$. For $f \in A(\overline{D})$ and $|z| = 1$ we can write

$$f(z) = U(\cos u, \sin u) + iV(\cos u, \sin u), \quad \forall z = e^{iu} \in \partial D.$$

Denoting $F(u) = U(\cos u, \sin u)$, $G(u) = V(\cos u, \sin u)$, $u \in \mathbb{R}$, by [1, relations (35) and (36)] we get

$$\begin{aligned}
\left|P_{n,\left(\frac{s}{2},0,\ldots,0,\frac{s}{2}\right)}(F)(u) - F(u)\right| &\le \frac{3}{2}\omega_2\left(F; \frac{s}{\sqrt{2(n+1)(n+2)}}\right)_{\mathbb{R}}, \ u \in \mathbb{R},\\
\left|P_{n,\left(\frac{s}{2},0,\ldots,0,\frac{s}{2}\right)}(G)(u) - G(u)\right| &\le \frac{3}{2}\omega_2\left(G; \frac{s}{\sqrt{2(n+1)(n+2)}}\right)_{\mathbb{R}}, \ u \in \mathbb{R},\\
\left|P_{n,\left(\frac{s}{n},\ldots,\frac{s}{n}\right)}(F)(u) - F(u)\right| &\le \frac{3}{2}\omega_2\left(F; \frac{s}{\sqrt{12n}}\right)_{\mathbb{R}}, \ u \in \mathbb{R},\\
\left|P_{n,\left(\frac{s}{n},\frac{s}{n},\ldots,\frac{s}{n}\right)}(G)(u) - G(u)\right| &\le \frac{3}{2}\omega_2\left(G; \frac{s}{\sqrt{12n}}\right)_{\mathbb{R}}, \ u \in \mathbb{R}.
\end{aligned}$$

But for $|z| = 1$, $z = e^{iu}$, we have

$$P_{n,(\delta_1,\ldots,\delta_n)}(f)(z) = P_{n,(\delta_1,\ldots,\delta_n)}(F)(u) + iP_{n,(\delta_1,\ldots,\delta_n)}(G)(u)$$

and by the obvious inequalities

$$\begin{aligned}
|F(x+u) - 2F(u) + F(x-u)| &\le |f(e^{i(x+u)}) - 2f(e^{iu}) + f(e^{i(x-u)})|\\
|G(x+u) - 2G(u) + G(x-u)| &\le |f(e^{i(x+u)}) - 2f(e^{iu}) + f(e^{i(x-u)})|,
\end{aligned}$$

it follows

$$\omega_2(F;\delta)_{\mathbb{R}} \le \omega_2(f;\delta)_{\partial D}, \quad \omega_2(G;\delta)_{\mathbb{R}} \le \omega_2(f;\delta)_{\partial D}.$$

Therefore, we immediately conclude: for $z = e^{iu} \in \partial D$

$$\begin{aligned}
&\left|P_{n,\left(\frac{s}{2},0,\ldots,0,\frac{s}{2}\right)}(f)(z) - f(z)\right|\\
&\quad\le \left|P_{n,\left(\frac{s}{2},0,\ldots,0,\frac{s}{2}\right)}(F)(u) - F(u)\right| + \left|P_{n,\left(\frac{s}{2},0,\ldots,0,\frac{s}{2}\right)}(G)(u) - G(u)\right|\\
&\quad\le 3\omega_2\left(f; \frac{s}{\sqrt{2(n+1)(n+2)}}\right)_{\partial D}.
\end{aligned}$$

Similarly, we get

$$\left|P_{n,\left(\frac{s}{n},\frac{s}{n},\ldots,\frac{s}{n}\right)}(f)(z) - f(z)\right| \le 3\omega_2\left(f; \frac{s}{\sqrt{12n}}\right)_{\partial D}, \quad z \in \partial D,$$

which proves the theorem. ■

Now, concerning the geometric properties, we present

Theorem 3.3. *Let $f \in A(\overline{D})$ and*

$$P_{n,(\delta_1,\ldots,\delta_n)}(f)(z) = \sum_{k=0}^{\infty} B_{k,n} z^k, \quad z \in D.$$

Suppose δ_k, $k = \overline{1,n}$ are chosen such that $B_{1,n} \neq 0$, for all $n \in \mathbb{N}$. Then we have

$$\frac{1}{B_{1,n}} P_{n,(\delta_1,\ldots,\delta_n)}\left(S_{3,B_{1,n}}\right) \subset S_3$$

and

$$\frac{1}{B_{1,n}} P_{n,(\delta_1,\ldots,\delta_n)}(S_M) \subset S_{M/|B_{1,n}|},$$

where S_3 and S_M are those defined in Theorem 2.3.

Proof. The proof is identical with the proof of Theorem 2.3. ■

Remarks. 1) Denote $B = \inf\{|B_{1,n}|;\ n \in \mathbb{N}\}$, with $\delta_1,\ \delta_2, \ldots, \delta_n$ fixed. If $B > 0$ then by Theorem 3.3 we get the following geometric properties:

if $f \in S_{3,B}$ then $P_{n,(\delta_1,\ldots,\delta_n)}(f) \in S_3$, for all $n \in \mathbb{N}$

and

if $f \in S_M$, $M > 1$, then $P_{n,(\delta_1,\ldots,\delta_n)}(f)$ is univalent in $\left\{z \in \mathbb{C};\ |z| < \frac{B}{M}\right\}$, for all $n \in \mathbb{N}$.

Therefore, would remain to calculate B, to check $B > 0$, problems which are left to the reader as open questions.

2) Reasoning as in the proof of Theorem 2.4, it follows that for sufficiently large n (depending on f), the complexified splines $P_{n,(s/2,0,\ldots,0,s/2)}(f)(z)$ and $P_{n,(s/n,s/n,\ldots,s/n)}(f)(z)$ preserve the starlikeness and the convexity of $f(z)$.

References

[1] J.A. Adell and C. Sanguesa, Approximation by B-spline convolution operators. A probabilistic approach, *J. Comp. Appl. Math.*, **174**, No. 1 (2005), 79–99.

[2] G.A. Anastassiou and S.G. Gal, Geometric and approximation properties of some singular integrals in the unit disk, *J. Ineq. Appl.*, vol. 2006, Article ID 17231, 19 pages, 2006.

[3] G.A. Anastassiou and S.G. Gal, Geometric and approximation properties of generalized singular integrals in the unit disk, *J. Korean Math. Soc.*, **43**(2006), No. 2, 425-443.

[4] G.A. Anastassiou and S.G. Gal, Geometric and approximation properties of a complex Post-Widder operator in the unit disk, *Applied Mathematics Letters*, **19**(2006), No. 4, 393-402.

[5] G.A. Anastassiou and S.G. Gal, Geometric and approximation properties of some complex rotation-invariant integral operators in the unit disk, *J. Comp. Anal. Appl.*, **8**(2006), No. 4, 357-368.

[6] S.G. Gal, Geometric and approximate properties of convolution polynomials in the unit disk, *Bull. Inst. Math. Acad. Sinica, New Series*, **1**(2006), No. 2, 307-336.

[7] P.T. Mocanu, T. Bulboaca, and Gr. St. Salagean, *Geometric Theory of Univalent Functions* (in Romanian), Casa Cartii de Stiinta, Cluj, 1999.

[8] M. Obradovic, Simple sufficient conditons for univalence, *Mat. Vesnik*, **49** (1997), 241–244.

[9] P.C. Sikkema, Approximation with convolution operators, in *Second Edmonton Conference in Approximation Theory*, Edmonton, Alberta, 1983, pp. 353–368.

[10] P.C. Sikkema, Fast approximation by means of convolution operators, *Indag. Math.*, **43**, No. 4 (1981), 431–434.

[11] P.C. Sikkema, Approximation with convolution operators depending on two parameters, in *Approximation Theory* IV (College Station, Texas), Academic Press, New York, 1983, pp. 679–684.

[12] P.C. Sikkema, On the exact degree of local approximation with convolution operators, *Indag. Math.*, **41**, No. 3 (1979), 337–351.

[13] V. Totik, Approximation by convolution operators, *Analysis Mathematics*, **8**, No. 2 (1982), 151–163.

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On best approximation in intuitionistic fuzzy metric spaces

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Abstract

In this paper, we study the problem of best approximation in intuitionistic fuzzy metric spaces. For this, we introduce the concept of strong intuitionistic fuzzy metric space, t-best approximation and the study the existence of t-best approximation in intuitionistic fuzzy metric spaces. We give the notion of t-approximately compact set in an intuitionistic fuzzy metric space to study the existence of t-best approximations. We give the concept of t-boundedly compact set and also study some properties of t-approximately compact sets.

Keywords. Intuitionistic fuzzy metric space; best approximation; compact sets.

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1. Introduction

In 1965, the concept of fuzzy set was introduced by Zadeh [17]. Many authors have introduced the concept of fuzzy metric space in different ways [2-4, 6, 7, 9, 10]. George and Veeramani [4, 5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [10] and defined a Hausdorff topology on this fuzzy metric space. They also showed that every metric induces a fuzzy metric. Veeramani [16] introduced the concept of t-best approximation in fuzzy metric spaces..

Park [12] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space due to George and Veeramani and introduced the notion of Cauchy sequences in an intuitionistic fuzzy metric space and proved the Baire's theorem and finding a necessary and sufficent condition for an intuitionistic fuzzy metric space to be complete and shown that every separable intuitionistic fuzzy metric space is second countable and that every subspace of an intuitionistic fuzzy metric space is separable and proved the Uniform limit theorem for intuitionistic fuzzy metric spaces. Alaca et al. [1] defined the completions of intuitionistic fuzzy metric spaces. A complete intuitionistic fuzzy metric space Y is said to be an intuitionistic fuzzy completion of a given intuitionistic fuzzy metric space X if X is isometric to a dense subspace of Y. They gave an example of an intuitionistic fuzzy metric space that does not admit any intuitionistic fuzzy metric completion. Many authors studied the concept of intuitionistic fuzzy metric space and its applications [8, 13].

The purpose of this paper to introduce the concept of strong intuitionistic fuzzy metric space, t-best approximation and the study the existence of t-best approximation in intuitionistic fuzzy metric spaces. We give the notion of t-approximately compact set in an intuitionistic fuzzy metric space to study the existence of t-best approximations. We give the concept of t-boundedly compact set and also study some properties of t-approximately compact sets. As every metric induces an intuitionistic fuzzy metric the results obtained in this paper are more general than the corresponding results of the metric spaces.

2. Intuitionistic fuzzy metric spaces

Definition 1 ([14]). *A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:*

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2 ([14]). *A binary operation $\Diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-conorm if $\Diamond$ is satisfying the following conditions:*

(i) $\Diamond$ is commutative and associative;
(ii) $\Diamond$ is continuous;
(iii) $a \Diamond 0 = a$ for all $a \in [0,1]$;
(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 3 ([12]). *A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all x, y, $z \in X$, s, $t > 0$,*

(IFM-1) $M(x,y,t) + N(x,y,t) \leq 1$;
(IFM-2) $M(x,y,t) > 0$;
(IFM-3) $M(x,y,t) = 1$ if and only if $x = y$;
(IFM-4) $M(x,y,t) = M(y,x,t)$;
(IFM-5) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(IFM-6) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous;
(IFM-7) $N(x,y,t) > 0$;
(IFM-8) $N(x,y,t) = 0$ if and only if $x = y$;
(IFM-9) $N(x,y,t) = N(y,x,t)$;
(IFM-10) $N(x,y,t) \Diamond N(y,z,s) \geq N(x,z,t+s)$;
(IFM-11) $N(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 1. *Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are assosiated [11], i.e. $x \Diamond y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in [0, 1]$.*

Remark 2. *In intuitionistic fuzzy metric space X, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.*

Example 1 (Induced intuitionistic fuzzy metric [12])**.** *Let (X, d) be a metric space. Denote $a * b = ab$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:*

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)},\ N_d(x, y, t) = \frac{md(x, y)}{ht^n + md(x, y)}$$

for all $h, m, n \in \mathbb{R}^+$. Then $(X, M_d, N_d, *, \Diamond)$ is an intuitionistic fuzzy metric space.

Remark 3. *Note the above example holds even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \Diamond b = \max\{a, b\}$ and hence (M, N) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h = m = n = 1$, we get*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},\ N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Example 2. *Let $X = \mathbb{N}$. Define $a * b = \max\{0, a + b - 1\}$ and $a \Diamond b = a + b - ab$ for all $a, b \in [0, 1]$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ as follows:*

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \le y, \\ \frac{y}{x} & \text{if } y \le x, \end{cases},\ N(x, y, t) = \begin{cases} \frac{y-x}{y} & \text{if } x \le y, \\ \frac{x-y}{x} & \text{if } y \le x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space.

Remark 4. *Note that, in the above example, t-norm $*$ and t-conorm $\Diamond$ are not associated. And there exists no metric d on X satisfying*

$$M(x, y, t) = \frac{t}{t + d(x, y)},\ N(x, y, t) = \frac{d(x, y)}{t + d(x, y)},$$

where $M(x, y, t)$ and $N(x, y, t)$ are as defined in above example. Also note the above functions (M, N) is not an intuitionistic fuzzy metric *with the t-norm and t-conorm defined as* $a * b = \min\{a, b\}$ *and* $a\Diamond b = \max\{a, b\}$.

Definition 4 ([12]). *Let* $(X, M, N, *, \Diamond)$ *be an intuitionistic fuzzy metric space and let* $r \in (0,1)$, $t > 0$ *and* $x \in X$. *The set*

$$B_{(M,N)}(x, r, t) = \{y \in X : M(x, y, t) > 1 - r,\ N(x, y, t) < r\}$$

is called the open ball with center x *and radius* r *with respect to* t.

Theorem 1 ([12]). *Every open ball* $B_{(M,N)}(x, r, t)$ *is an open set.*

Remark 5. *Let* $(X, M, N, *, \Diamond)$ *be an intuitionistic fuzzy metric space. Define*

$\tau_{(M,N)} = \{A \subset X :$ for each $x \in X$, there exist $t > 0$, $r \in (0, 1)$ such that $B_{(M,N)}(x, r, t) \subset A\}$. Then $\tau_{(M,N)}$ is a topology on X.

Remark 6. *(i) Since* $\{B_{(M,N)}(x, \frac{1}{n}, \frac{1}{n}) : n = 1, 2, ...\}$ *is a local base at* x, *the topology* $\tau_{(M,N)}$ *is first countable.*

(ii) Every intuitionistic fuzzy metric space is Hausdorff.

(iii) Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ be the topology on X induced by the fuzzy metric. Then for a sequence $(x_n)_n$ in X, $x_n \longrightarrow x$ if and only if $M(x_n, x, t) \longrightarrow 1$ and $N(x_n, x, t) \longrightarrow 0$ as $n \longrightarrow \infty$.

3. Best approximation

Definition 5. *An intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$ *is said to be a strong intuitionistic fuzzy metric space if for* x *in* X, $t > 0$, $y \longrightarrow M(y, x, t)$ *and* $y \longrightarrow N(y, x, t)$ *are continuous map on* X.

Remark 7. *Every standard intuitionistic fuzzy metric induced by a metric is also a strong intuitionistic fuzzy metric. We call this metric as the standard strong intuitionistic fuzzy metric induced by the metric.*

Example 3. *Let* $X = \mathbb{R}$, *the set of all real numbers. Then* M *and* N *defined as*

$$M(x, y, t) = e^{-\frac{|x-y|}{t}} \text{ and } N(x, y, t) = e^{-\frac{|x-y|}{t}} \left(e^{\frac{|x-y|}{t}} - 1 \right)$$

for all $x, y \in \mathbb{R}$ *and* $t > 0$. *Then* (M, N) *is a strong intuitionistic fuzzy metric, where* $a*b = \min\{a, b\}$ *and* $a\Diamond b = \max\{a, b\}$ *for all* $a, b \in [0, 1]$.

Definition 6. *Let* A *be a nonempty subset of an intuitionistic fuzzy metric space* $(X, M, N, *, \Diamond)$. *For* $x \in X$, $t > 0$, *let*

$$M(A, x, t) = \sup\{M(y, x, t) : y \in A\} \text{ and } N(A, x, t) = \inf\{N(y, x, t) : y \in A\}.$$

An element $y_0 \in A$ *is said to be t-best approximation to* x *from* A *if*

$$M(y_0, x, t) = M(A, x, t) \text{ and } N(y_0, x, t) = N(A, x, t).$$

Example 4. *Let $X = \mathbb{N}$. Let $a * b = ab$ and $a\Diamond b = \min\{1, a+b\}$ for all $a, b \in [0,1]$. Let M and N be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:*

$$M(x,y,t) = \begin{cases} \frac{x+t}{y+t} & \text{if } x \le y, \\ \frac{y+t}{x+t} & \text{if } y \le x, \end{cases} \quad \text{and } N(x,y,t) = \begin{cases} \frac{y-x}{y+t} & \text{if } x \le y, \\ \frac{x-y}{x+t} & \text{if } y \le x. \end{cases}$$

Then it is easy to prove that $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space. Also it is interesting to note that (M, N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a * b = \min\{a, b\}$ and $a\Diamond b = \max\{a, b\}$.

Let $A = \{2, 4, 6, ...\}$. Then

$$\begin{aligned} M(A,3,t) &= \max\left\{\frac{2+t}{3+t}, \frac{3+t}{4+t}\right\} = \frac{3+t}{4+t} = M(3,4,t), \\ N(A,3,t) &= \min\left\{\frac{1}{3+t}, \frac{1}{4+t}\right\} = \frac{1}{4+t} = N(3,4,t). \end{aligned}$$

Hence for each $t > 0$, 4 is a t-best approximation to 3 from A. As $M(3,4,t) > M(2,3,t)$ and $N(3,4,t) < N(2,3,t)$, 2 is not a t-best approximation to 3.

In fact for each odd number $k \in X$, $k+1 \in A$ is unique t-best approximation for each $t > 0$.

Remark 8. *Let (X, d) be a metric space and $(M_d,\, N_d)$ be the induced standard intuitionistic fuzzy metric. Then $y_0 \in A$ is a best approximation to $x \in X$ in the metric space if and only if y_0 is a t-best approximation to x in the induced intuitionistic fuzzy metric space $(X, M_d, N_d, *, \Diamond)$, for each $t > 0$.*

Definition 7. *Let $(X, M, N, *, \Diamond)$ is intuitionistic fuzzy metric space. A nonempty subset A of X is said to be t- approximatively compact if for each $x \in X$ and each sequence $(y_n)_n$ in X with $M(y_n, x, t) \longrightarrow M(A, x, t)$ and $N(y_n, x, t) \longrightarrow N(A, x, t)$ there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ converging to an element y in A.*

Remark 9. *(i) If A is approximatively compact in a metric space (X, d) then for each $t > 0$, A is t-approximatively compact in the induced intuitionistic fuzzy metric space.*

(ii) If A is a compact subset of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ then A is t-approximatively compact for each $t > 0$. Obviously the converse is not true.

Lemma 1. *Let A be a nonempty subset of an intuitionistic fuzzy metric space X. Then $x \in \overline{A}$, the closure of A, if and only if $M(A, x, t) = 1$ and $N(A, x, t) = 0$, for all $t > 0$.*

Proof. Suppose $x \in \overline{A}$. Since X is first countable then there exists a sequence $(x_n)_n$ in A such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. This implies that

$$M(x_n, x, t) \longrightarrow 1 \ \text{ and } \ N(x_n, x, t) \longrightarrow 0$$

as $n \longrightarrow \infty$ and for all $t > 0$. Fix a $t > 0$. Then for each $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ and $N(x_n, x, t) < \varepsilon$ for all $n \geq n_0$. Since

$$M(A, x, t) \geq M(x_n, x, t) > 1 - \varepsilon \ \text{ and } N(A, x, t) \leq N(x_n, x, t) < \varepsilon$$

then

$$1 \geq M(A, x, t) > 1 - \varepsilon \ \text{ and } \ 0 \leq N(A, x, t) < \varepsilon$$

for all $n \geq n_0$. Hence for each $\varepsilon > 0$,

$$M(A, x, t) = 1 \ \text{ and } \ N(A, x, t) = 0$$

for all $t > 0$.

Conversely suppose that $M(A, x, t) = 1$ and $N(A, x, t) = 0$ for all $t > 0$. It is easy to see that $B_{(M,N)}(x, \frac{1}{n}, \frac{1}{n}) \subseteq B_{(M,N)}(x, r, t)$ whenever $\frac{1}{n} < r, t$. Also $\{B_{(M,N)}(x, r, t) : 0 < r < 1,\ t > 0\ \}$ is also lacal base at $x \in X$. Since $M(A, x, \frac{1}{n}) = 1$ and $N(A, x, \frac{1}{n}) = 0$ for each $n \in \mathbb{N}$, then there exists a sequence $(x_n)_n$ in A such that

$$M(x_n, x, t) > 1 - \frac{1}{n} \ \text{ and } \ N(x_n, x, t) < \frac{1}{n}.$$

This implies $x_n \in B_{(M,N)}(x, \frac{1}{n}, \frac{1}{n}) \subseteq B_{(M,N)}(x, r, t)$. Hence $x_n \in B_{(M,N)}(x, r, t)$ which implies $B_{(M,N)}(x, r, t) \cap A \neq \varnothing$. Thus, $x \in \overline{A}$ for all $n \in \mathbb{N}$ and $t > 0$ which completes the proof. □

Theorem 2. *For $t > 0$, let A be a nonempty t-approximatively compact subset of a strong intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. Then for each x in X there exists y_0 in A such that*

$$M(A, x, t) = M(y_0, x, t) \text{ and } N(A, x, t) = N(y_0, x, t).$$

That is y_0 is a t-best approximation to x from A.

Proof. Since $M(A, x, t) = \sup\{M(y, x, t) : y \in A\}$ and $N(A, x, t) = \inf\{N(y, x, t) : y \in A\}$, then there exists a sequence $(y_n)_n$ in A such that

$$M(y_n, x, t) \longrightarrow M(A, x, t) \ \text{ and } \ N(y_n, x, t) \longrightarrow N(A, x, t)$$

for all $x \in X$. On the other hand A is a t-approximatively compact set implies that there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ and some y_0 in A such that $y_{n_k} \longrightarrow y_0$. Since $(X, M, N, *, \Diamond)$ is a strong intuitionistic fuzzy metric space then $y \longrightarrow M(y, x, t)$ and $y \longrightarrow N(y, x, t)$ are continuous functions which implies that

$$M(y_{n_k}, x, t) \longrightarrow M(y_0, x, t) \ \text{ and } \ N(y_{n_k}, x, t) \longrightarrow N(y_0, x, t).$$

But

$$M(y_n, x, t) \longrightarrow M(A, x, t) \ \text{ and } \ N(y_n, x, t) \longrightarrow N(A, x, t)$$

then

$$M(y_{n_k}, x, t) \longrightarrow M(A, x, t) \text{ and } N(y_{n_k}, x, t) \longrightarrow N(A, x, t).$$

Hence $M(A, x, t) = M(y_0, x, t)$ and $N(A, x, t) = N(y_0, x, t)$ which implies y_0 is a t- best approximation to x from A. □

Theorem 3. *Suppose, for some $t > 0$, A is a t-approximatively compact subset of a strong intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. Then A is closed in X.*

Proof. Let $x \in \overline{A}$. Then from Lemma 1 $M(A, x, t) = 1$ and $N(A, x, t) = 0$. Now A is t-approximatively compact implies, by Theorem 2, there exists $y \in A$ such that $M(y, x, t) = M(A, x, t)$ and $N(y, x, t) = N(A, x, t)$. Then $M(y, x, t) = 1$ and $N(y, x, t) = 0$ which implies $x = y \in A$. Therefore A is closed in X. □

Remark 10. *If A is a t-approximatively compact set then for each x in X*

$$P_A^t(x) = \{y \in A : M(A, x, t) = M(y, x, t) \text{ and } N(A, x, t) = N(y, x, t)\}$$

is a compact set.

Definition 8. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then we define a closed ball with centre $x \in X$, and radius r, $0 < r < 1$, $t > 0$ as*

$$B_{(M,N)}[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r,\ N(x, y, t) \leq r\}.$$

Lemma 2. *Every closed ball $B_{(M,N)}[x, r, t]$ is a closed set.*

Proof. Let $y \in \overline{B_{(M,N)}[x, r, t]}$. Since X is first countable, then there exist a sequence $(y_n)_n$ in $B_{(M,N)}[x, r, t]$ such that $y_n \to y$. Therefore $M(y_n, y, t) \longrightarrow 1$ and $N(y_n, y, t) \longrightarrow 0$ for all t. For a given $\varepsilon > 0$,

$$M(x, y, t + \varepsilon) \geq M(x, y_n, t) * M(y_n, y, \varepsilon)$$

and

$$N(x, y, t + \varepsilon) \leq N(x, y_n, t) \Diamond N(y_n, y, \varepsilon).$$

Hence

$$\begin{aligned} M(x, y, t + \varepsilon) &\geq \lim_n M(x, y_n, t) * \lim_n M(y_n, y, \varepsilon) \\ &\geq (1 - r) * 1 = 1 - r \end{aligned}$$

and

$$\begin{aligned} N(x, y, t + \varepsilon) &\leq \lim_n N(x, y_n, t) \Diamond \lim_n N(y_n, y, \varepsilon) \\ &\leq r \Diamond 0 = r. \end{aligned}$$

(If $M(x, y_n, t)$ and $N(x, y_n, t)$ are bounded, $(y_n)_n$ has a subsequence, which we again denote by $(y_n)_n$ for which $\lim_n M(x, y_n, t)$ and $\lim_n N(x, y_n, t)$

are exist.) In particular for $n \in \mathbb{N}$, take $\varepsilon = \frac{1}{n}$. Then $M(x, y, t + \frac{1}{n}) \geq 1 - r$ and $N(x, y, t + \frac{1}{n}) \leq r$. Hence from (IFM-6) and (IFM-11),

$$M(x, y, t) = \lim_n M\left(x, y, t + \frac{1}{n}\right) \geq 1 - r$$

and

$$N(x, y, t) = \lim_n N\left(x, y, t + \frac{1}{n}\right) \leq r.$$

Thus $y \in B_{(M,N)}[x, r, t]$ which completes the proof. □

Definition 9. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. For $t > 0$, a nonempty closed subset A of X is said to be t-boundedly compact if for each x in X, $0 < r < 1$, $B_{(M,N)}[x, r, t] \cap A$ is a compact subset of X.*

Remark 11. *Let (X, d) be a metric space and (M_d, N_d) be the standard intuitionistic fuzzy metric induced by d. Then, for $x \in X$, $0 < r < 1$, $t > 0$, $B[x, r] = \{y \in X : d(x, y) \leq r\} = B_{(M_d,N_d)}[x, 1 - \frac{t}{t+r}, t] = \{y \in X : M_d(x, y, t) \geq \frac{t}{t+r}$ and $N_d(x, y, t) \leq \frac{r}{t+r}\}$. Hence a nonempty closed set A is boundedly compact in the metric space (X, d) if and only if A is t-boundedly compact in the induced intuitionistic fuzzy metric space $(X, M_d, N_d, *, \Diamond)$ for some $t > 0$.*

Theorem 4. *If A is nonempty t-boundedly compact subset of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ then A is a t-approximatively compact set.*

Proof. For $x \in X$, let $(x_n)_n$ be a sequence in A such that

$$M(x_n, x, t) \longrightarrow M(A, x, t) \text{ and } N(x_n, x, t) \longrightarrow N(A, x, t).$$

Since $M(A, x, t) > 0$ and $N(A, x, t) < 1$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} M(A, x, t) - M(x_n, x, t) &< \frac{M(A, x, t)}{2} \text{ and} \\ N(x_n, x, t) - N(A, x, t) &< \frac{1 - N(A, x, t)}{2} \end{aligned}$$

for all $n \geq n_0$. Hence

$$M(x_n, x, t) > \frac{M(A, x, t)}{2} = 1 - r \text{ and } N(x_n, x, t) < \frac{1 + N(A, x, t)}{2} = r,$$

where $r = 1 - \frac{M(A,x,t)}{2} = \frac{1+N(A,x,t)}{2}$ $(0 < r < 1)$. Hence $x_n \in B_{(M,N)}[x, r, t] \cap A$. Since A is a t-boundedly compact set implies $B_{(M,N)}[x, r, t] \cap A$ is a compact set. Hence $(x_n)_n$ has a convergent subsequence $(x_{n_k})_k$ which converges to an element in A. Therefore A is t-approximatively compact. □

Remark 12. *In a metric space an approximatively compact set need not be compact [15]. Hence from Remark 11 it is clear that a t-approximatively compact set need not be a t-boundedly compact set in an intuitionistic fuzzy metric space.*

References

[1] C.Alaca, H.Efe, and C.Yildiz, On completion of intuitionistic fuzzy metric spaces, *Chaos, Solitons & Fractals* in press. $doi: 10.1016/j.chaos.2006.01.039$.

[2] Z. K.Deng, Fuzzy pseudo-metric spaces, *J. Math. Anal. Appl.* 86, $74-95$ (1982).

[3] M. A.Erceg, Metric spaces in fuzzy set theory, *J. Math. Anal. Appl.* 69, $205-230$ (1979).

[4] A.George and P.Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems* 64, $395-399$ (1994).

[5] A.George and P.Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy Sets and Systems* $90, 365-368$ (1997).

[6] M.Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Sets and Systems* 27, $385-389$ (1988).

[7] V.Gregori and S.Romaguera, Some Properties of fuzzy metric spaces, *Fuzzy Sets and Systems* 115, $485-489$ (2000).

[8] V.Gregori, S. Romaguera, and P.Veeramani, A note on intuitionistic fuzzy metric spaces, *Chaos, Solitons & Fractals* 28, $902-905$ (2006).

[9] O.Kaleva and S.Seikkala, On fuzzy metric spaces, *Fuzzy Sets and Systems* 12, $225-229$ (1984).

[10] O.Kramosil and J.Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica* 11, $326-334$ (1975).

[11] R.Lowen, *Fuzzy Set Theory*, Kluwer Academic Pub., Dordrecht (1996).

[12] J. H.Park, Intuitionistic fuzzy metric spaces, *Chaos, Solitons & Fractals* 22, $1039-1046$ (2004).

[13] R.Saadati and J. H.Park, On the intuitionistic topological spaces, *Chaos, Solitons & Fractals* 27, $331-344$ (2006).

[14] B.Schweizer and A.Sklar, Statistical metric spaces, *Pacific J. Math.* 10, $314-334$ (1960).

[15] I.Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag (1970).

[16] P.Veeramani, Best approximation in fuzzy metric spaces, *J. Fuzzy Math.* 9, $75-80$ (2001).

[17] L. A.Zadeh, Fuzzy sets, *Inform and Control* 8, $338-353$ (1965).

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Volterra Type Operators from Zygmund Space into Bloch Spaces

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Abstract. The boundedness and compactness of two Volterra type operators from the Zygmund space into the Bloch space and the little Bloch space are characterized in this paper.

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1 Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ its boundary. Denote by $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the α-Bloch space, denoted by $\mathcal{B}^\alpha$, if

$$B(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The space $\mathcal{B}^\alpha$ becomes a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + B(f)$. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ for which

$$(1 - |z|^2)^\alpha |f'(z)| \to 0$$

as $|z| \to 1$. This space is called the little α-Bloch space. For $\alpha = 1$, we obtain the well known classical Bloch space, simply denoted by $\mathcal{B}$. For $0 < \alpha < 1$, $\mathcal{B}^\alpha$ can be identified with the analytic Lipschitz space $\Lambda_{1-\alpha}$ (see, for example, [5, Theorem 5.1]).

Let Λ_1 denote the space of all $f \in H(\mathbb{D}) \bigcap C(\overline{\mathbb{D}})$ such that

$$\|f\|_{\Lambda_1} = \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty, \tag{1}$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$. By a theorem of Zygmund (see [5, Theorem 5.3]) and the Closed Graph Theorem we have the that $f \in \Lambda_1$ if and only if

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)|f''(z)| < \infty.$$

Moreover, the following asymptotic relation holds:

$$\|f\|_{\Lambda_1} \asymp \sup_{z\in\mathbb{D}}(1 - |z|^2)|f''(z)|. \tag{2}$$

Therefore, Λ_1 is called the Zygmund class. Since the quantities in (2) are semi norms (they do not distinguish between functions differing by a linear polynomial), it is natural to add them the quantity $|f(0)| + |f'(0)|$, to obtain two equivalent norms on the Zygmund class of functions. Zygmund class with such defined norm will be called Zygmud space. This norm will be again denoted by $\| \cdot \|_{\Lambda_1}$.

By (2) we have

$$|f'(z) - f'(0)| \le C\|f\|_{\Lambda_1} \log \frac{1}{1 - |z|}. \tag{3}$$

Also, we have

$$\begin{aligned}
|f(z) - f(0) - zf'(0)| &= \left| \int_0^z \int_0^1 f''(t\zeta)\zeta dt d\zeta \right| \\
&\le \|f\|_{\Lambda_1} \left| \int_0^z \int_0^1 \frac{|\zeta| dt}{1 - t|\zeta|} |d\zeta| \right| \\
&\le \|f\|_{\Lambda_1} \left| \int_0^{|z|} \ln \frac{1}{(1 - s)} ds \right| \\
&= \|f\|_{\Lambda_1} \Big(|z| + (|z| - 1) \ln \frac{1}{1 - |z|} \Big),
\end{aligned}$$

for every $z \in \mathbb{D}$. From this and since the quantity

$$\sup_{x\in[0,1)} \Big(x + (x - 1) \ln \frac{1}{1 - x} \Big)$$

is bounded, it follows that

$$\|f\|_\infty \le C\|f\|_{\Lambda_1}, \tag{4}$$

for every $f \in \Lambda_1$, and for some positive constant C independent of f.

Suppose that $g : \mathbb{D} \to \mathbb{C}$ is a holomorphic map, $f \in H(\mathbb{D})$. The integral operator, called a Volterra type operator,

$$J_g f(z) = \int_0^z f dg = \int_0^1 f(tz)zg'(tz)dt = \int_0^z f(\xi)g'(\xi)d\xi, \quad z \in \mathbb{D} \tag{5}$$

was introduced by Pommerenke in [12].

The companion operator I_g with symbol g are defined by

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi,$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. The importance of the operators J_g and I_g comes from the fact that

$$J_g f + I_g f = M_g f - f(0)g(0),$$

where M_g is the multiplication operator

$$(M_g f)(z) = g(z)f(z), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

In addition, J_g can be viewed as a generalization of the Cesàro operator.

In [12] Pommerenke showed that J_g is a bounded operator on the Hardy space H^2 if and only if $g \in BMOA$. The boundedness and compactness of J_g and I_g and their n-dimensional extensions, between various spaces of holomorphic functions have been studied recently (see, for example, [1, 2, 6, 7, 8, 9, 13, 14, 15] and the related references therein).

Here we study the boundedness and compactness of integral operators J_g and I_g from the Zygmund space into the Bloch space and little Bloch space.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \leq Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2 Main Results

In order to investigate the compactness of operators J_g and I_g, which map a space into $\mathcal{B}_0$, we need the following lemma, which was proved in [10] (see, also, [11]).

Lemma 1. *A closed set K in $\mathcal{B}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1 - |z|^2)|f'(z)| = 0.$$

The following lemma can be proved in a standard way (see, for example, [4, Theorem 3.11]).

Lemma 2. *The operator J_g (or I_g) : $\Lambda_1 \to \mathcal{B}$ is compact if and only if J_g (or I_g) is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in Λ_1 which converges to* *zero uniformly on compact subsets of* $\mathbb{D}$ *as* $k \to \infty$*, we have* $\|J_g f_k\|_{\mathcal{B}} \to 0$ $($*or* $\|I_g f_k\|_{\mathcal{B}} \to 0)$ *as* $k \to \infty$.

Now we are in a position to formulate and prove the main results of this section.

2.1 Boundedness of the operators $J_g; I_g : \Lambda_1 \to \mathcal{B}\,(\text{or}\,\mathcal{B}_0)$

In this subsection we study the boundedness of the operators $J_g; I_g : \Lambda_1 \to \mathcal{B}\,(\text{or}\,\mathcal{B}_0)$.

Theorem 1. *Suppose that* g *is an analytic function on* $\mathbb{D}$*. Then*

(a) $I_g : \Lambda_1 \to \mathcal{B}$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|g(z)|\log\frac{1}{1-|z|^2} < \infty. \tag{6}$$

(b) $I_g : \Lambda_1 \to \mathcal{B}_0$ *is bounded if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)|g(z)|\log\frac{1}{1-|z|^2} = 0. \tag{7}$$

Proof. (a) Suppose that $f \in \Lambda_1$ and (6) holds. Then by (3) we have

$$\begin{aligned}\|I_g f\|_{\mathcal{B}} &= \sup_{z\in\mathbb{D}}(1-|z|^2)|(I_g f)'(z)| = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)||g(z)|\\ &\le C\|f\|_{\Lambda_1}\sup_{z\in\mathbb{D}}(1-|z|^2)|g(z)|\log\frac{1}{1-|z|^2},\end{aligned}$$

which implies that the operator $I_g : \Lambda_1 \to \mathcal{B}$ is bounded.

Conversely, assume that $I_g : \Lambda_1 \to \mathcal{B}$ is bounded. Let

$$h(z) = (z-1)\left[\left(1+\log\frac{1}{1-z}\right)^2 + 1\right]$$

and

$$f_a(z) = \frac{h(\bar{a}z)}{\bar{a}}\left(\log\frac{1}{1-|a|^2}\right)^{-1} \tag{8}$$

for $a \in \mathbb{D}$. Then, we have

$$f_a'(z) = \left(\log\frac{1}{1-\bar{a}z}\right)^2\left(\log\frac{1}{1-|a|^2}\right)^{-1} \tag{9}$$

and

$$f_a''(z) = \frac{2\bar{a}}{1-\bar{a}z}\left(\log\frac{1}{1-\bar{a}z}\right)\left(\log\frac{1}{1-|a|^2}\right)^{-1}.$$

Thus for $1/\sqrt{2} < |a| < 1$, we have

$$|f_a''(z)| \le \frac{2}{1-|z|}\left(\log\frac{1}{1-|a|} + C\right)\left(\log\frac{1}{1-|a|^2}\right)^{-1} \le \frac{C}{1-|z|} \tag{10}$$

and consequently

$$M_1 = \sup_{1/\sqrt{2}<|a|<1} \|f_a\|_{\Lambda_1} < \infty.$$

From (9) and the following well known estimate (see, for example, [16])

$$\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \asymp \frac{1}{(1-|z|^2)^2} \asymp \frac{1}{(1-|a|^2)^2} \asymp \frac{1}{|D(a,r)|},$$

when $z \in D(a,r)$ and where $|D(a,r)|$ denotes the area of the Bergman disk $D(a,r)$, we have that

$$\begin{aligned}
& \left(\log\frac{1}{1-|a|^2}\right)^2 |g(a)|^2 = |f_a'(a)g(a)|^2 \\
\le\ & \frac{C}{(1-|a|^2)^2}\int_{D(a,r)} |f_a'(z)|^2|g(z)|^2 dA(z) \\
=\ & \frac{C}{(1-|a|^2)^2}\int_{D(a,r)} |f_a'(z)|^2|g(z)|^2(1-|z|^2)^2\frac{1}{(1-|z|^2)^2} dA(z) \\
\le\ & C\int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^4} \sup_{z\in D(a,r)} (1-|z|^2)^2|f_a'(z)|^2|g(z)|^2 \\
\le\ & \frac{C}{(1-|a|^2)^2}\|I_g f_a\|_{\mathcal{B}}^2,
\end{aligned}$$

which implies that

$$(1-|a|^2)|g(a)|\log\frac{1}{1-|a|^2} \le C\|I_g f_a\|_{\mathcal{B}} \le CM_1\|I_g\|_{\Lambda_1\to\mathcal{B}} \tag{11}$$

for every $1/\sqrt{2} < |a| < 1$.

On the other hand, we have that

$$\begin{aligned}
& \sup_{|a|\le 1/\sqrt{2}} (1-|a|^2)|g(a)|\log\frac{1}{1-|a|^2} \\
\le\ & \frac{4}{3}\ln 2 \max_{|a|=1/\sqrt{2}} |g(a)| \\
\le\ & \sup_{1/\sqrt{2}\le|a|<1} (1-|a|^2)|g(a)|\log\frac{1}{1-|a|^2}.
\end{aligned} \tag{12}$$

From (11) and (12) we obtain (6).

(b) First assume that condition (7) does not hold. If it were, then it would exist $\varepsilon_0 > 0$ and a sequence $(z_k)_{k\in\mathbb{N}} \in \mathbb{D}$, such that $\lim_{k\to\infty}|z_k| = 1$ and

$$(1-|z_k|^2)|g(z_k)|\log\frac{1}{1-|z_k|^2} \ge \varepsilon_0 > 0$$

for sufficiently large k. We may assume also

$$\frac{1-|z_{k-1}|}{2} > 1-|z_k|, \qquad k \in \mathbb{N}.$$

Then, for every non-negative integer s there is at most one z_k such that $1-\frac{1}{2^s} \leq |z_k| < 1-\frac{1}{2^{s+1}}$. Hence, there is $m_0 \in \mathbb{N}$ such that for any Carleson window

$$Q = \{re^{i\theta} \mid 0 < 1-r < l(Q),\ |\theta-\theta_0| < l(Q)\}$$

and $s \in \mathbb{N}$, there is at most m_0 elements in

$$\{z_k \in Q \mid 2^{-(s+1)}l(Q) < 1-|z_k| < 2^{-s}l(Q)\}.$$

Therefore, $(z_k)_{k\in\mathbb{N}}$ is an interpolating sequence for $\mathcal{B}$, in sense of [3].

By [3] we have that there is an $h \in \mathcal{B}$ such that

$$h(z_k) = \log\frac{1}{1-|z_k|^2}, \qquad k \in \mathbb{N}.$$

Let $f(z) = \int_0^z h(\xi)d\xi$. Then from the definition of Bloch functions and Zygmund functions, we see that $f \in \Lambda_1$. We obtain

$$\begin{aligned}(1-|z_k|^2)|(I_gf)'(z_k)| &= (1-|z_k|^2)|f'(z_k)||g(z_k)| \\ &= (1-|z_k|^2)|h(z_k)||g(z_k)| \\ &= (1-|z_k|^2)|g(z_k)|\log\frac{1}{1-|z_k|^2} \geq \varepsilon_0.\end{aligned}$$

Thus, $I_gf \notin \mathcal{B}_0$, which is a contradiction.

Conversely, suppose that (7) holds. Then by (a) we see that $I_g : \Lambda_1 \to \mathcal{B}$ is bounded. Since, for any $f \in \Lambda_1$, in view of (3), we have

$$(1-|z|^2)|(I_gf)'(z)| \leq C\|f\|_{\Lambda_1}(1-|z|^2)|g(z)|\log\frac{1}{1-|z|^2},$$

by (7), it follows that $I_gf \in \mathcal{B}_0$. Since $\mathcal{B}_0$ is closed subset of $\mathcal{B}$, we obtain $I_g\Lambda_1 \subset \mathcal{B}_0$. Therefore $I_g : \Lambda_1 \to \mathcal{B}_0$ is bounded.

Theorem 2. *Suppose that g is an analytic function on $\mathbb{D}$. Then,*

(a) $J_g : \Lambda_1 \to \mathcal{B}$ is bounded if and only if $g \in \mathcal{B}$.

(b) $J_g : \Lambda_1 \to \mathcal{B}_0$ is bounded if and only if $g \in \mathcal{B}_0$.

Proof. *(a)* By (4), we have that

$$\begin{aligned}\|J_gf\|_{\mathcal{B}} &= \sup_{z\in\mathbb{D}}(1-|z|^2)|(J_gf)'(z)| = \sup_{z\in\mathbb{D}}(1-|z|^2)|f(z)||g'(z)| \\ &\leq C\sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)|\|f\|_{\Lambda_1}.\end{aligned}$$

Therefore $g \in \mathcal{B}$ implies that J_g is a bounded operator from Λ_1 to $\mathcal{B}$.

Conversely, suppose J_g is a bounded operator from Λ_1 to $\mathcal{B}$. Taking the function $f = 1$, we obtain $g \in \mathcal{B}$.

(b) If we assume that $J_g : \Lambda_1 \to \mathcal{B}_0$ is bounded, then by taking $f = 1$ we obtain $g \in \mathcal{B}_0$.

Conversely, suppose that $g \in \mathcal{B}_0$. Then from (a) we see that J_g is a bounded operator from Λ_1 to $\mathcal{B}$. Since $g \in \mathcal{B}_0$, for any $f \in \Lambda_1$ we have that

$$\begin{aligned}\lim_{|z|\to 1}(1-|z|^2)|(J_g f)'(z)| &= \lim_{|z|\to 1}(1-|z|^2)|g'(z)|\,|f(z)| \\ &\leq \|f\|_{\Lambda_1} \lim_{|z|\to 1}(1-|z|^2)|g'(z)| = 0,\end{aligned}$$

i.e. $J_g f \in \mathcal{B}_0$. Since $\mathcal{B}_0$ is closed subset of $\mathcal{B}$, we obtain $J_g \Lambda_1 \subset \mathcal{B}_0$. Therefore $J_g : \Lambda_1 \to \mathcal{B}_0$ is bounded. This finishes the proof of the theorem.

2.2 Compactness of the operators $J_g; I_g : \Lambda_1 \to \mathcal{B}\,(\text{or}\,\mathcal{B}_0)$

Now, we investigate the compactness of the operators $J_g; I_g : \Lambda_1 \to \mathcal{B}\,(\text{or}\,\mathcal{B}_0)$.

Theorem 3. *Assume that g is an analytic function on $\mathbb{D}$. Then the following statements are equivalent:*

(a) $I_g : \Lambda_1 \to \mathcal{B}$ *is compact;*

(b) $I_g : \Lambda_1 \to \mathcal{B}_0$ *is compact;*

(c)

$$\lim_{|z|\to 1}(1-|z|^2)|g(z)| \log \frac{1}{1-|z|^2} = 0. \tag{13}$$

Proof. $(c) \Rightarrow (b)$. From Lemma 1, we know that $I_g : \Lambda_1 \to \mathcal{B}_0$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{\Lambda_1}\leq 1}(1-|z|^2)|(I_g f)'(z)| = 0. \tag{14}$$

We have

$$(1-|z|^2)|(I_g f)'(z)| \leq C\|f\|_{\Lambda_1}(1-|z|^2)|g(z)| \log \frac{1}{1-|z|^2}. \tag{15}$$

Taking the supremum in (15) over the the unit ball of the space Λ_1, then letting $|z| \to 1$, we obtain that (14) holds, as desired.

$(b) \Rightarrow (a)$ is clear.

$(a) \Rightarrow (c)$ Now we assume that the operator $I_g : \Lambda_1 \to \mathcal{B}$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|z_k| \to 1$ as $k \to \infty$. Set

$$f_k(z) = \frac{(1-\bar{z}_k z)}{\bar{z}_k}\left[\left(1+\log\frac{1}{1-\bar{z}_k z}\right)^2 + 1\right]\left(\log\frac{1}{1-|z_k|^2}\right)^{-1}, \tag{16}$$

$k \in \mathbb{N}$. Then in view of (10), we have that $\sup_{k\in\mathbb{N}} \|f_k\|_{\Lambda_1} \le C$ and that f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $I_g : \Lambda_1 \to \mathcal{B}$ is compact, we have

$$\|I_g f_k\|_{\mathcal{B}} \to 0 \quad \text{as} \quad k \to \infty$$

by Lemma 2. Thus

$$\begin{aligned}(1-|z_k|^2)|g(z_k)| \log \frac{1}{1-|z_k|^2} &\le \sup_{z\in\mathbb{D}} (1-|z|^2)|g(z)||f_k'(z)| \\ &= \sup_{z\in\mathbb{D}} (1-|z|^2)|(I_g f_k)'(z)| \\ &= \|I_g f_k\|_{\mathcal{B}} \to 0\end{aligned}$$

as $k \to \infty$. Since $(z_k)_{k\in\mathbb{N}}$ is an arbitrary sequence such that $|z_k| \to 1$ as $k \to \infty$, (13) follows.

Theorem 4. *Assume that g is an analytic function on $\mathbb{D}$. Then,*

(a) $J_g : \Lambda_1 \to \mathcal{B}_0$ is compact if and only if $g \in \mathcal{B}_0$.

(b) $J_g : \Lambda_1 \to \mathcal{B}$ is compact if and only if $g \in \mathcal{B}$.

Proof. *(a)* From Lemma 1, we know that $J_g : \Lambda_1 \to \mathcal{B}_0$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{\Lambda_1} \le 1} (1-|z|^2)|(J_g f)'(z)| = 0. \tag{17}$$

By (4) we have that

$$(1-|z|^2)|(J_g f)'(z)| \le C(1-|z|^2)|g'(z)|\|f\|_{\Lambda_1}. \tag{18}$$

Taking the supremum in (18) over the unit ball of the space Λ_1, then letting $|z| \to 1$, we obtain that condition (17) holds, from which the compactness of $J_g : \Lambda_1 \to \mathcal{B}_0$ follows.

Conversely, suppose that $J_g : \Lambda_1 \to \mathcal{B}_0$ is compact. Then it is clear that $J_g : \Lambda_1 \to \mathcal{B}_0$ is bounded. From this and Theorem 2 we see that $g \in \mathcal{B}_0$.

(b) Suppose that $J_g : \Lambda_1 \to \mathcal{B}$ is compact. Then it is clear that J_g is bounded, and by Theorem 2 we have that $g \in \mathcal{B}$.

Conversely, assume $g \in \mathcal{B}$. Let $(f_k)_{k\in\mathbb{N}}$ be a bounded sequence in Λ_1 and convergent to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Now note that $f_k \in \Lambda_1 \subset A(\mathbb{D})$, $k \in \mathbb{N}$, where $A(\mathbb{D})$ is the disk algebra, consisting of all functions in $H(\mathbb{D})$ that are continuous up to the boundary of the unit disk. It can be easily proved that $\lim_{k\to\infty} \sup_{z\in\mathbb{D}} |f_k(z)| = 0$. Hence, we have that

$$\|J_g f_k\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} (1-|z|^2)|g'(z)||f_k(z)| \le \|g\|_{\mathcal{B}} \sup_{z\in\mathbb{D}} |f_k(z)| \to 0, \tag{19}$$

as $k \to \infty$. Therefore, $J_g : \Lambda_1 \to \mathcal{B}$ is compact.

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References

[1] A. Aleman and A. G. Siskakis, An integral operator on H^p, *Complex Variables*, **28** (1995), 140-158.

[2] A. Aleman and A. G. Siskakis, Integral operators on Bergman spaces, *Indiana Univ. Math. J.* **46** (1997), 337-356.

[3] B. Boe and A. Nikolau, Interpolation by functions in the Bloch space, *J. Anal. Math.* **94** (2004), 171-194.

[4] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Math., CRC Press, Boca Raton, 1995.

[5] P. L. Duren, *Theory of H^p Spaces*, Academic press, New York, 1970.

[6] S. Li, Riemann-Stieltjes operators between Bergman type spaces and α-Bloch spaces, *Int. J. Math. Math. Sci.* Volume 2006, Article ID 86259, Pages 1-17.

[7] S. Li, Riemann-Stieltjes operators from $F(p, q, s)$ to Bloch space on the unit ball, *J. Ineq. Appl.* Volume 2006, Article ID 27874, Pages 1-14.

[8] S. Li and S. Stević, Reimann-Stieltjies type integral operators on the unit ball in $\mathbb{C}^n$, *Complex Variables Elliptic Functions* **2** (2007) (to appear).

[9] S. Li and S. Stević, Reimann-Stieltjies operators on Hardy spaces in the unit ball of $\mathbb{C}^n$, *Bull. Belg. Math. Soc. Simon Stevin* (to appear).

[10] K. Madigan and A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347** (7) (1995), 2679-2687.

[11] S. Ohno, K. Stroethoff and R. Zhao, Weighted composition operators between Bloch-type spaces. *Rocky Mountain J. Math.* **33** (1) (2003), 191-215.

[12] C. Pommerenke, Schlichte funktionen und analytische funktionen von beschränkter mittlerer oszillation, *Comment. Math. Helv.* **52** (1977), 591-602.

[13] A. G. Siskakis and R. Zhao, A Volterra type operator on spaces of analytic functions, *Contemp. Math.* **232** (1999), 299-311.

[14] S. Stević, On an integral operator on the unit ball in $\mathbb{C}^n$, *J. Inequal. Appl.* **2005** (1) (2005), 81-88.

[15] S. Stević, Boundedness and compactness of an integral operator on a weighted space on the polydisc, *Indian J. Pure Appl. Math.* **37** (6) (2006), 343-355.

[16] K. Zhu, *Operator Theory on Function Spaces*, Marcel Dekker, Inc. Pure and Applied Mathematics 139, New York and Basel, 1990.

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Localizing Entropies via an Open Cover†

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Abstract. For a given topological dynamical system (X, T), we introduce and study some entropies of open covers. The main result as follows: (1)the inequality

$$h_p(T,\mathcal{U}) \leqslant h_m(T,\mathcal{U}) \leqslant h_{top}(T,\mathcal{U}) \leqslant h_m(T,\mathcal{U}) + h_b(T),$$

relating pointwise pre-image entropies of open covers $h_p(T,\mathcal{U}), h_m(T,\mathcal{U})$, branch pre-image entropy $h_b(T)$ and the topological entropy of open covers $h_{top}(T,\mathcal{U})$. (2) pseudo-entropy of open covers equals topological entropy of open covers. (3)pseudo-point-entropies of open covers equal topological entropy of open covers.
Keywords. Topological entropy, Pointwise preimage entropy, Pseudo-orbits
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1 Introduction

Entropies are fundamental to our current understanding of dynamical systems. The topological entropy $h_{top}(T)$ for a continuous map T of a compact metric space X into itself is a measure of its dynamical complexity, which was first defined by Adler, Konheim and McAndrew [1], and later given several equivalent definitions by Bowen and others [2,3]. The topological entropy measures the maximal exponential growth rate of orbits for an arbitrary topological dynamical systems.

When the mapping T under consideration is a homeomorphism it is well known that T and the inverse mapping T^{-1} have the same topological entropy. However, when the map is not invertible, different ways of extending the procedure into the past lead to several new entropy-like invariants for non-invertible maps.

Recently, the pre-image relation entropy $h_r(T)$ of a compact metric space has been introduced by Langevin and Walczak [5], which was shown to be a new tool for studying the topology and dynamics of compact metric spaces. Later, several important pre-image entropy invariants, such as pointwise pre-image entropy, pointwise branch entropy, partial pre-image entropy and bundle-like pre-image entropy, etc., have been introduced and their relationships with topological entropy have been established(see [6-11]). In particular M.Hurley [7] established a beautiful inequality relating pointwise, branch and topological entropy. In section 2 of this paper, we study the localization of Hurley's inequalities, which is a direct generalization.

A pseudo-orbit is an important concept in the theory of dynamical systems. In reality, it is difficult to find a real orbit in the system. Usually a pseudo-orbit can be used to approximate the real orbit, and so it is a useful tool to study the dynamics of dynamical systems.

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In an earlier paper with M.Barge and R.Swanson [12], they have given one of the equivalent definitions of topological entropy involves counting segments of pseudo-orbits instead of orbit segments. In section 3, we shall define and study pseudo-entropy of an open cover and show that it equals topological entropy of the open cover. In [7], M.Huley considered pseudo-orbits for inverse and shown that point entropy with pseudo-orbits equals topological entropy. In section 4, we shall consider the localization of pseudo-point-entropies.

2 The localization of Hurley's inequalities

Let (X,T) be a topological dynamical system. This means that X is a compact metric space, that d denotes the metric, and $T : X \to X$ is a continuous map from X to itself. A open cover of X is a family of open sets of X, whose union is X. Denote the set of open covers by $\mathcal{C}_X^o$. Given two open covers $\mathcal{U},\mathcal{V} \in \mathcal{C}_X^o$, let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. If $M, N \in \mathbb{N}$(where $\mathbb{N}$ denote the set of all positive integers) with $M \leqslant N$ and $\mathcal{U} \in \mathcal{C}_X$, we use the notation $\mathcal{U}_M^N = \bigvee\limits_{n=M}^{N} T^{-n}\mathcal{U}$.

Given $\mathcal{U} \in \mathcal{C}_X^o$ and $K \subset X$, put

$$N(\mathcal{U}|K) = \min\{card(\mathcal{F}) : \mathcal{F} \subset \mathcal{U}, \bigcup_{F\in\mathcal{F}} F \supset K\}.$$

We write $N(\mathcal{U}|X)$ simply by $N(\mathcal{U})$. Then the topological entropy of $\mathcal{U}$ with respect to T is defined as

$$h_{top}(T,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}) = \inf_{n\geqslant 1} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}),$$

and the pointwise preimage entropies of $\mathcal{U}$ with respect to T is defined as

$$h_p(T,\mathcal{U}) = \sup_{x\in X} \limsup_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}|T^{-n}(x)),$$

$$h_m(T,\mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} N(\mathcal{U}_0^{n-1}|T^{-n}(x)).$$

We have the trivial inequalities $h_p(T,\mathcal{U}) \leqslant h_m(T,\mathcal{U}) \leqslant h_{top}(T,\mathcal{U})$. Moreover, it is not hard to see that $h_p(T,\mathcal{U}) = h_m(T,\mathcal{U}) = 0$ when T is a homeomorphism.

Given $x \in X$ let $T_n(x)$ denote the tree of inverse images of x up to order n, which is defined by

$$T_n(x) = \{[z_0, z_1, \cdots, z_n] : z_0 = x, T(z_j) = z_{j-1}, \forall 1 \leqslant j \leqslant n\}.$$

The root point of the tree $T_n(x)$ is x. Each ordered set $[z_0, z_1, \cdots, z_n]$ contained in $T_n(x)$ is called a branch of $T_n(x)$. Let $T_n = \bigcup\limits_{x\in X} T_n(x)$. We define a metrics on T_n as follows. Let $A = [a_0, a_1, \cdots, a_n], B = [b_0, b_1, \cdots, b_n] \in T_n$, the branch distance is defined as

$$d_n(A,B) = \max_{0\leqslant j\leqslant n} d(a_j, b_j).$$

Let $O_n = \{T_n(x) : x \in X\}$. For each two elements $T_n(x)$ and $T_n(y)$ of O_n, denote by $d_H(T_n(x), T_n(y))$ the usual Hausdorff metric between them based upon metric d_n. A subset Z_n of O_n is said to be d_H-(n,ϵ)-separated if for any two distinct trees $T_n(x), T_n(y) \in Z_n$, we have $d_H(T_n(x), T_n(y)) > \epsilon$. Let $t(n,\epsilon)$ denote the maximal cardinality of any d_H-(n,ϵ)-separated subset of O_n. Define the entropy by

$$h_b(T) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log t(n,\epsilon),$$

which is called the branch preimage entropy of T.

In [7], M.Huley established a beautiful inequality relating pointwise, branch and topological entropy. Now we consider the localization of Hurley's inequalities.

Theorem 2.1 *Let (X,T) be a topological dynamical system. For each $\mathcal{U}\in\mathcal{C}_X^o$, we have*

$$h_p(T,\mathcal{U}) \leqslant h_m(T,\mathcal{U}) \leqslant h_{top}(T,\mathcal{U}) \leqslant h_m(T,\mathcal{U}) + h_b(T).$$

Proof. Obviously, $h_p(T,\mathcal{U}) \leqslant h_m(T,\mathcal{U}) \leqslant h_{top}(T,\mathcal{U})$. We show that $h_{top}(T,\mathcal{U}) \leqslant h_m(T,\mathcal{U})+ h_b(T)$. Let $\epsilon > 0$ be a Lebesgue number of $\mathcal{U}$. Let Y denote a maximal d_H-$(n,\epsilon/3)$-separated set of O_n and let Z denote the set of all root points of trees in Y. For each $z\in Z$ let $M(z,\mathcal{U})$ be a subcover of $\mathcal{U}_0^{n-1}$ with respect to $T^{-n}(z)$ and the cardinality of it is $N(\mathcal{U}_0^{n-1}|T^{-n}(z))$. Set

$$M = \bigcup_{z\in Z} M(z,\mathcal{U}).$$

Claim. *M is an open cover of X.*

In fact, let $x\in X$ and let $w = f^n(x)$. Since Y is a maximal d_H-$(n,\epsilon/3)$-separated set of O_n, there is a tree $T_n(y)\in Y$ such that $d_H(T_n(w), T_n(y)) \leqslant \epsilon/3$. Now consider the branch B_1 of $T_n(w)$ whose end is x, i.e.

$$B_1 = [w = f^n(x), f^{n-1}(x), \cdots, f(x), x].$$

Hence there exists a branch $B_2 = [y_0 = y, y_1, \cdots, y_n]$ of $T_n(y)$ such that $d_n(B_1, B_2) \leqslant \epsilon/3$. This means that $d(T^j(y_n), T^j(x)) \leqslant \epsilon/3$ for each $0 \leqslant j \leqslant n$. By the definition of ϵ, there exists $A\in M(y,\mathcal{U})$ such that $x\in A$. It means M is an open cover of X. This completes the proof of the claim.

Using the claim, we get

$$N(\mathcal{U}_0^{n-1}) \leqslant |M|,$$

where $|M|$ denotes the cardinality of M. Hence

$$\begin{aligned} N(\mathcal{U}_0^{n-1}) &\leqslant |M| \leqslant |Z|\cdot \sup_{x\in X} N(\mathcal{U}_0^{n-1}|T^{-n}(x)) \\ &= t(n,\epsilon/3)\cdot \sup_{x\in X} N(\mathcal{U}_0^{n-1}|T^{-n}(x)). \end{aligned}$$

Therefore,

$$\begin{aligned} h_{top}(T,\mathcal{U}) &\leqslant h_m(T,\mathcal{U}) + \limsup_{n\to\infty}\frac{1}{n}\log t(n,\epsilon/3) \\ &\leqslant h_m(T,\mathcal{U}) + h_b(T). \end{aligned}$$

This completes the proof. $\square$

By Theorem 2.1, we can direct obtain Hurley's inequalities.

Theorem 2.2 *(Hurley's inequalities[7]) For any topological dynamical system* (X,T),

$$h_m(T) \leqslant h_{top}(T) \leqslant h_m(T) + h_b(T).$$

3 The localization of pseudo-entropy

Topological entropy has been characterized by Swanson (see [11]) and Misiurewicz (see [13]) in terms of growth rates of pseudo-orbit. In this section, we consider the localization of pseudo-entropy.

Let (X,T) be a topological dynamical system. Let X^n denote the n-fold Cartesian product of $X (n \in \mathbb{N})$. For $\alpha > 0$, an α-pseudo-orbit for T of length n is a point $x = (x_0, x_1, \cdots, x_{n-1}) \in X^n$ with the property that $d(T(x_{j-1}), x_j) < \alpha$ for all $1 \leqslant j \leqslant n-1$. Let $\Psi_n(\alpha) \subset X^n$ denote all α-pseudo-orbit of length n. Given $\mathcal{U} \in \mathcal{C}_X^o$ and $\alpha > 0$. Let

$$\mathcal{U}^n = \{[U_0, U_1, \cdots, U_{n-1}] : U_j \in \mathcal{U}, 0 \leqslant j \leqslant n-1\},$$

where

$$[U_0, U_1, \cdots, U_{n-1}] = \{(u_0, u_1, \cdots, u_{n-1}) \in X^n : u_j \in U_j, 0 \leqslant j \leqslant n-1\}.$$

Obviously, $\mathcal{U}^n$ is an open cover of X^n. Set $N(\mathcal{U}, n, \alpha) = N(\mathcal{U}^n | \Psi_n(\alpha))$. It is easy to see that $N(\mathcal{U}, n+m, \alpha) \leqslant N(\mathcal{U}, n, \alpha) \cdot N(\mathcal{U}, m, \alpha)$. Now we put

$$h_\Psi(T, \mathcal{U}, \alpha) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}, n, \alpha) = \inf_{n\geqslant 1} \frac{1}{n} \log N(\mathcal{U}, n, \alpha),$$

and define the pseudo-entropy of $\mathcal{U}$ with respect to T as

$$h_\Psi(T, \mathcal{U}) = \lim_{\alpha\to 0} h_\Psi(T, \mathcal{U}, \alpha) = \inf_{\alpha>0} h_\Psi(T, \mathcal{U}, \alpha).$$

Theorem 3.1 *Let* (X,T) *be a topological dynamical system and* $\mathcal{U} \in \mathcal{C}_X^o$. *Then*

$$h_{top}(T, \mathcal{U}) = h_\Psi(T, \mathcal{U}).$$

Proof. It is obviously that $h_{top}(T, \mathcal{U}) \leqslant h_\Psi(T, \mathcal{U})$. Now we show the reverse inequality.

It follows from the definition of the pseudo-orbit entropy of a cover that, for each $\alpha > 0$ and $n \in \mathbb{N}$,

$$h_\Psi(T, \mathcal{U}, \alpha) \leqslant \frac{1}{n} \log N(\mathcal{U}, n, \alpha).$$

Since $N(\mathcal{U}, n, \alpha)$ decreases as $\alpha \to 0$ and equals $N(\mathcal{U}_0^{n-1})$, we get

$$\inf_{0<\alpha\leqslant 1} h_\Psi(T, \mathcal{U}, \alpha) \leqslant \inf_{0<\alpha\leqslant 1} \frac{1}{n} \log N(\mathcal{U}, n, \alpha) = \frac{1}{n} \log N(\mathcal{U}_0^{n-1}).$$

Letting $n \to \infty$, we have $h_\Psi(T, \mathcal{U}) \leqslant h_{top}(T, \mathcal{U})$. This complete the proof of theorem. □

Remark 3.2: We define the pseudo-entropy of T as $h_\Psi(T) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_\Psi(T, \mathcal{U})$. In [12], Pseudo-entropy was defined using separating set. Using the standard techniques of topological entropy(see [4], for example), we can get that open covers and separate set define same pseudo-entropy. Therefore, we can give a simpler proof of Theorem 1 in [12] by Theorem 3.1.

4 The localization of pseudo-point-entropies

In [7], Hurley showed that point entropy with pseudo-orbit equals topological entropy. In this section, we localize the point entropy with pseudo-orbit via an open cover. To make this precise, consider the following definitions.

Given $x \in X$. Let $\Psi_n(\alpha, x) \subset X^{n+1}$ denote the collection of all α-pseudo-orbits of length $n+1$ that end at x. That is an element of $\Psi_n(\alpha, x)$ is an α-pseudo-orbit $(y_0, y_1, \cdots, y_n)$ with $y_n = x$. Set

$$N(\mathcal{U}, n, \alpha, x) = N(\mathcal{U}^{n+1}|\Psi_n(\alpha, x)),$$

and let $\widetilde{N}(\mathcal{U}, n, \alpha)$ denote the quantity $\max_{x\in X} N(\mathcal{U}, n, \alpha, x)$. Clearly,

$$N(\mathcal{U}, n, \alpha, x) \leqslant \widetilde{N}(\mathcal{U}, n, \alpha) \leqslant N(\mathcal{U}, n, \alpha)$$

for each $x \in X, \alpha > 0$ and $\mathcal{U} \in \mathcal{C}_X^o$. Define the entropies

$$\begin{aligned} h_{p,\Psi}(T, \mathcal{U}) &= \sup_{x\in X} \lim_{\alpha\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}, n, \alpha, x) \\ &= \sup_{x\in X} \inf_{\alpha>0} \limsup_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}, n, \alpha, x), \end{aligned}$$

and

$$\begin{aligned} h_{m,\Psi}(T, \mathcal{U}) &= \lim_{\alpha\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \widetilde{N}(\mathcal{U}, n, \alpha) \\ &= \inf_{\alpha>0} \limsup_{n\to\infty} \frac{1}{n} \log \widetilde{N}(\mathcal{U}, n, \alpha), \end{aligned}$$

which are called pseudo-point-entropies of $\mathcal{U}$ with respect to T.

Theorem 4.1 *Let (X, T) be a topological dynamical system and $\mathcal{U} \in \mathcal{C}_X^o$. Then*

$$h_{top}(T, \mathcal{U}) = h_{m,\Psi}(T, \mathcal{U}).$$

Proof. Let $\epsilon > 0$ is a Lebesgue number of $\mathcal{U}$ and $0 < \alpha \leqslant \epsilon/6$.

Claim: *There is a constant $K = K(\alpha)$ such that for each $n \in \mathbb{N}$,*

$$N(\mathcal{U}, n, \alpha) \leqslant K\widetilde{N}(\mathcal{U}, n, \alpha).$$

In fact, selecting a finite α-dense subset of X, $\{x_1, x_2, \cdots, x_K\}$. For each x_i let $M(x_i, \mathcal{U}, \alpha)$ be a subcover of $\mathcal{U}^{n+1}$ with respect to $\Psi_n(\alpha, x_i)$ and the cardinality of it is $N(\mathcal{U}, n, \alpha, x_i)$. Set

$$M = \bigcup_{i=1}^{K} M(x_i, \mathcal{U}, \alpha).$$

Now we slow that M is an open cover of $\Psi_{n+1}(\alpha)$.

Let $y = (y_0, y_1, \cdots, y_n) \in \Psi_{n+1}(\alpha)$ and select one of the points x_i satisfying $d(T(y_{n-1}), x_i) < \alpha$. Thus

$$z = (z_0, z_1, \cdots, z_n) = (y_0, y_1, \cdots, y_{n-1}, x_i)$$

is an α-pseudo-orbit ending at x_i. Since $d(T(y_{n-1}), y_n) < \alpha$ and $d(T(y_{n-1}), x_i) < \alpha$, we have

$$d(y_n, x_i) \leqslant d(T(y_{n-1}), y_n) + d(T(y_{n-1}), x_i) < 2\alpha \leqslant \epsilon/3.$$

By the definition of ϵ, there is a $A \in M(x_i, \mathcal{U}, \alpha)$ such that $y \in A$.

Therefore, we have

$$N(\mathcal{U}, n, \alpha) \leqslant N(\mathcal{U}, n+1, \alpha) \leqslant K\widetilde{N}(\mathcal{U}, n, \alpha).$$

This complete the proof of the claim.

By the claim,

$$\widetilde{N}(\mathcal{U}, n, \alpha) \leqslant N(\mathcal{U}, n, \alpha) \leqslant K\widetilde{N}(\mathcal{U}, n, \alpha).$$

It implies that

$$\limsup_{n\to\infty} \frac{1}{n} \log \widetilde{N}(\mathcal{U}, n, \alpha) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}, n, \alpha).$$

Using Theorem 3.1, we have

$$\begin{aligned} h_{top}(T, \mathcal{U}) &= \lim_{\alpha\to 0} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}, n, \alpha) \\ &= \lim_{\alpha\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \widetilde{N}(\mathcal{U}, n, \alpha) \\ &= h_{m,\Psi}(T, \mathcal{U}). \end{aligned}$$

This complete the proof. □

Theorem 4.2 *Let (X, T) be a topological dynamical system and $\mathcal{U} \in \mathcal{C}_X^o$. Then*

$$h_{top}(T, \mathcal{U}) = h_{p,\Psi}(T, \mathcal{U}).$$

Proof. It follows from the definition and Theorem 4.1 that,

$$h_{p,\Psi}(T, \mathcal{U}) \leqslant h_{m,\Psi}(T, \mathcal{U}) = h_{top}(T, \mathcal{U}).$$

Now we consider the reverse inequality.

Let $\alpha > 0$. There exists $y_i(\alpha) \in X$ and $n_i = n_i(\alpha)$ going to infinity with

$$\limsup_{n\to\infty} \frac{1}{n} \log \widetilde{N}(\mathcal{U}, n, \alpha) = \lim_{i\to\infty} \frac{1}{n_i} \log N(\mathcal{U}, n_i, \alpha, y_i(\alpha)).$$

We may assume without loss of generality that $y_i(\alpha)$ converges to a limit $y(\alpha)$. If $0 < \beta \leqslant \epsilon/3$, where $\epsilon > 0$ is a Lebesgue number of $\mathcal{U}$, then for all large i,

$$N(\mathcal{U}, n_i, \alpha + \beta, y(\alpha)) \geqslant N(\mathcal{U}, n_i, \alpha, y_i(\alpha)).$$

Choosing a sequence $\alpha_j \to 0$ such that $y(\alpha_j)$ converges to some point $y \in X$. Hence when $d(y(\alpha_j), y) \leqslant \beta$, for each $n \in \mathbb{N}$ have

$$N(\mathcal{U}, n, \alpha_j + 2\beta, y) \geqslant N(\mathcal{U}, n, \alpha_j + \beta, y(\alpha_j)).$$

For a fix j with $d(y(\alpha_j), y) \leqslant \beta$, we can see that

$$N(\mathcal{U}, n_i, \alpha_j + 2\beta, y) \geqslant N(\mathcal{U}, n_i, \alpha_j, y_i(\alpha_j))$$

for each large i. Hence we have,

$$\limsup_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}, n, \alpha_j + 2\beta, y) \geqslant \limsup_{n\to\infty} \frac{1}{n} \log \widetilde{N}(\mathcal{U}, n, \alpha_j).$$

Letting $j \to \infty$, we can get,

$$\limsup_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}, n, 3\beta, y) \geqslant \inf_{\alpha>0} \limsup_{n\to\infty} \frac{1}{n} \log \widetilde{N}(\mathcal{U}, n, \alpha).$$

Therefore,

$$\begin{aligned} h_{p,\Psi}(T,\mathcal{U}) &\geqslant \inf_{\beta>0} \limsup_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}, n, \beta, y) \\ &\geqslant \inf_{\alpha>0} \limsup_{n\to\infty} \frac{1}{n} \log \widetilde{N}(\mathcal{U}, n, \alpha). \\ &= h_{m,\Psi}(T,\mathcal{U}) = h_{top}(T,\mathcal{U}). \end{aligned}$$

This complete the proof. $\square$

Remark 4.3 We define the pseudo-point-entropies of T by $h_{p,\Psi}(T) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_{p,\Psi}(T,\mathcal{U})$ and $h_{m,\Psi}(T) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_{m,\Psi}(T,\mathcal{U})$. There is a fact that use open covers and separated set can define same pseudo-point-entropies. Therefore, Theorem 4.1 and Theorem 4.2 are generalizations of [7, Proposition 4.2 and Theorem 4.3].

References

[1] R.L.Adler, A.G.Konheim and M.H.Mcandrew, Topological entropy, Trans. Amer. Math. Soc., 114(1965),309-319.

[2] R. Bowen, Entropy for group endomorphisms and homogeneous spaces, Trans. Amer. Math. Soc. 153(1971), 401-414.

[3] E.I.Dinaburg, The relation between topological entropy and metric entropy, Soviet Math. Dokl. 11(1970), 13-16.

[4] Peter Walters. An introduction to ergodic theory. Graduate Texts in Mathematics,no.79, Spinger, New York and Berlin,1982.

[5] R.Langevin and P.Walczak, Entropie d'une dynamique. C. R. Acad. Sci. Paris Ser. I, 312(1991),141-144.

[6] R.Langevin and F.Przytycki, Entropie de l'image inverse d'une application, Bull. Soc. Math. France. 120(1991),237-250.

[7] Mike Hurley, On topological entropy of maps, Ergodic Theory and Dynamical Systems,1995,557-568.

[8] Z.Nitecki and F.Przytycki, Preimage entropy for mappings, Int.J. of Bifurcation and Chaos, 9(9)(1999), 1815-1843.

[9] Doris Fiebig,Ulf Fiebig and Zbigniew Nitecki, Entropy and preimage sets, Ergodic Theory and Dynamical Systems, 23(2003),1785-1806.

[10] Zeng Fanping. Partial entropy, Bundle-like entropy and Topological entropy. Dynamical Systems, Proceedings of the International Conference in Honor of Professor Liao Shantao, World Scientific Publishing Co. , 1999, 329-333.

[11] F.Zeng, K.Yan and G.Zhang, Pre-image pressure and invariant measure. Ergod. Th. and Dynam. Sys., 27(2007), 1037-1052.

[12] M.Barge and R.Swanson, Pseudo-orbits and topological entropy, Proc. Amer. Math. Soc. 109(1990), 559-566.

[13] M.Misiurewicz, Remark on the definition of topological entropy, pp. 65-67 of Dynamical Systems and Partial Differential Equations. L. Lara-Carrero and J.Lewowicz, eds. Equinoccio:Caracas, 1986.

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Stability of Lyness' Equation with Period-Two Coefficient via KAM theory

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Abstract

By using the KAM theory we investigate the stability of period-two solution of Lyness' equation with period-two coefficient:

$$x_{n+1} = \frac{a_n + x_n}{x_{n-1}}, \qquad n = 0, 1, \ldots$$

where

$$a_n = \begin{cases} \alpha, & \text{for} \quad n = 2k \\ \beta, & \text{for} \quad n = 2k+1, \quad k = 0, 1, , \ldots \end{cases}$$

Key Words: KAM, Lyness, normal form, periodic coefficient, stability
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1 Introduction

Consider the following difference equation

$$x_{n+1} = \frac{a_n + x_n}{x_{n-1}}, \qquad n = 0, 1, \ldots \tag{1}$$

where

$$a_n = \begin{cases} \alpha, & \text{for} \quad n = 2k \\ \beta, & \text{for} \quad n = 2k+1, \quad k = 0, 1, \ldots \end{cases}$$

and $\alpha, \beta > 0, x_{-1}, x_0 > 0$. This equation is reduced to the following two equations:

$$\begin{aligned} x_{2k+1} &= \frac{\alpha + x_{2k}}{x_{2k-1}}, \\ x_{2k+2} &= \frac{\beta + x_{2k+1}}{x_{2k}}, \quad k = 0, 1, \ldots . \end{aligned}$$

By setting

$$u_k = x_{2k-1}, \quad v_k = x_{2k}$$

we obtain the following system of difference equations

$$\begin{aligned} u_{k+1} &= \frac{\alpha + v_k}{u_k}, \\ v_{k+1} &= \frac{\alpha + \beta u_k + v_k}{u_k v_k}, \quad k = 0, 1, \ldots \end{aligned} \tag{2}$$

where the parameters α and β and the initial conditions u_0 and v_0 are in $(0, \infty)$. We assume that $\alpha \neq \beta$.

Several authors have studied Lyness' equation

$$x_{n+1} = \frac{a + x_n}{x_{n-1}}, \qquad n = 0, 1, \ldots \tag{3}$$

and obtained numerous results concerning the stability of the equilibrium, non-existence of solutions that converge to the equilibrium, invariants, etc., see [1], [8], [9], [11], [12], and [15]. See also [10] for the application of the KAM theory to Lyness' equation (3). See [2]-[4] and [5] for the results on the feasible periods for solutions of (3) and the existence of non-periodic solutions of (3).

Here we will extend the result on the stability of the equilibrium, which was proven in [10] and [11] for Eq.(3) to the case of Eq.(1). The stability of the equilibrium of Eq.(3) has been proved in [7] for period-two coefficient and in [13] for period-three coefficient. These proofs were based on the construction of the corresponding Lyapunov functions associated with the invariants of the equation. Here our method is based on the KAM theory (Kolmogorov-Arnold-Moser), which brings the considered equation to certain normal form which, in addition to investigation of stability of an equilibrium, can be used to find different periodic solutions, chaotic solutions etc.

First, we present the basic results that will be used in the sequel.

Theorem 1.1 (Birkhoff Normal Form) *Let* $\mathbf{F} : R^2 \to R^2$ *be an area-preserving* C^n *map (n-times continuously differentiable) with a fixed point at the origin whose complex-conjugate eigenvalues* λ *and* $\overline{\lambda}$ *are on the unit disk (elliptic fixed point).*

Suppose there exists an integer l *such that*

$$4 \leq l \leq n+1,$$

and suppose that the eigenvalues satisfy

$$\lambda^k \neq 1 \quad \textit{for} \quad k = 3, 4, \ldots, l.$$

Let $r = [\frac{l}{2}]$ *be the integer part of* $\frac{l}{2}$.

Then there exists a smooth function $g(z, \overline{z})$ *that vanishes with its derivatives up to order* $r-1$ *at* $z = 0$*, and there exists a real polynomial*

$$\alpha(w) = \alpha_1 w + \alpha_2 w^2 + \cdots + \alpha_r w^r$$

such that the map $\mathbf{F}$ *can be reduced to the normal form by suitable change of complex coordinates*

$$z \to \mathbf{F}(z, \overline{z}) = \lambda\, z e^{i\alpha(z\overline{z})} + g(z, \overline{z}).$$

In other words the corresponding system of difference equations

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$$

can be reduced to the form

$$\begin{bmatrix} r_{n+1} \\ s_{n+1} \end{bmatrix} = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} r_n \\ s_n \end{bmatrix} + \begin{bmatrix} O_l \\ O_l \end{bmatrix} \tag{4}$$

where

$$\omega = \sum_{k=0}^{M} \gamma_k (r_n^2 + s_n^2)^k, \quad M = \left\lceil \frac{l}{2} \right\rceil - 1. \tag{5}$$

Here O_l *denotes a convergent power series in* r_n *and* s_n *with terms of order greater than or equal to* l *which vanishes at the origin and* $\lceil x \rceil$ *denotes the least integer greater than or equal to* x.

Using Theorem 1.1 we can state the main stability result for an elliptic fixed point, known as the the KAM Theorem (or Kolmogorov-Arnold-Moser Theorem), see [6], [12], and [14].

Theorem 1.2 (KAM Theorem) *Let* $\mathbf{F} : R^2 \to R^2$ *be an area-preserving map with an elliptic fixed point at the origin satisfying the conditions of Theorem 1.1. If the polynomial* $\alpha(|z|^2)$ *is not identically zero, then the origin is a stable equilibrium point. In other words if for some* $k \in \{1, \ldots, M\}$ *we have* $\gamma_k \neq 0$ *in (5), then the origin is a stable equilibrium point.*

2 Equilibrium Solution and Linearized Stability Analysis

An equilibrium solution (p, q) of system (2) satisfies the system of equations

$$p = \frac{\alpha + q}{p}, \quad q = \frac{\beta p + \alpha + q}{pq}, \tag{6}$$

which implies

$$\left.\begin{array}{l} p^2 = \alpha + q, \\ q^2 = \beta + p. \end{array}\right\} \tag{7}$$

This implies that there exists a unique positive equilibrium $E = (p, q)$ which satisfies

$$pq = \frac{\beta p + \alpha + q}{q} = \frac{\beta p}{q} + \frac{\alpha}{q} + 1 > 1. \tag{8}$$

Clearly, $p \neq q$ if $\alpha \neq \beta$. Straightforward calculation shows that (p, q) is the unique period-two solution of Eq.(1).

Theorem 2.1 *The equilibrium $E = (p, q)$ of system (2) is nonhyperbolic point of elliptic type.*

Proof. System (2) can be written in the form

$$u_{n+1} = f\left(u_n, v_n\right), \quad v_{n+1} = g\left(u_n, v_n\right) \quad n = 0, 1, 2, \ldots ,$$

where

$$f\left(u, v\right) = \frac{\alpha + v}{u}, \quad g\left(u, v\right) = \frac{\alpha + \beta u + v}{uv}.$$

The Jacobian matrix of the associated map $T\left(u, v\right) = \left(\frac{\alpha+v}{u}, \frac{\alpha+\beta u+v}{uv}\right)$ evaluated at the equilibrium point $E = (p, q)$ has the form:

$$J_T\left(E\right) = \begin{bmatrix} -1 & \frac{1}{p} \\ -\frac{1}{q} & -1 + \frac{1}{p\,q} \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix is

$$\lambda^2 - \left(\frac{1}{p\,q} - 2\right)\lambda + 1 = 0.$$

The eigenvalues of the Jacobian matrix are

$$\lambda_{1,2} = \frac{\frac{1}{p\,q} - 2 \pm \sqrt{\left(\frac{1}{p\,q} - 2\right)^2 - 4}}{2} = \frac{\frac{1}{p\,q} - 2 \pm i\,\sqrt{4 - \left(\frac{1}{p\,q} - 2\right)^2}}{2},$$

where $\left(\frac{1}{p\,q} - 2\right)^2 - 4 < 0$, which holds in view of (8).

This implies

$$|\lambda_{1,2}| = \frac{\left(\frac{1}{p\,q} - 2\right)^2 + \left[4 - \left(\frac{1}{p\,q} - 2\right)^2\right]}{4} = 1.$$

Consequently, the equilibrium $E = (p, q)$ is nonhyperbolic point of elliptic type. $\square$

3 The KAM theory applied to equation (2)

In this section we will apply the KAM theory to establish the stability of the equilibrium (p,q) of system (2). We assume that $\alpha \neq \beta$.

Taking logarithms in (2) and using the substitution:

$$\begin{aligned} U_n &= \ln \frac{u_n}{p}, \\ V_n &= \ln \frac{v_n}{q}, \end{aligned}$$

we obtain the following system:

$$\begin{aligned} U_{n+1} &= \ln\left(\alpha + qe^{V_n}\right) - U_n - 2\ln p, \\ V_{n+1} &= \ln\left(\beta pe^{U_n} + \alpha + qe^{V_n}\right) - U_n - V_n - \ln p - 2\ln q. \end{aligned} \tag{9}$$

Now we define the map T as:

$$T : \begin{pmatrix} U \\ V \end{pmatrix} \to \begin{pmatrix} \ln\left(\alpha + qe^{V}\right) - U - 2\ln p \\ \ln\left(\beta pe^{U} + \alpha + qe^{V}\right) - U - V - \ln p - 2\ln q \end{pmatrix}. \tag{10}$$

Jacobian matrix of this map has the form:

$$J_T = \begin{pmatrix} -1 & \frac{qe^V}{\alpha+qe^V} \\ -1 + \frac{\beta pe^U}{\beta pe^U+\alpha+qe^V} & -1 + \frac{qe^V}{\beta pe^U+\alpha+qe^V} \end{pmatrix}.$$

The determinant of Jacobian matrix has the form:

$$\begin{aligned} \det J_T &= -1\left(-1 + \frac{qe^V}{\beta pe^U + \alpha + qe^V}\right) - \frac{qe^V}{\alpha + qe^V}\left(-1 + \frac{\beta pe^U}{\beta pe^U + \alpha + qe^V}\right) \\ &= 1 - \frac{qe^V}{\beta pe^U + \alpha + qe^V} + \frac{qe^V}{\alpha + qe^V} - \frac{\beta qpe^U e^V}{\left(\alpha + qe^V\right)\left(\beta pe^U + \alpha + qe^V\right)} = 1. \end{aligned}$$

System (9) has the equilibrium $(0,0)$.

The characteristic equation of the linearized system of system (9) at the equilibrium point $(0,0)$

$$\det\left(J_T(0,0) - \lambda I\right) = 0$$

takes the form:

$$\begin{vmatrix} -1-\lambda & \frac{q}{\alpha+q} \\ -1 + \frac{\beta p}{\beta p+\alpha+q} & -1 + \frac{q}{\beta p+\alpha+q} - \lambda \end{vmatrix} = 0,$$

that is,

$$\lambda^2 - \left(\frac{1}{pq} - 2\right)\lambda + 1 = 0.$$

The roots of this equation are:

$$\lambda = \frac{\frac{1}{pq}-2+i\sqrt{\left|\left(\frac{1}{pq}-2\right)^2-4\right|}}{2} = \frac{1}{2pq}\left[(1-2pq)+i\sqrt{4pq-1}\right],$$
$$\overline{\lambda} = \frac{\frac{1}{pq}-2-i\sqrt{\left|\left(\frac{1}{pq}-2\right)^2-4\right|}}{2} = \frac{1}{2pq}\left[(1-2pq)-i\sqrt{4pq-1}\right],$$

and so $|\lambda| = 1$. Straightforward calculations show that $\lambda^3 \neq 1$ and $\lambda^4 \neq 1$ and so $l = 4$.

3.1 The first transformation

In order to obtain the Birkhoff normal form of system (9) we will expand the right hand sides of the equations of system (9) at the equilibrium point $(0,0)$ up to the order $l-1=3$:

$$\left.\begin{aligned}
U_{n+1} &= -U_n + \tfrac{q}{p^2}V_n + \tfrac{(p^2-q)q}{2p^4}V_n^2 + \tfrac{(p^2-q)q(p^2-2q)}{6p^6}V_n^3 + O_4\\
V_{n+1} &= -\tfrac{p}{q^2}U_n + \left(\tfrac{1}{pq}-1\right)V_n\\
&\quad +\tfrac{1}{2}\left[\tfrac{(q^2-p)p}{q^4}U_n^2 - 2\tfrac{(q^2-p)}{pq^3}U_nV_n + \tfrac{pq-1}{p^2q^2}V_n^2\right]\\
&\quad +\tfrac{1}{6}\left[\begin{array}{l}\tfrac{(q^2-p)p(2p-q^2)}{q^6}U_n^3 - 3\tfrac{(q^2-p)(2p-q^2)}{pq^5}U_n^2V_n\\ +3\tfrac{(q^2-p)(2-pq)}{p^2q^4}U_nV_n^2 + \tfrac{(pq-1)(pq-2)}{p^3q^3}V_n^3\end{array}\right] + O_4.
\end{aligned}\right\} \tag{11}$$

The first transformation is given as:

$$\begin{bmatrix} U_{n+1} \\ V_{n+1}\end{bmatrix} = P\cdot\begin{bmatrix} x_n \\ y_n\end{bmatrix} \tag{12}$$

where P is a 2×2 matrix such that

$$diag\left(\lambda,\overline{\lambda}\right) = P^{-1}J_0P$$

and J_0 is the Jacobian matrix of map T of (10) evaluated at $(0,0)$. In other words, matrix P consists of the eigenvectors of matrix J_0 that correspond to the eigenvalues λ and $\overline{\lambda}$. Taking into account that J_0 has the form

$$J_0 = \begin{bmatrix} -1 & \frac{q}{p^2} \\ -\frac{p}{q^2} & -1+\frac{1}{pq}\end{bmatrix},$$

matrix P is given by

$$P=\begin{bmatrix} 1 & 1 \\ \frac{p^2}{q}(1+\lambda) & \frac{p^2}{q}\left(1+\overline{\lambda}\right) \end{bmatrix}.$$

The substitution (12) takes the form:

$$\begin{aligned} U_n &= x_n+y_n, \\ V_n &= \frac{p^2}{q}\left((1+\lambda)x_n+\left(1+\overline{\lambda}\right)y_n\right)=\frac{p^2}{q}\left(sx_n+\overline{s}y_n\right)=\frac{p^2}{q}\left(sx_n+ty_n\right). \end{aligned}$$

After simplification system (11) becomes:

$$\left.\begin{aligned} & x_{n+1}+y_{n+1}=-\left(x_n+y_n\right)+\left(sx_n+ty_n\right)+\tfrac{\left(p^2-q\right)}{2q}\left(sx_n+ty_n\right)^2 \\ & +\tfrac{\left(p^2-q\right)\left(p^2-2q\right)}{6q^2}\left(sx_n+ty_n\right)^3+O_4, \\ & \tfrac{p^2}{q}\left(sx_{n+1}+ty_{n+1}\right)=-\tfrac{p}{q^2}\left(x_n+y_n\right)+\left(\tfrac{1}{pq}-1\right)\tfrac{p^2}{q}\left(sx_n+ty_n\right) \\ & +\tfrac{p\left(q^2-p\right)}{2q^4}\left(x_n+y_n\right)^2-\tfrac{p\left(q^2-p\right)}{q^4}\left(x_n+y_n\right)\left(sx_n+ty_n\right) \\ & +\tfrac{p^2(pq-1)}{2q^4}\left(sx_n+ty_n\right)^2+\tfrac{\left(q^2-p\right)p\left(2p-q^2\right)}{6q^6}\left(x_n+y_n\right)^3 \\ & -\tfrac{p\left(q^2-p\right)\left(2p-q^2\right)}{2q^6}\left(x_n+y_n\right)^2\left(sx_n+ty_n\right) \\ & +\tfrac{p^2\left(q^2-p\right)(2-pq)}{2q^6}\left(x_n+y_n\right)\left(sx_n+ty_n\right)^2 \\ & +\tfrac{p^3(pq-1)(pq-2)}{6q^6}\left(sx_n+ty_n\right)^3+O_4. \end{aligned}\right\} \qquad (13)$$

Combining the equations of (13) we obtain

$$\left.\begin{aligned} (t-s)\,y_{n+1}= & \; s\left(x_n+y_n\right)-s\left(sx_n+ty_n\right)-s\tfrac{\left(p^2-q\right)}{2q}\left(sx_n+ty_n\right)^2 \\ & -s\tfrac{\left(p^2-q\right)\left(p^2-2q\right)}{6q^2}\left(sx_n+ty_n\right)^3-\tfrac{1}{pq}\left(x_n+y_n\right) \\ & +\left(\tfrac{1}{pq}-1\right)\left(sx_n+ty_n\right)+\tfrac{\left(q^2-p\right)}{2pq^3}\left(x_n+y_n\right)^2 \\ & -\tfrac{\left(q^2-p\right)}{pq^3}\left(x_n+y_n\right)\left(sx_n+ty_n\right)+\tfrac{(pq-1)}{2q^3}\left(sx_n+ty_n\right)^2 \\ & +\tfrac{\left(q^2-p\right)\left(2p-q^2\right)}{6pq^5}\left(x_n+y_n\right)^3 \\ & -\tfrac{\left(q^2-p\right)\left(2p-q^2\right)}{2pq^5}\left(x_n+y_n\right)^2\left(sx_n+ty_n\right) \\ & +\tfrac{\left(q^2-p\right)(2-pq)}{2q^5}\left(x_n+y_n\right)\left(sx_n+ty_n\right)^2 \\ & +\tfrac{p(pq-1)(pq-2)}{6q^5}\left(sx_n+ty_n\right)^3+O_4. \end{aligned}\right\} \qquad (14)$$

The term that multiplies x_n has the form:

$$\begin{aligned} s-s^2-\frac{1}{pq}+s\left(\frac{1}{pq}-1\right) &= -s^2-\frac{1}{pq}+s\frac{1}{pq} \\ &= -\lambda^2-2\lambda-1+\frac{1}{pq}\lambda=0, \end{aligned}$$

and the term that multiplies y_n becomes:

$$\begin{aligned} s-st-\frac{1}{pq}+\left(\frac{1}{pq}-1\right)t &= (1+\lambda)-(1+\lambda)\left(1+\overline{\lambda}\right)-\frac{1}{pq}+\left(\frac{1}{pq}-1\right)\left(1+\overline{\lambda}\right) \\ &= \overline{\lambda}^2-1=\overline{\lambda}^2-\lambda\overline{\lambda}=\overline{\lambda}\left(\overline{\lambda}-\lambda\right). \end{aligned}$$

Now, (14) is simplified as:

$$\begin{aligned} y_{n+1} &= \overline{\lambda}y_n-s\frac{\left(p^2-q\right)}{2q}\frac{1}{t-s}\left(sx_n+ty_n\right)^2-s\frac{\left(p^2-q\right)\left(p^2-2q\right)}{6q^2}\frac{1}{t-s}\left(sx_n+ty_n\right)^3 \\ &+\frac{\left(q^2-p\right)}{2pq^3}\frac{1}{t-s}\left(x_n+y_n\right)^2-\frac{\left(q^2-p\right)}{pq^3}\frac{1}{t-s}\left(x_n+y_n\right)\left(sx_n+ty_n\right) \\ &+\frac{(pq-1)}{2q^3}\frac{1}{t-s}\left(sx_n+ty_n\right)^2+\frac{\left(q^2-p\right)\left(2p-q^2\right)}{6pq^5}\frac{1}{t-s}\left(x_n+y_n\right)^3 \\ &-\frac{\left(q^2-p\right)\left(2p-q^2\right)}{2pq^5}\frac{1}{t-s}\left(x_n+y_n\right)^2\left(sx_n+ty_n\right) \\ &+\frac{\left(q^2-p\right)(2-pq)}{2q^5}\frac{1}{t-s}\left(x_n+y_n\right)\left(sx_n+ty_n\right)^2 \\ &+\frac{p(pq-1)(pq-2)}{6q^5}\frac{1}{t-s}\left(sx_n+ty_n\right)^3+O_4. \end{aligned} \tag{15}$$

Using (15) in (13) we obtain:

$$\begin{aligned} x_{n+1} &= -\overline{\lambda}y_n+s\frac{\left(p^2-q\right)}{2q}\frac{1}{t-s}\left(sx_n+ty_n\right)^2 \\ &+s\frac{\left(p^2-q\right)\left(p^2-2q\right)}{6q^2}\frac{1}{t-s}\left(sx_n+ty_n\right)^3 \\ &-\frac{\left(q^2-p\right)}{2pq^3}\frac{1}{t-s}\left(x_n+y_n\right)^2+\frac{\left(q^2-p\right)}{pq^3}\frac{1}{t-s}\left(x_n+y_n\right)\left(sx_n+ty_n\right) \\ &-\frac{(pq-1)}{2q^3}\frac{1}{t-s}\left(sx_n+ty_n\right)^2 \\ &-\frac{\left(q^2-p\right)\left(2p-q^2\right)}{6pq^5}\frac{1}{t-s}\left(x_n+y_n\right)^3 \\ &+\frac{\left(q^2-p\right)\left(2p-q^2\right)}{2pq^5}\frac{1}{t-s}\left(x_n+y_n\right)^2\left(sx_n+ty_n\right) \\ &-\frac{\left(q^2-p\right)(2-pq)}{2q^5}\frac{1}{t-s}\left(x_n+y_n\right)\left(sx_n+ty_n\right)^2 \\ &-\frac{p(pq-1)(pq-2)}{6q^5}\frac{1}{t-s}\left(sx_n+ty_n\right)^3 \\ &-\left(x_n+y_n\right)+\left(sx_n+ty_n\right)+\frac{\left(p^2-q\right)}{2q}\left(sx_n+ty_n\right)^2 \\ &+\frac{\left(p^2-q\right)\left(p^2-2q\right)}{6q^2}\left(sx_n+ty_n\right)^3+O_4. \end{aligned}$$

In view of

$$s=1+\lambda \quad \text{i} \quad t=1+\overline{\lambda},$$

we obtain that

$$-\overline{\lambda}y_n-x_n-y_n+sx_n+ty_n=\lambda x_n.$$

Now we have

$$\begin{aligned}
x_{n+1} &= \lambda x_n+s\tfrac{\left(p^2-q\right)}{2q}\frac{1}{t-s}\left(s^2x_n^2+2sx_nty_n+t^2y_n^2\right)\\
&\quad+s\tfrac{\left(p^2-q\right)\left(p^2-2q\right)}{6q^2}\frac{1}{t-s}\left(s^3x_n^3+3s^2x_n^2ty_n+3sx_nt^2y_n^2+t^3y_n^3\right)\\
&\quad-\tfrac{\left(q^2-p\right)}{2pq^3}\frac{1}{t-s}\left(x_n^2+2x_ny_n+y_n^2\right)\\
&\quad+\tfrac{\left(q^2-p\right)}{pq^3}\frac{1}{t-s}\left(sx_n^2+x_nty_n+y_nsx_n+ty_n^2\right)\\
&\quad-\tfrac{(pq-1)}{2q^3}\frac{1}{t-s}\left(s^2x_n^2+2sx_nty_n+t^2y_n^2\right)\\
&\quad-\tfrac{\left(q^2-p\right)\left(2p-q^2\right)}{6pq^5}\frac{1}{t-s}\left(x_n^3+3x_n^2y_n+3x_ny_n^2+y_n^3\right)\\
&\quad+\tfrac{\left(q^2-p\right)\left(2p-q^2\right)}{2pq^5}\frac{1}{t-s}\left(x_n^3s+x_n^2ty_n+2x_n^2y_ns+2x_ny_n^2t+y_n^2sx_n+y_n^3t\right)\\
&\quad-\tfrac{\left(q^2-p\right)(2-pq)}{2q^5}\frac{1}{t-s}\left(s^2x_n^3+2sx_n^2ty_n+x_nt^2y_n^2+y_ns^2x_n^2+2sx_nty_n^2+t^2y_n^3\right)\\
&\quad-\tfrac{p(pq-1)(pq-2)}{6q^5}\frac{1}{t-s}\left(s^3x_n^3+3s^2x_n^2ty_n+3sx_nt^2y_n^2+t^3y_n^3\right)\\
&\quad+\tfrac{\left(p^2-q\right)}{2q}\left(s^2x_n^2+2sx_nty_n+t^2y_n^2\right)\\
&\quad+\tfrac{\left(p^2-q\right)\left(p^2-2q\right)}{6q^2}\left(s^3x_n^3+3s^2x_n^2ty_n+3sx_nt^2y_n^2+t^3y_n^3\right)+O_4.
\end{aligned}$$

Thus we have

$$x_{n+1}=\lambda x_n+\sigma\left(ax_n^2+bx_ny_n+cy_n^2+dx_n^3+ex_n^2y_n+fx_ny_n^2+gy_n^3\right)+O_4, \tag{16}$$

where $\sigma a, \sigma b, \sigma c$, and σe can be simplified as follows:

$$\begin{aligned}
\sigma a &= \tfrac{p^2-q}{2q}\frac{1}{t-s}s^3-\tfrac{q^2-p}{2pq^3}\frac{1}{t-s}+\tfrac{q^2-p}{pq^3}\frac{1}{t-s}s-\tfrac{pq-1}{2q^3}\frac{1}{t-s}s^2+\tfrac{p^2-q}{2q}s^2\\
&= \frac{1}{2}\frac{1}{t-s}\left[\tfrac{p^2-q}{q}s^3-\tfrac{q^2-p}{pq^3}+2s\tfrac{q^2-p}{pq^3}-\tfrac{pq-1}{q^3}s^2+\tfrac{p^2-q}{q}s^2\left(t-s\right)\right]\\
&= \frac{1}{2}\frac{1}{t-s}\left[-\tfrac{q^2-p}{pq^3}+2s\tfrac{q^2-p}{pq^3}-\tfrac{pq-1}{q^3}s^2+\tfrac{p^2-q}{q}ts^2\right]\\
&= \frac{1}{2}\frac{1}{t-s}\left[-\tfrac{q^2-p}{pq^3}+2\left(\lambda+1\right)\frac{q^2-p}{pq^3}-\frac{pq-1}{pq^4}\lambda+\frac{p^2-q}{pq^2}\left(\lambda+1\right)\right]\\
&= \tfrac{1}{2pq^4}\frac{1}{t-s}\left[-q\left(q^2-p\right)+2\left(\lambda+1\right)q\left(q^2-p\right)-\left(pq-1\right)\lambda+q^2\left(p^2-q\right)\left(\lambda+1\right)\right]\\
&= \tfrac{1}{2pq^4}\frac{1}{t-s}\left[\left(q^3-3qp+1+q^2p^2\right)\lambda+\left(-pq+p^2q^2\right)\right]\\
&= \tfrac{1}{4p^2q^5}\tfrac{1}{\lambda-\overline{\lambda}}\left[\left(-q^3+3qp-1-q^2p^2\right)\left(1-2pq+i\sqrt{4pq-1}\right)+2p^2q^2\left(1-pq\right)\right],
\end{aligned}$$

that is

$$\sigma a = \frac{\left(-q^3+2q^4p+5pq-5p^2q^2-1\right)+i\left(-q^3+3qp-1-q^2p^2\right)\sqrt{4pq-1}}{4p^2q^5\left(\lambda-\overline{\lambda}\right)}. \tag{17}$$

Similarly

$$\sigma b = \frac{1}{2p^2q^3}\frac{1}{\lambda-\overline{\lambda}}\left[\left(2pq^2-p^2-q\right)+i\left(p^2-q\right)\sqrt{4pq-1}\right] \tag{18}$$

and

$$\sigma c = \frac{\left(2p^2q^4-p^3q^2-5pq^3+4p^2q-p+q^2\right)-i\left(-3pq^3+2p^2q-p+q^2+p^3q^2\right)\sqrt{4pq-1}}{4p^3q^5\left(\lambda-\overline{\lambda}\right)}. \tag{19}$$

In a similar way we obtain

$$\sigma e = \frac{1}{4p^2q^5}\frac{\left(10p^2q^2-q^3-6pq-p^3-2q^4p+2-2p^4q\right)+i\left(-4pq+q^3+2+p^3\right)\sqrt{4pq-1}}{\lambda-\overline{\lambda}}$$

$$= \frac{1}{4pq^4}\frac{\left(10p^2q^2-q^3-6pq-p^3-2q^4p+2-2p^4q\right)+i\left(-4pq+q^3+2+p^3\right)\sqrt{4pq-1}}{i\sqrt{4pq-1}}$$

which implies

$$Re\left(\sigma e\right) = \frac{1}{4pq^4}\left(-4pq+p^3+q^3+2\right) \tag{20}$$

In the above calculation we made use of:

$$\left.\begin{array}{l}
t-s=\overline{\lambda}-\lambda=-\frac{1}{pq}\left[i\sqrt{4pq-1}\right],\\
st=\left(1+\lambda\right)\left(1+\overline{\lambda}\right)=\frac{1}{pq},\\
t+s=\frac{1}{pq},\\
s^2=1+2\lambda+\lambda^2=\frac{1}{pq}\lambda,\\
t^2=1+2\overline{\lambda}+\overline{\lambda}^2=\frac{1}{pq}\overline{\lambda},\\
\sigma=\frac{1}{\lambda-\overline{\lambda}}\\
\lambda=\frac{1}{2pq}\left[\left(1-2pq\right)+i\sqrt{4pq-1}\right].
\end{array}\right\} \tag{21}$$

Thus, the linear terms of (15) and (16) are transformed into normal form.

4 The second transformation

The objective of second transformation is to obtain the nonlinear terms up to order $l-1$ in normal form.

The change of variables

$$\left.\begin{array}{l}
x_n=\xi_n+\phi_2\left(\xi_n,\eta_n\right)+\phi_3\left(\xi_n,\eta_n\right)\\
y_n=\eta_n+\psi_2\left(\xi_n,\eta_n\right)+\psi_3\left(\xi_n,\eta_n\right)
\end{array}\right\},\quad n=0,1\ldots \tag{22}$$

where

$$\phi_k\left(\xi,\eta\right)=\sum_{j=0}^{k}a_{kj}\xi^{k-j}\eta^j \qquad and \qquad \psi_k\left(\xi,\eta\right)=\sum_{j=0}^{k}\overline{a}_{kj}\xi^j\eta^{k-j}$$

for $k=2$ and $k=3$, gives

$$\left.\begin{aligned}
x_n &= \xi_n + \left(a_{20}\xi_n^2 + a_{21}\xi_n\eta_n + a_{22}\eta_n^2\right)\\
&\quad + \left(a_{30}\xi_n^3 + a_{31}\xi_n^2\eta_n + a_{32}\xi_n\eta_n^2 + a_{33}\eta_n^3\right)\\
y_n &= \eta_n + \left(\overline{a_{20}}\eta_n^2 + \overline{a_{21}}\xi_n\eta_n + \overline{a_{22}}\xi_n^2\right)\\
&\quad + \left(\overline{a_{30}}\eta_n^3 + \overline{a_{31}}\xi_n\eta_n^2 + \overline{a_{32}}\xi_n^2\eta_n + \overline{a_{33}}\xi_n^3\right)\\
x_n^2 &= \xi_n^2 + 2a_{20}\xi_n^3 + 2a_{21}\xi_n^2\eta_n + 2a_{22}\xi_n\eta_n^2 + O_4\\
y_n^2 &= \eta_n^2 + 2\overline{a_{20}}\eta_n^3 + 2\overline{a_{21}}\xi_n\eta_n^2 + 2\overline{a_{22}}\xi_n^2\eta_n + O_4\\
x_n^3 &= \xi_n^3 + O_4\\
y_n^3 &= \eta_n^3 + O_4\\
x_n^2 y_n &= \xi_n^2\eta_n + O_4\\
x_n y_n^2 &= \xi_n\eta_n^2 + O_4\\
x_n y_n &= \overline{a_{22}}\xi_n^3 + \left(a_{20} + \overline{a_{21}}\right)\xi_n^2\eta_n\\
&\quad + \left(a_{21} + \overline{a_{20}}\right)\xi_n\eta_n^2 + \xi_n\eta_n + a_{22}\eta_n^3 + O_4
\end{aligned}\right\} \tag{23}$$

and

$$\left.\begin{aligned}
\xi_{n+1} &= \lambda\xi_n + \alpha_2\xi_n^2\eta_n + O_4\\
\eta_{n+1} &= \overline{\lambda}\eta_n + \overline{\alpha}_2\xi_n\eta_n^2 + O_4
\end{aligned}\right\}, \quad n = 0, 1 \dots . \tag{24}$$

Using (24) in the first two relations of (23), we obtain:

$$\left.\begin{aligned}
x_{n+1} &= \lambda\xi_n + a_{20}\lambda^2\xi_n^2 + a_{30}\lambda^3\xi_n^3 + a_{21}\lambda\overline{\lambda}\xi_n\eta_n\\
&\quad + \left(\alpha_2 + a_{31}\lambda^2\overline{\lambda}\right)\xi_n^2\eta_n + a_{32}\lambda\overline{\lambda}^2\xi_n\eta_n^2 + a_{22}\overline{\lambda}^2\eta_n^2 + a_{33}\overline{\lambda}^3\eta_n^3,\\
y_{n+1} &= \overline{\lambda}\eta_n + \overline{a_{20}}\overline{\lambda}^2\eta_n^2 + \overline{a_{30}}\overline{\lambda}^3\eta_n^3 + \overline{a_{21}}\lambda\overline{\lambda}\xi_n\eta_n\\
&\quad + \left(\overline{\alpha_2} + \overline{a_{31}}\overline{\lambda}^2\lambda\right)\xi_n\eta_n^2 + \overline{a_{32}}\lambda^2\overline{\lambda}\xi_n^2\eta_n + \overline{a_{22}}\overline{\lambda}^2\xi_n^2 + \overline{a_{33}}\lambda^3\xi_n^3.
\end{aligned}\right\} \tag{25}$$

Substituting (25) in the left-hand side of (16) and substituting (24) in the right-hand side of (16) we obtain after simplification:

$$\begin{aligned}
&a_{20}\lambda^2\xi_n^2 + a_{30}\lambda^3\xi_n^3 + a_{21}\xi_n\eta_n + \left(\alpha_2 + a_{31}\lambda\right)\xi_n^2\eta_n\\
&+a_{32}\overline{\lambda}\xi_n\eta_n^2 + a_{22}\overline{\lambda}^2\eta_n^2 + a_{33}\overline{\lambda}^3\eta_n^3\\
= \quad &\lambda\left[\left(a_{20}\xi_n^2 + a_{21}\xi_n\eta_n + a_{22}\eta_n^2\right) + \left(a_{30}\xi_n^3 + a_{31}\xi_n^2\eta_n + a_{32}\xi_n\eta_n^2 + a_{33}\eta_n^3\right)\right]\\
&+\sigma a\left[\xi_n^2 + 2a_{20}\xi_n^3 + 2a_{21}\xi_n^2\eta_n + 2a_{22}\xi_n\eta_n^2\right]\\
&+\sigma b\left[\overline{a_{22}}\xi_n^3 + \left(a_{20} + \overline{a_{21}}\right)\xi_n^2\eta_n + \left(a_{21} + \overline{a_{20}}\right)\xi_n\eta_n^2 + \xi_n\eta_n + a_{22}\eta_n^3\right]\\
&+\sigma c\left[\eta_n^2 + 2\overline{a_{20}}\eta_n^3 + 2\overline{a_{21}}\xi_n\eta_n^2 + 2\overline{a_{22}}\xi_n^2\eta_n\right]\\
&+\sigma d\xi_n^3 + \sigma e\xi_n^2\eta_n + \sigma f\xi_n\eta_n^2 + \sigma g\eta_n^3 + O_4.
\end{aligned}$$

Equationg the corresponding coefficients on both sides of the last identity we obtain:

$$\begin{array}{rcl}
\xi_n^2 & : & a_{20}\lambda^2 = \lambda a_{20} + \sigma a \\
\xi_n^3 & : & a_{30}\lambda^3 = \lambda a_{30} + \sigma a 2a_{20} + \sigma b\overline{a_{22}} + \sigma d \\
\xi_n \eta_n & : & a_{21} = \lambda a_{21} + \sigma b \\
\xi_n^2 \eta_n & : & \alpha_2 + a_{31}\lambda = \lambda a_{31} + 2\sigma a a_{21} + \sigma b\left(a_{20} + \overline{a_{21}}\right) + 2\sigma c\overline{a_{22}} + \sigma e \\
\xi_n \eta_n^2 & : & a_{32}\overline{\lambda} = \lambda a_{32} + 2\sigma a a_{22} + \sigma b\left(a_{21} + \overline{a_{20}}\right) + 2\sigma c\overline{a_{21}} + \sigma f \\
\eta_n^2 & : & a_{22}\overline{\lambda}^2 = \lambda a_{22} + \sigma c \\
\eta_n^3 & : & a_{33}\overline{\lambda}^3 = \lambda a_{33} + \sigma b a_{22} + 2\sigma c\overline{a_{20}} + \sigma g.
\end{array}$$

This implies

$$a_{20} = \frac{\sigma a}{\lambda^2 - \lambda}, \quad a_{21} = \frac{-\sigma b}{\lambda - 1}, \quad a_{22} = \frac{\sigma c}{\overline{\lambda}^2 - \lambda},$$

$$a_{30} = \frac{\sigma}{\lambda^3 - \lambda}\left[\left(2aa_{20} + b\overline{a_{22}}\right) + d\right], \quad a_{31} = \text{arbitrary},$$

$$a_{32} = \frac{-\sigma}{\lambda - \overline{\lambda}}\left[\left(2aa_{22} + b\overline{a_{20}}\right) + \left(ba_{21} + 2c\overline{a_{21}}\right) + f\right],$$

$$a_{33} = \frac{\sigma}{\overline{\lambda}^3 - \lambda}\left[\left(ba_{22} + 2c\overline{a_{20}}\right) + g\right],$$

$$\alpha_2 = \sigma\left(ba_{20} + 2c\overline{a_{22}}\right) + \sigma\left(2aa_{21} + b\overline{a_{21}}\right) + \sigma e. \tag{26}$$

5 The third transformation

The objective of third transformation consists in expressing the terms in (24) as real values. This is achieved by using the transformation:

$$\xi_n = r_n + is_n,$$

$$\eta_n = r_n - is_n.$$

Comparing the system obtained with (4) and using (5) for $l = 4$, we determine the twist coefficients γ_0 and γ_1. We have

$$\cos\gamma_0 = Re(\lambda) \quad \text{and} \quad \gamma_1 = \frac{-1}{\sin\gamma_0} \cdot Re(\alpha_2). \tag{27}$$

In order to apply the KAM theorem we have to determine $Re(\alpha_2)$ and to show that $\gamma_1 \neq 0$.

The following formulas will be useful:

$$\lambda^2 = \left(-1 - 2\lambda + \frac{1}{pq}\lambda\right) \quad \text{and} \quad \overline{\lambda}^2 = \left(-1 - 2\overline{\lambda} + \frac{1}{pq}\overline{\lambda}\right)$$

$$\overline{\lambda}^2 + \lambda^2 = -2 - 2\left(\overline{\lambda} + \lambda\right) + \frac{1}{pq}\left(\overline{\lambda} + \lambda\right) = -2 + \left(\overline{\lambda} + \lambda\right)\left(\frac{1}{pq} - 2\right) = 2 + \frac{1}{p^2q^2} - 4\frac{1}{pq},$$

$$\begin{aligned}
\sigma &= \frac{1}{\lambda-\overline{\lambda}}, \qquad \overline{\sigma} = \frac{1}{\overline{\lambda}-\lambda} = -\sigma, \\
\sigma\overline{\sigma} &= \frac{1}{\left(\lambda-\overline{\lambda}\right)}\frac{1}{\left(\overline{\lambda}-\lambda\right)} = \frac{1}{\left(1-\lambda^2-\overline{\lambda}^2+1\right)} \\
&= \frac{1}{\left(2-\left(\lambda^2+\overline{\lambda}^2\right)\right)} = \frac{1}{\left(4\frac{1}{pq}-\frac{1}{p^2q^2}\right)} = \frac{p^2q^2}{(4pq-1)},
\end{aligned}$$

$$\sigma^2 = \left[\frac{1}{\left(\lambda-\overline{\lambda}\right)}\right]^2 = \left[\frac{1}{\frac{i\sqrt{4pq-1}}{pq}}\right]^2 = \frac{-p^2q^2}{(4pq-1)},$$

$$\overline{a_{21}} = \frac{-\overline{\sigma}\overline{b}}{\overline{\lambda}-1} = \frac{\sigma\overline{b}}{\overline{\lambda}-1}, \quad \overline{a_{22}} = \frac{\overline{\sigma}\overline{c}}{\lambda^2-\overline{\lambda}}.$$

Using the last identities (26) can be written in the form

$$\alpha_2 = 2\sigma a a_{21} + \sigma b\overline{a_{21}} + \sigma b a_{20} + 2\sigma c\overline{a_{22}} + \sigma e,$$

where

$$\sigma b a_{20} = \sigma b\frac{\sigma a}{\lambda^2-\lambda} = \frac{\sigma a\sigma b}{\lambda^2-\lambda}, \quad 2\sigma c\overline{a_{22}} = 2\sigma c\frac{\overline{\sigma c}}{\lambda^2-\overline{\lambda}} = \frac{2\sigma c\overline{\sigma c}}{\lambda^2-\overline{\lambda}},$$

$$2\sigma a a_{21} = 2\sigma a\frac{\sigma b}{\lambda-1} = -2\frac{\sigma a\sigma b}{\lambda-1}, \quad \sigma b\overline{a_{21}} = \sigma b\frac{\sigma\overline{b}}{\overline{\lambda}-1} = \frac{\sigma^2 b\overline{b}}{\overline{\lambda}-1}.$$

The formulas (17) and (18) imply

$$\sigma a\sigma b = \frac{-1}{8p^2q^6\,(4pq-1)}\left[A + iB\sqrt{4pq-1}\right],$$

where

$$\begin{aligned}
A &= 2q\,(p+1+q)\left(p^2-p-pq-q+q^2+1\right)\left(2p^2q^2+1-4pq\right) \\
B &= -2q\,(p+1+q)\,(2pq-1)\left(p^2-p-pq-q+q^2+1\right) \\
B-A &= -4pq^2\,(p+1+q)\,(pq-1)\left(p^2-p-pq-q+q^2+1\right).
\end{aligned}$$

Taking into account that

$$\frac{1}{\lambda-1} = -\frac{1}{2} - \frac{i}{2}\frac{\sqrt{4pq-1}}{4pq-1},$$

we obtain

$$Re\,(2\sigma a a_{21}) = Re\left(-2\sigma a\sigma b\frac{1}{\lambda-1}\right) = \frac{1}{(4pq-1)}\frac{1}{8p^2q^6}\,[B-A],$$

and

$$Re\,(2\sigma a a_{21}) = \frac{-\,(p+1+q)\,(pq-1)\left(p^2-p-pq-q+q^2+1\right)}{2pq^4\,(4pq-1)}. \tag{28}$$

Likewise,

$$Re\left(\sigma b\overline{a_{21}}\right) = Re\left(\sigma^2\overline{b}b\frac{1}{\overline{\lambda}-1}\right) = \sigma^2\overline{b}bRe\left(\frac{1}{\overline{\lambda}-1}\right) = -\frac{1}{2}\sigma^2\overline{b}b,$$

that is

$$Re\left(\sigma b\overline{a_{21}}\right) = \frac{1}{2q^3\left(4pq-1\right)}\left(p+1+q\right)\left(p^2-p-pq-q+q^2+1\right). \tag{29}$$

In view of the fact that

$$\begin{aligned}
\frac{1}{\lambda-1} &= \frac{1}{\frac{1}{2pq}\left[(1-2pq)+i\sqrt{4pq-1}\right]-1} \\
&= \frac{2pq}{\left[(1-2pq)+i\sqrt{4pq-1}\right]-2pq} = \frac{2pq\left[(1-4pq)-i\sqrt{4pq-1}\right]}{\left[(1-4pq)^2+4pq-1\right]} \\
&= \frac{2pq\left[(1-4pq)-i\sqrt{4pq-1}\right]}{\left[(1-8pq+16p^2q^2)+4pq-1\right]} = \frac{2pq\left[(1-4pq)-i\sqrt{4pq-1}\right]}{-4pq\left(1-4pq\right)},
\end{aligned}$$

we obtain

$$\frac{1}{\lambda-1} = -\frac{1}{2}-\frac{i}{2}\frac{\sqrt{4pq-1}}{4pq-1} \quad \text{and} \quad \frac{1}{\overline{\lambda}-1} = -\frac{1}{2}+\frac{i}{2}\frac{\sqrt{4pq-1}}{4pq-1}.$$

Next we have,

$$\lambda^2-\overline{\lambda} = \lambda^2-\frac{1}{\lambda} = \frac{\lambda^3-1}{\lambda} = \frac{(\lambda-1)\left(\lambda^2+\lambda+1\right)}{\lambda} = \frac{(\lambda-1)\left(\frac{1}{pq}\lambda-\lambda\right)}{\lambda} = \frac{1-pq}{pq}\left(\lambda-1\right),$$

$$\frac{1}{\lambda^2-\lambda} = \frac{\overline{\lambda}}{\lambda-1} = -\frac{1}{4pq}\left[1+\frac{i}{2}\frac{\sqrt{4pq-1}}{4pq-1}\right]\left[(1-2pq)-i\sqrt{4pq-1}\right].$$

Therefore,

$$\frac{1}{\lambda^2-\lambda} = \frac{1}{2pq}\left[(pq-1)+i\left(3pq-1\right)\frac{\sqrt{4pq-1}}{4pq-1}\right],$$

and so

$$Re\left(\frac{1}{\lambda^2-\lambda}\right) = \frac{pq-1}{2pq}.$$

Likewise,

$$\frac{1}{\lambda^2-\overline{\lambda}} = \frac{pq}{1-pq}\frac{1}{\lambda-1} = \frac{pq}{1-pq}\left(-\frac{1}{2}-\frac{i}{2}\frac{\sqrt{4pq-1}}{4pq-1}\right),$$

$$\sigma^2 ab\frac{1}{\lambda^2-\lambda} = \frac{-p^2q^2}{(4pq-1)}\frac{1}{8p^4q^8}\left[A+iB\sqrt{4pq-1}\right]\frac{1}{2pq}\left[(pq-1)+i\left(3pq-1\right)\frac{\sqrt{4pq-1}}{4pq-1}\right],$$

$$Re\left(\sigma b a_{20}\right) = Re\left(\sigma^2 ab\frac{1}{\lambda^2-\lambda}\right) = \frac{-\left[A\left(pq-1\right)-B\left(3pq-1\right)\right]}{16p^3q^7\left(4pq-1\right)}, \tag{30}$$

$$Re\left(2\sigma c\overline{a_{22}}\right) = Re\left(2\sigma\overline{\sigma}c\overline{c}\frac{1}{\lambda^2-\overline{\lambda}}\right) = -\frac{1}{2}\frac{pq}{1-pq}2\sigma\overline{\sigma}c\overline{c} = \frac{pq}{pq-1}\sigma\overline{\sigma}c\overline{c}. \tag{31}$$

Using (19) we obtain

$$\sigma c\overline{\sigma c} = \frac{-q^3 - 2p^2q^2 + pq - p^3 + 4p^4q + 4q^4p + q^2p^5 - 7p^3q^3 + q^5p^2}{4p^2q^5\,(4pq-1)}.$$

From (28), (29), (30), (31), and (20) we obtain

$$Re\,(\alpha_2) = Re\,(2\sigma a a_{21}) + Re\,(\sigma b\overline{a_{21}}) + Re\,(\sigma b a_{20}) + Re\,(2\sigma c\overline{a_{22}}) + Re\,(\sigma e)\,,$$

that is,

$$\begin{aligned} Re\,(\alpha_2) \quad = \quad & \frac{-1}{(4pq-1)}\frac{1}{2pq^4}\,(p+1+q)\,(pq-1)\left(p^2-p-pq-q+q^2+1\right) \\ & + \frac{1}{2q^3\,(4pq-1)}\,(p+1+q)\left(p^2-p-pq-q+q^2+1\right) \\ & + \frac{-1}{(4pq-1)}\frac{1}{16p^3q^7}\left[4p^3q^4\,(p+1+q)\left(p^2-p-pq-q+q^2+1\right)\right] \\ & + \frac{pq}{pq-1}\sigma\overline{\sigma}c\overline{c} + \frac{1}{4pq^4}\left(-4pq+p^3+q^3+2\right) \end{aligned}$$

and

$$Re\,(\alpha_2) = \frac{\left(p^4q - p^3 + 8p^2q^2 - 2pq - 10p^3q^3 + q^4p - q^3 + 2q^2p^5 + 2q^5p^2\right)}{2pq^4\,(4pq-1)\,(pq-1)}. \qquad (32)$$

Combining (27) and $Re\,(\lambda) = \dfrac{1-2pq}{2pq}$ with (32), we obtain

$$\gamma_1(p,q) = \frac{-\left(p^4q - p^3 + 8p^2q^2 - 2pq - 10p^3q^3 + q^4p - q^3 + 2q^2p^5 + 2q^5p^2\right)}{q^3\,(4pq-1)\,(pq-1)\,\sqrt{4pq-1}}. \qquad (33)$$

It follows from (33) that $\gamma_1(p,q) = 0$ if and only if

$$p^4q - p^3 + 8p^2q^2 - 2pq - 10p^3q^3 + q^4p - q^3 + 2q^2p^5 + 2q^5p^2 = 0. \qquad (34)$$

In the special case $p = q$ we have

$$2p^5 - 2p^3 + 8p^4 - 2p^2 - 10p^6 + 4p^7 = p^2(2p+1)(p^2-p-1)(p-1)^2 = 0.$$

We conclude that either $p = 1$ or $p = \frac{1+\sqrt{5}}{2}$. In view of $p > 1$ we have that $p = \frac{1+\sqrt{5}}{2}$, which leaves the problem of stability of the equilibrium unanswered for this value of parameter as in [10]. This shows that in a more general case under the investigation we can not hope for more. In this case the problem of stability remains open along the curve $\gamma_1 = 0$. The role of a single point $p = \frac{1+\sqrt{5}}{2}$ in a constant coefficient case is played by the curve $\gamma_1 = 0$ in periodic case.

In our case $\alpha \neq \beta$ which implies $p \neq q, pq > 1, p > 1$ and $q > 1$.

If $\alpha < \beta$ (respectively $\alpha > \beta$) then $p < q$ (respectively $p > q$). If $\alpha < \beta$ (respectively $\alpha > \beta$) we have that $\gamma_1 = 0$ for all (p,q) located above (below) the line

$p = q$ in Figure 1. For such (p, q) (because of $\alpha = p^2 - q > 0$, $\beta = q^2 - p > 0$) the KAM theorem does not give any information about stability for all points that belong to the set F consisting of the arc of curve $\gamma_1 = 0$ between the points $A(\frac{1}{2}\sqrt[3]{12}, \frac{1}{2}\sqrt[3]{18})$ and $C(\frac{1}{2}\sqrt[3]{18}, \frac{1}{2}\sqrt[3]{12})$. The points A and C are obtained as intersections of $q^2 = p$ and $p^2 = q$ and curve $\gamma_1 = 0$. Specifically, $q^2 = p$ and $\gamma_1 = 0$ intersect at $(p, q) = (\frac{1}{2}\sqrt[3]{18}, \frac{1}{2}\sqrt[3]{12})$ and parabola $p^2 = q$ and $\gamma_1 = 0$ intersect at $(p, q) = (\frac{1}{2}\sqrt[3]{12}, \frac{1}{2}\sqrt[3]{18})$.

Indeed, for $q^2 = p$ we have that $\gamma_1 = 0$ if

$$p^4\sqrt{p} - p^3 + 8p^3 - 2p\sqrt{p} - 10p^4\sqrt{p} + p^3 - p\sqrt{p} + 2p^6 + 2p^4\sqrt{p} = 0.$$

that is

$$8p^3 + 2p^6 - 3p^{\frac{3}{2}} - 7p^{\frac{9}{2}} = 0,$$

whose solutions are: $\frac{1}{4}4^{\frac{2}{3}}\sqrt[3]{9}$, 0, 1.
For all other points the stability of the equilibrium follows from the KAM theorem. Thus we have proved the following result:

Theorem 5.1 *The period-two solution of (1) is stable for all points outside the set F.*

The region F is depicted on the following picture:

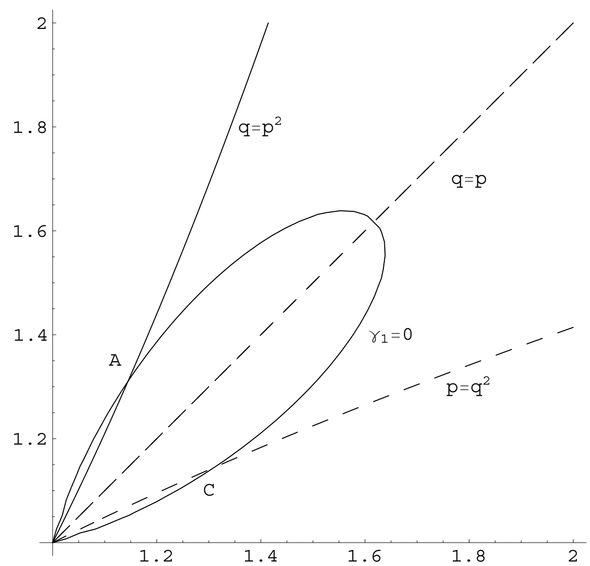

Figure 1: The region F is the part of the curve $\gamma_1 = 0$ between the parabolas $p^2 = q$ and $q^2 = p$ that is between the points A and C.

References

[1] E. Barbeau, B. Gelbord, and S. Tanny, Periodicity of solutions of the generalized Lyness recursion, *J. Difference Equ. Appl.* 1(1995), 291-306.

[2] G. Bastien and M. Rogalski, Global behavior of the solutions of Lyness' difference equation $u_{n+2}u_n = u_{n+1} + a$, *J. Difference Equ. Appl.* 10(2004), 977-1003.

[3] G. Bastien and M. Rogalski, New method for the study of solutions of the difference equation $u_{n+2} = \frac{u_{n+1}+a}{u_n+b}$, *Rad. Mat.* 12(2004), 135-152.

[4] G. Bastien and M. Rogalski, On some algebraic difference equations $u_{n+2}u_n = g(u_{n+1})$ related to families of conics or cubics: generalization of the Lyness' sequences, *J. Math. Anal. Appl.* 300(2004), 303-333.

[5] F. Beukers and R. Cushman, Zeeman's monotonicity conjecture. *J. Differential Equations* 143(1998), 191-200.

[6] J. K. Hale and H. Kocak, *Dynamics and Bifurcation*, Springer-Verlag, New York, (1991).

[7] E. Janowski, M. R. S. Kulenović, and Z. Nurkanović, Stability of the k-th order Lyness' equation with a period-k coefficient, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* 17 (2007), 143-152.

[8] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordreht/Boston/London, 1993.

[9] V. L. Kocic, G. Ladas, and I. W. Rodrigues, On the rational recursive sequences, *J. Math. Anal. Appl.* 173(1993), 127–157.

[10] V. L. Kocic, G. Ladas, G. Tzanetopoulos, and E. Thomas, On the stability of Lyness' equation, *Dynam. Contin. Discrete Impuls. Systems*, 1(1995), 245–254.

[11] M. R. S. Kulenović, Invariants and related Liapunov functions for difference equations, *Appl. Math. Lett.* 13(2000), 1-8.

[12] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman and Hall/CRC, Boca Raton, London, 2002.

[13] M. R. S. Kulenović and Z. Nurkanović, Stability of Lyness' Equation with Period-three Coefficient, *Rad. Mat.*, 12 (2004), 153-161.

[14] C. Siegel and J. Moser, *Lectures on Celestial Mechanics*, Springer-Varlag, New York, 1971.

[15] E. C. Zeeman, Geometric unfolding of a difference equation, Preprint, (Hertford College, Oxford) (1996).

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Intuitionistic Fuzzy BCC-Subalgebras of BCC-Algebras

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Abstract

Using the idea of intuitionistic fuzzy set due to Atanassov [2], we define the notion of intuitionistic fuzzy BCC-subalgebras as a natural generalization of fuzzy BCC-subalgebras due to Dudek et al. [7] and obtain some related results. Using a t-norm and a t-conorm, the direct product, and T-product and S-product of intuitionistic fuzzy BCC-subalgebras are discussed and their properties are investigated.

Keywords: t-norm; t-conorm; BCC-Algebra.

AMS Subject Classifications: 06F35, 03G25, 94D05

1 Introduction and preliminaries

A BCK-algebra is an important class of logical algebras introduced by Iseki [12] and was extensively investigated by several researchers. The class of all BCK-algebras is a quasivariety. Iseki [12] posed the interesting problem of whether the class of BCK-algebras is a variety. In connection with this problem, Komori [16] introduced the notion of BCC-algebras and Dudek [5,6] modified the notion of BCC-algebras by using a dual form of the definition in the sense of Komori [16]. Fuzzy structures in BCC-algebras were introduced by Dudek et al. [7-10] and fuzzy structures in BCK-algebras were introduced by Xi [21].

In the present paper, using the idea of intuitionistic fuzzy sets due to Atanassov [2,3] as a generalization of fuzzy sets [22], we define the notion of intuitionistic fuzzy BCC-subalgebras of the BCC-algebras with the help of t-conorms and t-norms as a generalization of fuzzy BCC-subalgebras due to Dudek et al. [7]. We also investigate the direct product and product of intuitionistic fuzzy BCC-subalgebras of BCC-algebras with the help of a t-conorm and a t-norm.

Let us recall [16] that a BCC-algebra is a nonempty set X with a constant 0 and a binary operation $*$ which satisfies the following conditions, for all $x, y, z \in X$: (i) $((x*y)*(z*y))*(x*z) = 0$; (ii) $x*x = 0$; (iii) $0*x = 0$; (iv) $x*0 = x$; (v) $x*y = 0$ and $y*x = 0$ imply $x = y$. A nonempty subset Y of a BCC-algebra X is called a BCC-subalgebra of X if $x*y \in Y$ for all $x, y \in Y$ (see also [13,21]).

By a t-norm T, we mean a binary operation on the unit interval $[0,1]$ which satisfies the following conditions, for all $x, y, z \in [0,1]$: (i) $T(x,1) = x$; (ii) $T(x,y) \leq T(x,z)$ if $y \leq z$; (iii) $T(x,y) = T(y,x)$; (iv) $T(x, T(y,z)) = T(T(x,y), z)$. For a t-norm T, we have $T(x,y) \leq$

$\min\{x, y\}$ for all $x, y \in [0, 1]$. Examples of t-norms are $T_L(x, y) = \max\{x + y - 1, 0\}$ and $T_M(x, y) = \min\{x, y\}$.

By a t-conorm S, we mean a binary operation on the unit interval $[0, 1]$ which satisfies the following conditions, for all $x, y, z \in [0, 1]$: (i) $S(x, 0) = x$; (ii) $S(x, y) \leq S(x, z)$ if $y \leq z$; (iii) $S(x, y) = S(y, x)$; (iv) $S(x, S(y, z)) = S(S(x, y), z)$. For a t-conorm S, we have $S(x, y) \geq \max\{x, y\}$ for all $x, y \in [0, 1]$. Examples of t-conorms are $S_L(x, y) = \min\{x + y, 1\}$ and $S_M(x, y) = \max\{x, y\}$.

A t-norm T and a t-conorm S are called associated [18], i.e. $S(x, y) = 1 - T(1 - x, 1 - y)$ for all $x, y \in [0, 1]$.

Note that, the concepts of t-conorms and t-norms are known as the axiomatic skeletons that we use for characterizing fuzzy unions and intersections, respectively. These concepts were originally introduced by Menger [19] and several properties and examples for these concepts were proposed by many authors (see [1,4,11,14,15,17-20]).

A fuzzy set A in an arbitrary non-empty set X is a function $\mu_A : X \to [0, 1]$. The complement of μ_A, denoted by μ_A^c, is the fuzzy set in X given by $\mu_A^c(x) = 1 - \mu_A(x)$ for all $x \in X$.

For any fuzzy set μ_A in X and any $\alpha \in [0, 1]$, Dudek et al. [10] defined two sets

$$U(\mu_A; \alpha) = \{x \in X : \mu_A(x) \geq \alpha\} \text{ and } V(\mu_A; \alpha) = \{x \in X : \mu_A(x) \leq \alpha\},$$

which are called an upper and lower α-level cut of μ_A and can be used to the characterization of μ_A.

Definition 1.1 (7). *A fuzzy set A in a BCC-algebra X is called a fuzzy BCC-subalgebra of X if*

$$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

for all $x, y \in X$.

Definition 1.2 (7). *A fuzzy set A in a BCC-algebra X is called a fuzzy BCC-subalgebra of X with respect to a t-norm T (or simply, a T-fuzzy BCC-subalgebra of X) if*

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$$

for all $x, y \in X$. Every BCC-algebra is a fuzzy BCC-algebra and so a T-fuzzy BCC-subalgebra but converse is not true (see [5,8,9]).

As a generalization of the notion of fuzzy sets in X, Atanassov [2] introduced the concept of intuitionistic fuzzy sets defined on X as objects having the form

$A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$,

where the functions $\mu_A : X \to [0, 1]$ and $\lambda_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

For every two intuitionistic fuzzy sets A and B in X, we have [3],

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,

(ii) $\Box A = \{(x, \mu_A(x), \mu_A^c(x)) : x \in X\}$,

(iii) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) : x \in X\}$.

2 (T,S)-intuitionistic fuzzy BCC-subalgebras

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$ as Dudek et al. [10].

DEFINITION 2.1. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in a BCC-algebra X is said to be an intuitionistic fuzzy BCC-subalgebra of X if*

(i) $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(ii) $\lambda_A(x * y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$

for all $x, y \in X$.

REMARK 2.1. *If $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, then every intuitionistic fuzzy BCC-subalgebra of a BCC-algebra X is a fuzzy BCC-subalgebra of X.*

DEFINITION 2.2. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in a BCC-algebra X is said to be an intuitionistic fuzzy BCC-subalgebra of X with respect to a t-norm T and a t-conorm S (or simply, a (T, S)-intuitionistic fuzzy BCC-subalgebra of X) if*

(i) $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$,

(ii) $\lambda_A(x * y) \leq S(\lambda_A(x), \lambda_A(y))$

for all $x, y \in X$.

Example. Let $X = \{0, 1, 2, 3, 4\}$ be a proper BCC-algebra with the Cayley table as follows

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Define intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X by

$$\mu_A(x) = \begin{cases} 1, & x \in \{0,1,2\} \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \lambda_A(x) = \begin{cases} 0, & x \in \{0,1,2\} \\ 1/2, & \text{otherwise.} \end{cases}$$

It is easy to check that $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, $\mu_A(x * y) \geq T_L(\mu_A(x), \mu_A(y))$ and $\lambda_A(x * y) \leq S_M(\lambda_A(x), \lambda_A(y))$ for all $x, y \in X$. Hence $A = (\mu_A, \lambda_A)$ is a (T_L, S_M)-intuitionistic fuzzy BCC-subalgebra of X. Also note that t-norm T_L and t-conorm S_M are not associated.

REMARK 2.2. *Note that, the above example holds even with the t-norm T_L and t-conorm S_L, and hence $A = (\mu_A, \lambda_A)$ is a (T_L, S_L)-intuitionistic fuzzy BCC-subalgebra of X.*

LEMMA 2.1. *If $A = (\mu_A, \lambda_A)$ is a (T, S)-intuitionistic fuzzy BCC-subalgebra of a BCC-algebra X, then so is $\Box A = (\mu_A, \mu_A^c)$ such that t-norm T and t-conorm S are associated.*

Proof. For all $x, y \in X$, we have

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)),$$

and so

$$1 - \mu_A^c(x * y) \geq T(1 - \mu_A^c(x), 1 - \mu_A^c(y)),$$

which implies

$$1 - T(1 - \mu_A^c(x), 1 - \mu_A^c(y)) \geq \mu_A^c(x * y).$$

Since T and S are associated, we have

$$S(\mu_A^c(x), \mu_A^c(y)) \geq \mu_A^c(x * y).$$

This completes the proof.

Lemma 2.2. *If $A=(\mu_A,\lambda_A)$ is a (T,S)-intuitionistic fuzzy BCC-subalgebra of a BCC-algebra X, then so is $\Diamond A=(\lambda_A^c,\lambda_A)$ such that t-norm T and t-conorm S are associated.*

Proof. The proof is similar to the proof of Lemma 2.1.

Combining the above two lemmas, it is easy to see that the following theorem is valid.

Theorem 2.1. *$A=(\mu_A,\lambda_A)$ is a (T,S)-intuitionistic fuzzy BCC-subalgebra of a BCC-algebra X if and only if $\Box A$ and $\Diamond A$ are (T,S)-intuitionistic fuzzy BCC-subalgebra of X such that t-norm T and t-conorm S are associated.*

Corollary 2.1. *$A=(\mu_A,\lambda_A)$ is a (T,S)-intuitionistic fuzzy BCC-subalgebra of a BCC-algebra X if and only if μ_A and λ_A^c are T-fuzzy BCC-subalgebra of X such that t-norm T and t-conorm S are associated.*

Theorem 2.2. *Let $A=(\mu_A,\lambda_A)$ be a (T,S)-intuitionistic fuzzy BCC-subalgebra of a BCC-algebra X and let $\alpha',\alpha\in[0,1]$. Then*

(i) if $\alpha'=0$ and $\alpha=1$, then $U(\mu_A;\alpha)\cap V(\lambda_A;\alpha')$ is either empty or a BCC-subalgebra of X.

(ii) if $T=T_M$ and $S=S_M$, then $U(\mu_A;\alpha)\cap V(\lambda_A;\alpha')$ is either empty or a BCC-subalgebra of X, moreover $\lambda_A(0)\le\lambda_A(x)$ and $\mu_A(0)\ge\mu_A(x)$ for all $x\in X$.

Proof. (i) Assume that $\alpha'=0$, $\alpha=1$ and $x,y\in U(\mu_A;\alpha)\cap V(\lambda_A;\alpha')$. Then $\mu_A(x)\ge\alpha=1,\mu_A(y)\ge\alpha=1,\lambda_A(x)\le\alpha'=0$ and $\lambda_A(y)\le\alpha'=0$. It follows from Definition 2.2

$\mu_A(x*y)\ge T(\mu_A(x),\mu_A(y))\ge T(1,1)=1$ and

$\lambda_A(x*y)\le S(\lambda_A(x),\lambda_A(y))\le S(0,0)=0$.

Hence, $x*y\in U(\mu_A;\alpha)\cap V(\lambda_A;\alpha')$. Thus, $U(\mu_A;\alpha)\cap V(\lambda_A;\alpha')$ is a BCC-subalgebra of X.

(ii) Assume that $T=T_M$, $S=S_M$ and $x,y\in U(\mu_A;\alpha)\cap V(\lambda_A;\alpha')$. Then

$\mu_A(x*y)\ge T(\mu_A(x),\mu_A(y))=\min\{\mu_A(x),\mu_A(y)\}\ge\min\{\alpha,\alpha\}=\alpha$ and

$\lambda_A(x*y)\le S(\lambda_A(x),\lambda_A(y))=\max\{\lambda_A(x),\lambda_A(y)\}\le\max\{\alpha',\alpha'\}=\alpha'$

for all $\alpha',\alpha\in[0,1]$. Hence, $x*y\in U(\mu_A;\alpha)\cap V(\lambda_A;\alpha')$. Thus $U(\mu_A;\alpha)\cap V(\lambda_A;\alpha')$ is a BCC-subalgebra of X. Moreover, since $x*x=0$ for all $x\in X$, we have

$\mu_A(0)=\mu_A(x*x)\ge T(\mu_A(x),\mu_A(x))=\min\{\mu_A(x),\mu_A(x)\}=\mu_A(x)$ and

$\lambda_A(0)=\lambda_A(x*x)\le S(\lambda_A(x),\lambda_A(x))=\max\{\lambda_A(x),\lambda_A(x)\}=\lambda_A(x)$.

This completes the proof.

Theorem 2.3. *Let $A=(\mu_A,\lambda_A)$ be a (T,S)-intuitionistic fuzzy BCC- subalgebra of a BCC-algebra X. If there exists a sequence $\{x_n\}$ in X such that*

$\lim_{n\to\infty}T(\mu_A(x_n),\mu_A(x_n))=1$ and $\lim_{n\to\infty}S(\lambda_A(x_n),\lambda_A(x_n))=0$

then $\mu_A(0)=1$ and $\lambda_A(0)=0$.

Proof. Let $x\in X$. Then

$\mu_A(0)=\mu_A(x*x)\ge T(\mu_A(x),\mu_A(x))$ and

$\lambda_A(0)=\lambda_A(x*x)\le S(\lambda_A(x),\lambda_A(x))$.

Therefore,

$\mu_A(0)\ge T(\mu_A(x_n),\mu_A(x_n))$ and $\lambda_A(0)\le S(\lambda_A(x_n),\lambda_A(x_n))$

for each $n\in\mathbb{N}$. Since

$1\ge\mu_A(0)\ge\lim_{n\to\infty}T(\mu_A(x_n),\mu_A(x_n))=1$ and

$0\le\lambda_A(0)\le\lim_{n\to\infty}S(\lambda_A(x_n),\lambda_A(x_n))=0$.

It follows that $\mu_A(0)=1$ *and* $\lambda_A(0)=0$.

If $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy set in a BCC-algebra X and f is a self mapping of X, we define mappings

$\mu_A[f] : X \to [0,1]$ by $\mu_A[f](x) = \mu_A(f(x))$
and
$\lambda_A[f] : X \to [0,1]$ by $\lambda_A[f](x) = \lambda_A(f(x))$
for all $x \in X$, respectively.

PROPOSITION 2.1. *If $A = (\mu_A, \lambda_A)$ is a (T,S)-intuitionistic fuzzy BCC- subalgebra of a BCC-algebra X and f is an endomorphism of X, then $(\mu_A[f], \lambda_A[f])$ is a (T,S)-intuitionistic fuzzy BCC-subalgebra of X.*

Proof. For any given $x, y \in X$, we have

$$\begin{aligned} \mu_A[f](x*y) &= \mu_A(f(x*y)) = \mu_A(f(x)*f(y)) \geq T(\mu_A(f(x)), \mu_A(f(y))) \\ &= T(\mu_A[f](x), \mu_A[f](y)), \end{aligned}$$

$$\begin{aligned} \lambda_A[f](x*y) &= \lambda_A(f(x*y)) = \lambda_A(f(x)*f(y)) \leq S(\lambda_A(f(x)), \lambda_A(f(y))) \\ &= S(\lambda_A[f](x), \lambda_A[f](y)). \end{aligned}$$

This completes the proof.

If f is a self mapping of a BCC-algebra X and $B = (\mu_B, \lambda_B)$ is an intuitionistic fuzzy set in $f(X)$, then the intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X defined by $\mu_A = \mu_B \circ f$ and $\lambda_A = \lambda_B \circ f$ is called the preimage of B under f.

THEOREM 2.4. *An onto homomorphic preimage of a (T,S)-intuitionistic fuzzy BCC-subalgebra is a (T,S)-intuitionistic fuzzy BCC-subalgebra.*

Proof. Let $f : X \to X'$ be an onto homomorphism of BCC-algebras, $B = (\mu_B, \lambda_B)$ be a (T,S)-intuitionistic fuzzy BCC-subalgebra of X', and $A = (\mu_A, \lambda_A)$ be preimage of B under f. Then, we have

$$\begin{aligned} \mu_A(x*y) &= \mu_B(f(x*y)) = \mu_B(f(x)*f(y)) \geq T(\mu_B(f(x)), \mu_B(f(y))) \\ &= T(\mu_A(x), \mu_A(y)), \end{aligned}$$

$$\begin{aligned} \lambda_A(x*y) &= \lambda_B(f(x*y)) = \lambda_B(f(x)*f(y)) \leq S(\lambda_B(f(x)), \lambda_B(f(y))) \\ &= S(\lambda_A(x), \lambda_A(y)) \end{aligned}$$

for all $x, y \in X$. Hence, $A = (\mu_A, \lambda_A)$ is a (T,S)-intuitionistic fuzzy BCC-subalgebra of X.

If f is a self mapping of a BCC-algebra X and $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set in X, then the intuitionistic fuzzy set $A^f = (\mu_A^f, \lambda_A^f)$ in $f(X)$ defined by

$$\mu_A^f(y) = \sup_{x \in f^{-1}(y)} \mu_A(x) \text{ and } \lambda_A^f(y) = \inf_{x \in f^{-1}(y)} \lambda_A(x) \text{ for all } y \in f(x)$$

is called image of $A = (\mu_A, \lambda_A)$ under f.

An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in X is said to have sup property and inf property if there exists a $t_0 \in T$ such that $\mu_A(t_0) = \sup_{t \in T} \mu_A(t)$ and $\lambda_A(t_0) = \inf_{t \in T} \lambda_A(t)$ for every subset $T \subseteq X$.

PROPOSITION 2.2. *An onto homomorphic image of an intuitionistic fuzzy BCC-subalgebra with sup property and inf property is an intuitionistic fuzzy BCC-subalgebra.*

Proof. Let $f : X \to X'$ be an onto homomorphism of BCC-algebras and $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy BCC-subalgebra of X with sup property and inf property. For given $x', y' \in X'$, let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ such that $\mu_A(x_0) = \sup_{t \in f^{-1}(x')} \mu_A(t)$,

$\mu_A(y_0) = \sup_{t\in f^{-1}(y')} \mu_A(t)$, $\lambda_A(x_0) = \inf_{t\in f^{-1}(x')} \lambda_A(t)$ and $\lambda_A(y_0) = \inf_{t\in f^{-1}(y')} \lambda_A(t)$, respectively. Then

$$\begin{aligned}
\mu_A^f(x' * y') &= \sup_{z\in f^{-1}(x'*y')} \mu_A(z) \geq \min\{\mu_A(x_0), \mu_A(y_0)\} \\
&= \min\left\{ \sup_{t\in f^{-1}(x')} \mu_A(t), \sup_{t\in f^{-1}(y')} \mu_A(t) \right\} \\
&= \min\left\{ \mu_A^f(x'), \mu_A^f(y') \right\},
\end{aligned}$$

$$\begin{aligned}
\lambda_A^f(x' * y') &= \inf_{z\in f^{-1}(x'*y')} \lambda_A(z) \leq \max\{\lambda_A(x_0), \lambda_A(y_0)\} \\
&= \max\left\{ \inf_{t\in f^{-1}(x')} \lambda_A(t), \inf_{t\in f^{-1}(y')} \lambda_A(t) \right\} \\
&= \max\left\{ \lambda_A^f(x'), \lambda_A^f(y') \right\}.
\end{aligned}$$

Hence, $A^f = (\mu_A^f, \lambda_A^f)$ is an intuitionistic fuzzy BCC-subalgebra of X'.

Definition 2.3 (7,11,15). *A t-norm T and a t-conorm S are called continuous t-norm and continuous t-conorm if T and S are continuous functions from $[0,1] \times [0,1]$ to $[0,1]$ with respect to the usual topology.*

Note that the functions "min" and "max" are continuous t-norm and t-conorm, respectively. Using Definition 2.3, Proposition 2.2 can be strengthened in the following way:

Theorem 2.5. *Let T be a continuous t-norm, S be a continuous t-conorm and f be a homomorphism on a BCC-algebra X. If $A = (\mu_A, \lambda_A)$ is a (T,S)-intuitionistic fuzzy BCC-subalgebra of X, then $A^f = (\mu_A^f, \lambda_A^f)$ is a (T,S)-intuitionistic fuzzy BCC-subalgebra of $f(X)$.*

Proof. Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1 * y_2)$, where $y_1, y_2 \in f(X)$. Consider the set

$$A_1 * A_2 = \{x \in X : x = x_1 * x_2 \text{ for some } x_1 \in A_1, x_2 \in A_2\}.$$

If $x \in A_1 * A_2$ then $x = x_1 * x_2$ for some $x_1 \in A_1, x_2 \in A_2$. Then, we have

$$f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2,$$

that is, $x \in f^{-1}(y_1 * y_2) = A_{12}$. So, $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned}
\mu_A^f(y_1 * y_2) &= \sup_{x\in f^{-1}(y_1*y_2)} \mu_A(x) = \sup_{x\in A_{12}} \mu_A(x) \geq \sup_{x\in A_1*A_2} \mu_A(x) \\
&\geq \sup_{x_1\in A_1, x_2\in A_2} \mu_A(x_1 * x_2) \geq \sup_{x_1\in A_1, x_2\in A_2} T(\mu_A(x_1), \mu_A(x_2))
\end{aligned}$$

$$\begin{aligned}
\lambda_A^f(y_1 * y_2) &= \inf_{x\in f^{-1}(y_1*y_2)} \lambda_A(x) = \inf_{x\in A_{12}} \lambda_A(x) \leq \inf_{x\in A_1*A_2} \lambda_A(x) \\
&\leq \inf_{x_1\in A_1, x_2\in A_2} \lambda_A(x_1 * x_2) \leq \inf_{x_1\in A_1, x_2\in A_2} S(\lambda_A(x_1), \lambda_A(x_2)).
\end{aligned}$$

Since S and T are continuous, for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $\sup_{x_1\in A_1} \mu_A(x_1) - \mu_A(x_1) \le \delta, \sup_{x_2\in A_2} \mu_A(x_2) - \mu_A(x_2) \le \delta, \lambda_A(x_1) - \inf_{x_1\in A_1} \lambda_A(x_1) \le \delta$ and $\lambda_A(x_2) - \inf_{x_2\in A_2} \lambda_A(x_2) \le \delta$ then

$$T(\sup_{x_1\in A_1} \mu_A(x_1), \sup_{x_2\in A_2} \mu_A(x_2)) - T(\mu_A(x_1), \mu_A(x_2)) \le \varepsilon \text{ and}$$
$$S(\lambda_A(x_1), \lambda_A(x_2)) - S(\inf_{x_1\in A_1} \lambda_A(x_1), \inf_{x_2\in A_2} \lambda_A(x_2)) \le \varepsilon.$$

Consequently

$$\begin{aligned}
\mu_A^f(y_1 * y_2) &\ge \sup_{x_1\in A_1, x_2\in A_2} T(\mu_A(x_1), \mu_A(x_2)) \\
&\ge T(\sup_{x_1\in A_1} \mu_A(x_1), \sup_{x_2\in A_2} \mu_A(x_2)) \\
&= T(\sup_{x_1\in f^{-1}(y_1)} \mu_A(x_1), \sup_{x_2\in f^{-1}(y_2)} \mu_A(x_2)) \\
&= T(\mu_A^f(y_1), \mu_A^f(y_2)),
\end{aligned}$$

$$\begin{aligned}
\lambda_A^f(y_1 * y_2) &\le \inf_{x_1\in A_1, x_2\in A_2} S(\lambda_A(x_1), \lambda_A(x_2)) \\
&\le S(\inf_{x_1\in A_1} \lambda_A(x_1), \inf_{x_2\in A_2} \lambda_A(x_2)) \\
&= S(\inf_{x_1\in f^{-1}(y_1)} \lambda_A(x_1), \inf_{x_2\in f^{-1}(y_2)} \lambda_A(x_2)) \\
&= S(\lambda_A^f(y_1), \lambda_A^f(y_2)).
\end{aligned}$$

This completes the proof.

LEMMA 2.3 (11). *Let T and S be a t-norm and a t-conorm, respectively. Then*

$$T(T(x,y), T(z,t)) = T(T(x,z), T(y,t))$$

and

$$S(S(x,y), S(z,t)) = S(S(x,z), S(y,t))$$

for all $x, y, z, t \in [0,1]$.

THEOREM 2.6. *Let T be a t-norm, S be a t-conorm and $X = X_1 \times X_2$ be the direct product BCC-algebra of BCC-algebras X_1 and X_2. If $A_1 = (\mu_{A_1}, \lambda_{A_1})$ (resp. $A_2 = (\mu_{A_2}, \lambda_{A_2})$) is a (T,S)-intuitionistic fuzzy BCC-subalgebra of X_1 (resp. X_2), then $A = (\mu_A, \lambda_A)$ is a (T,S)-intuitionistic fuzzy BCC-subalgebra of X defined by $\mu_A = \mu_{A_1} \times \mu_{A_2}, \lambda_A = \lambda_{A_1} \times \lambda_{A_2}$ and*

$$\begin{aligned}
\mu_A(x_1,x_2) &= (\mu_{A_1} \times \mu_{A_2})(x_1,x_2) = T(\mu_{A_1}(x_1), \mu_{A_2}(x_2)), \\
\lambda_A(x_1,x_2) &= (\lambda_{A_1} \times \lambda_{A_2})(x_1,x_2) = S(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2))
\end{aligned}$$

for all $(x_1, x_2) \in X$.

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of X. Then, we have

$$\begin{aligned}
\mu_A(x * y) &= \mu_A((x_1,x_2) * (y_1,y_2)) = \mu_A(x_1 * y_1, x_2 * y_2) \\
&= T(\mu_{A_1}(x_1 * y_1), \mu_{A_2}(x_2 * y_2)) \\
&\ge T(T(\mu_{A_1}(x_1), \mu_{A_1}(y_1)), T(\mu_{A_2}(x_2), \mu_{A_2}(y_2))) \\
&= T(T(\mu_{A_1}(x_1), \mu_{A_2}(x_2)), T(\mu_{A_1}(y_1), \mu_{A_2}(y_2))) \\
&= T(\mu_A(x_1,x_2), \mu_A(y_1,y_2)) \\
&= T(\mu_A(x), \mu_A(y)),
\end{aligned}$$

$$\begin{aligned}\lambda_A(x*y) &= \lambda_A((x_1,x_2)*(y_1,y_2)) = \lambda_A(x_1*y_1, x_2*y_2)\\ &= S(\lambda_{A_1}(x_1*y_1), \lambda_{A_2}(x_2*y_2))\\ &\leq S(S(\lambda_{A_1}(x_1),\lambda_{A_1}(y_1)), S(\lambda_{A_2}(x_2),\lambda_{A_2}(y_2)))\\ &= S(S(\lambda_{A_1}(x_1),\lambda_{A_2}(x_2)), S(\lambda_{A_1}(y_1),\lambda_{A_2}(y_2)))\\ &= S(\lambda_A(x_1,x_2),\lambda_A(y_1,y_2))\\ &= S(\lambda_A(x),\lambda_A(y)).\end{aligned}$$

This completes the proof.

Now, we will generalize the idea to the product of n (T,S)-intuitionistic fuzzy BCC-subalgebras. First of all, we need the followings:

DEFINITION 2.4 (11). *Let T be a t-norm and S be a t-conorm. The functions $T_n : \prod_{i=1}^{n}[0,1] \to [0,1]$ and $S_n : \prod_{i=1}^{n}[0,1] \to [0,1]$ are a t-norm and a t-conorm defined by*

$$T_n(x_1,x_2,...,x_n) = T(x_i, T(x_1,x_2,...,x_{i-1},x_{i+1},...,x_n))$$

and

$$S_n(x_1,x_2,...,x_n) = S(x_i, S(x_1,x_2,...,x_{i-1},x_{i+1},...,x_n))$$

for all $x_i \in [0,1]$ where $1 \leq i \leq n$, $n \geq 2, S_2 = S, T_2 = T, S_1 = id$ and $T_1 = id$ (identity), respectively.

LEMMA 2.4 (11). *Let T be a t-norm, S be a t-conorm. Then*

$$T_n(T(x_1,y_1),T(x_2,y_2),...,T(x_n,y_n)) = T(T_n(x_1,x_2,...,x_n),T_n(x_1,x_2,...,x_n))$$

and

$$S_n(S(x_1,y_1),S(x_2,y_2),...,S(x_n,y_n)) = S(S_n(x_1,x_2,...,x_n),S_n(x_1,x_2,...,x_n))$$

for all $x_i, y_i \in [0,1]$ where $1 \leq i \leq n$ and $n \geq 2$.

THEOREM 2.7. *Let T be a t-norm, S be a t-conorm, $\{X_i\}_{i=1}^{n}$ be the finite collection of BCC-algebras and $X = X_1 \times X_2 \times ... \times X_n$ be the direct product BCC-algebra of $\{X_i\}$. If $A_i = (\mu_{A_i}, \lambda_{A_i})$ is a (T,S)-intuitionistic fuzzy BCC-subalgebra of X_i, then $A = (\mu_A, \lambda_A)$ defined by*

$$\begin{aligned}\mu_A(x_1,x_2,...,x_n) &= (\prod_{i=1}^{n}\mu_{A_i})(x_1,x_2,...,x_n)\\ &= T_n(\mu_{A_1}(x_1),\mu_{A_2}(x_2),...,\mu_{A_n}(x_n))\end{aligned}$$

and

$$\begin{aligned}\lambda_A(x_1,x_2,...,x_n) &= (\prod_{i=1}^{n}\lambda_{A_i})(x_1,x_2,...,x_n)\\ &= S_n(\lambda_{A_1}(x_1),\lambda_{A_2}(x_2),...,\lambda_{A_n}(x_n))\end{aligned}$$

is a (T,S)-intuitionistic fuzzy BCC-subalgebra of the BCC-algebra X.

Proof. Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be any elements of X. Then, we have

$$\begin{aligned}
\mu_A(x * y) &= \mu_A(x_1 * y_1, x_2 * y_2, , ..., x_n * y_n) \\
&= T_n(\mu_{A_1}(x_1 * y_1), \mu_{A_2}(x_2 * y_2), ..., \mu_{A_n}(x_n * y_n)) \\
&\geq T_n(T(\mu_{A_1}(x_1), \mu_{A_1}(y_1)), T(\mu_{A_2}(x_2), \mu_{A_2}(y_2)) \\
&\quad , ..., T(\mu_{A_n}(x_n), \mu_{A_n}(y_n))) \\
&= T(T_n(\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)), T_n(\mu_{A_1}(y_1), \mu_{A_2}(y_2) \\
&\quad , ..., \mu_{A_n}(y_n))) \\
&= T(\mu_A(x_1, x_2, ..., x_n), \mu_A(y_1, y_2, ..., y_n)) \\
&= T(\mu_A(x), \mu_A(y)),
\end{aligned}$$

$$\begin{aligned}
\lambda_A(x * y) &= \lambda_A(x_1 * y_1, x_2 * y_2, , ..., x_n * y_n) \\
&= S_n(\lambda_{A_1}(x_1 * y_1), \lambda_{A_2}(x_2 * y_2), ..., \lambda_{A_n}(x_n * y_n)) \\
&\leq S_n(S(\lambda_{A_1}(x_1), \lambda_{A_1}(y_1)), S(\lambda_{A_2}(x_2), \lambda_{A_2}(y_2)) \\
&\quad , ..., S(\lambda_{A_n}(x_n), \lambda_{A_n}(y_n))) \\
&= S(S_n(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2), ..., \lambda_{A_n}(x_n)), S_n(\lambda_{A_1}(y_1), \lambda_{A_2}(y_2) \\
&\quad , ..., \lambda_{A_n}(y_n))) \\
&= S(\lambda_A(x_1, x_2, ..., x_n), \lambda_A(y_1, y_2, ..., y_n)) \\
&= S(\lambda_A(x), \lambda_A(y)).
\end{aligned}$$

Hence, $A = (\mu_A, \lambda_A)$ is a (T, S)-intuitionistic fuzzy BCC-subalgebra of X.

DEFINITION 2.5. *Let T be a t-norm, S be a t-conorm and $A = (\mu_A, \lambda_A)$, $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets in a BCC-algebra X. Then T-product of μ_A and μ_B, and S-product of λ_A and λ_B, written $[\mu_A.\mu_B]_T$ and $[\lambda_A.\lambda_B]_S$, are defined by*

$$[\mu_A.\mu_B]_T(x) = T(\mu_A(x), \mu_B(x))$$

and

$$[\lambda_A.\lambda_B]_S(x) = S(\lambda_A(x), \lambda_B(x))$$

for all $x \in X$, respectively.

THEOREM 2.8. *Let T be a t-norm, S be a t-conorm, and $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be (T, S)-intuitionistic fuzzy BCC-subalgebras of a BCC-algebra X. If T_1 is a t-norm which dominates T, that is,*

$$T_1(T(x, y), T(z, t)) \geq T(T_1(x, z), T_1(y, t))$$

and S_1 is a t-conorm which dominates S, that is,

$$S_1(S(x, y), S(z, t)) \leq S(S_1(x, z), S_1(y, t))$$

for all $x, y, z, t \in [0, 1]$, then $([\mu_A.\mu_B]_{T_1}, [\lambda_A.\lambda_B]_{S_1})$ is a (T, S)-intuitionistic fuzzy BCC-subalgebra of X.

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
[\mu_A.\mu_B]_{T_1}(x*y) &= T_1(\mu_A(x*y), \mu_B(x*y)) \\
&\geq T_1(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))) \\
&\geq T(T_1(\mu_A(x), \mu_B(x)), T_1(\mu_A(y), \mu_B(y))) \\
&= T([\mu_A.\mu_B]_{T_1}(x), [\mu_A.\mu_B]_{T_1}(y)),
\end{aligned}$$

$$\begin{aligned}
[\lambda_A.\lambda_B]_{S_1}(x*y) &= S_1(\lambda_A(x*y), \lambda_B(x*y)) \\
&\leq S_1(S(\lambda_A(x), \lambda_A(y)), S(\lambda_B(x), \lambda_B(y))) \\
&\leq S(S_1(\lambda_A(x), \lambda_B(x)), S_1(\lambda_A(y), \lambda_B(y))) \\
&= S([\lambda_A.\lambda_B]_{S_1}(x), [\lambda_A.\lambda_B]_{S_1}(y)).
\end{aligned}$$

This completes the proof.

Let $f : X \to X^{\prime}$ be an onto homomorphism of BCC-algebras. Let T and T_1 be t-norms such that T_1 dominates T, and S and S_1 be t-conorms such that S_1 dominates S. If $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ are (T,S)-intuitionistic fuzzy BCC-subalgebras $X^{\prime}$, then $([\mu_A.\mu_B]_{T_1}, [\lambda_A.\lambda_B]_{S_1})$ is a (T,S)- intuitionistic fuzzy BCC-subalgebra of $X^{\prime}$. Since every onto homomorphic preimage of a (T,S)-intuitionistic fuzzy BCC-subalgebra is a (T,S)-intuitionistic fuzzy BCC-subalgebra, then $f^{-1}(A)$, $f^{-1}(B)$ and $(f^{-1}([\mu_A.\mu_B]_{T_1}), f^{-1}([\lambda_A.\lambda_B]_{S_1}))$ are (T,S)-intuitionistic fuzzy BCC-subalgebras of X. The next theorem provides that the relation between $f^{-1}([\mu_A.\mu_B]_{T_1})$ and the T_1-product $[f^{-1}(\mu_A).\ f^{-1}(\mu_B)]_{T_1}$, and $f^{-1}([\lambda_A.\lambda_B]_{S_1})$ and the S_1-product $[f^{-1}(\lambda_A).f^{-1}(\lambda_B)]_{S_1}$, respectively.

Theorem 2.9. *Let $f : X \to X^{\prime}$ be an onto homomorphism of BCC-algebras, T and T_1 be t-norms such that T_1 dominates T, and S and S_1 be t-conorms such that S_1 dominates S. Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be (T,S)-intuitionistic fuzzy BCC-subalgebras of a BCC-algebra $X^{\prime}$. If $[\mu_A.\mu_B]_{T_1}$ is the T_1-product of μ_A and μ_B such that $[f^{-1}(\mu_A).f^{-1}(\mu_B)]_{T_1}$ is the T_1-product of $f^{-1}(\mu_A)$ and $f^{-1}(\mu_B)$, and $[\lambda_A.\lambda_B]_{S_1}$ is the S_1-product of λ_A and λ_B such that $[f^{-1}(\lambda_A).f^{-1}(\lambda_B)]_{S_1}$ is the S_1-product of $f^{-1}(\lambda_A)$ and $f^{-1}(\lambda_B)$ then*

$$f^{-1}([\mu_A.\mu_B]_{T_1}) = [f^{-1}(\mu_A).f^{-1}(\mu_B)]_{T_1},$$
$$f^{-1}([\lambda_A.\lambda_B]_{S_1}) = [f^{-1}(\lambda_A).f^{-1}(\lambda_B)]_{S_1}.$$

Proof. For any $x \in X$, we have

$$\begin{aligned}
[f^{-1}([\mu_A.\mu_B]_{T_1})](x) &= [\mu_A.\mu_B]_{T_1}(f(x)) = T_1(\mu_A(f(x)), \mu_B(f(x))) \\
&= T_1([f^{-1}(\mu_A)](x), [f^{-1}(\mu_B)](x)) \\
&= [f^{-1}(\mu_A).f^{-1}(\mu_B)]_{T_1}(x),
\end{aligned}$$

$$\begin{aligned}
[f^{-1}([\lambda_A.\lambda_B]_{S_1})](x) &= [\lambda_A.\lambda_B]_{S_1}(f(x)) = S_1(\lambda_A(f(x)), \lambda_B(f(x))) \\
&= S_1([f^{-1}(\lambda_A)](x), [f^{-1}(\lambda_B)](x)) \\
&= [f^{-1}(\lambda_A).f^{-1}(\lambda_B)]_{S_1}(x).
\end{aligned}$$

This completes the proof.

3 Conclusions

In this work, we generalize the notion of fuzzy BCC-subalgebras intraduced by Dudek et al. [7] to (T, S)-intuitionistic fuzzy BCC-subalgebras with the help of arbitrary t-norms and t-conorms. Furthermore, we obtain some results such as; every fuzzy BCC-subalgebra and T-fuzzy BCC-subalgebra of a BCC-algebra are (T, S)-intuitionistic fuzzy BCC-subalgebra but converse is not true and direct product of two (T, S)-intuitionistic fuzzy BCC-subalgebras of a BCC-algebra is a (T, S)-intuitionistic fuzzy BCC-subalgebra. This generalization also answers the question for intuitionistic fuzzy sets, settled by Iseki [12]. Moreover, using that generalization, one could define the notions intuitionistic fuzzy subgroups and intuitionistic fuzzy BCC-ideals in BCC-algebras with the respect to a t-norm and a t-conorm in the sense of [1] and [8], respectively. Using the idea of Dudek et al. [9], one could also generalize the notion of fuzzy topological BCC-algebras to intuitionistic fuzzy structures.

The notions given in this paper can be fundamental to other sciences. For instance, in the last decade, most of researchers are focused on Content Based Image Retrieval, shortly CBIR, and managing uncertainty becomes a fundamental topic in image database. Intuitionistic fuzzy set theory can be ideally suited to deal with this kind of uncertainty. This fuzziness is mainly due to similarity of media features, imperfection in the feature extraction algorithms, etc. Using the concept of this paper, one could develop an intuitionistic fuzzy model for image data and provide an intuitionistic fuzzy subalgebra for dealing with such data. Moreover, new intuitionistic fuzzy algebraic operators could be defined in order to capture the fuzziness related to the semantic descriptors of an image, and built thematic categorizations of multimedia documents using ontological information and intuitionistic fuzzy subalgebra in triangular norm systems.

References

[1] M.T. Abu Osman, On some product of fuzzy subgroups, Fuzzy Sets Syst. 24 (1987), 79-86.

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1986), 87-96.

[3] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets Syst. 61 (1994), 137-142.

[4] D. Dubois, H. Prade, New results about properties and semantics of fuzzy set-theoretic operators. In: Wang PP, Chang SK, editors. Fuzzy sets: theory and applications to policy analysis and information systems. New York: Plenum Press, 1980.

[5] W.A. Dudek, The number of subalgebras of finite BCC-algebras, Bull. Inst. Math. Acad. Sinica 20 (1992), 129-136.

[6] W.A. Dudek, On proper BCC-algebras, Bull. Inst. Math. Acad. Sinica 20 (1992), 137-150.

[7] W.A. Dudek, K.H. Kim, Y.B. Jun, Fuzzy BCC-subalgebras of BCC-algebras with respect to a t-norm, Sci. Math. 3 (2000), 99-106.

[8] W.A. Dudek, Y.B. Jun, Fuzzy BCC-ideals in BCC-algebras, Math. Montisnigri 10 (1999), 21-30.

[9] W.A. Dudek, Y.B. Jun, S.M. Hong, On fuzzy topological BCC-algebras, Discuss Math. Gen. Algebra Appl. 20(1) (2000), 77-86.

[10] W.A. Dudek, B. Davvaz, Y.B. Jun, On intuitionistic fuzzy sub-hyperquasigroups of hyperquasigroups. Inform. Sci. 170 (2005), 251-262.

[11] O. Hadzic, E. Pap, Fixed point theory in probabilistic metric spaces, Dordrecht: Kluwer. Acad.. Publishers, 2001.

[12] K. Iseki, BCK-algebras with condition (S), Math. Japon. 24(1) (1979), 107-119.

[13] Y.B. Jun, Q. Zhang, Fuzzy subalgebras of BCK-algebras with respect to a t-norm, Far East J. Math. Sci. 2(3) (2000), 489-495.

[14] E.P. Klement, Operations on fuzzy sets: an axiomatic approach, Inform. Sci. 27 (1984), 221-232.
[15] E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Dordrecht: Kluwer Acad. Publishers, 2000.
[16] Y. Komori, The class of BCC-algebras is not a veriety, Math. Japon. 29 (1984), 391-394.
[17] S. Kutukcu, Topics in intuitionistic fuzzy metric spaces, J. Comput. Anal. Appl. 9(2) (2007), 173-180.
[18] R. Lowen, Fuzzy set theory, Dordrecht: Kluwer Acad. Publishers, 1996.
[19] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. 28 (1942), 535-537.
[20] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 314-334.
[21] O.G. Xi, Fuzzy BCK-algebras, Math. Japon. 36(5) (1991), 935-942.
[22] L.A. Zadeh, Fuzzy sets, Inform Control 8 (1965), 338-353.

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A common fixed point theorem for contractive mappings

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Abstract

The purpose of this paper is to obtain a new contraction mapping and a common fixed point theorem by using a new continuity condition in intuitionistic fuzzy metric spaces. This gives a generalization of the results by Pant and Jha [11], and Balasubramaniam et al. [2]. We also give an answer the open problem of Rhoades [13] in intuitionistic fuzzy metric spaces.

Keywords: Common Fixed Points; R-weakly Commuting Maps; Contractive Conditions; Reciprocal Continuity.

AMS Subject Classifications: 47H10, 54H25

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [19]. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [5], and George and Veeramani [3] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Recently, many authors have proved fixed point theorems involving fuzzy sets [2,4,6-11,14-18]. Vasuki [18] investigated some fixed point theorems in fuzzy metric spaces for R-weakly commuting mappings and Pant [9] introduced the notion of reciprocal continuity of mappings in metric spaces. Balasubramaniam et al. [2] proved the open problem of Rhoades [13] on the existence of a contractive definition which generates a fixed point but does not force the mapping to be continuous at the fixed point possesses an affirmative answer. Pant and Jha [11] proved an analogue of the result given by Balasubramaniam et al. [2].

The purpose of this paper is to introduce R-weakly commutativity and to prove a common fixed point theorem in intuitionistic fuzzy metric spaces by studying the relationship between the continuity and reciprocal continuity. This gives a generalization of the results by Pant and Jha [11], and Balasubramaniam et al. [2]. We also give an answer the open problem of Rhoades [11] in intuitionistic fuzzy metric spaces and an example to illustrate the main theorem.

2 Preliminaries

Definition 2.1. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if* $*$ *is satisfying the following conditions:*

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

DEFINITION 2.2. *A binary operation* $\Diamond : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-conorm if* $\Diamond$ *is satisfying the following conditions:*
(a) $\Diamond$ is commutative and associative;
(b) $\Diamond$ is continuous;
(c) $a \Diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

DEFINITION 2.3 (12). *A 5-tuple* $(X, M, N, *, \Diamond)$ *is said to be an intuitionistic fuzzy metric space (shortly, IFM-space) if* X *is an arbitrary set,* $*$ *is a continuous t-norm,* $\Diamond$ *is a continuous t-conorm and* M, N *are fuzzy sets on* $X^2 \times (0, \infty)$ *satisfying the following conditions: for all* x, y, $z \in X$, s, $t > 0$,
(IFM-1) $M(x,y,t) + N(x,y,t) \leq 1$;
(IFM-2) $M(x,y,t) > 0$;
(IFM-3) $M(x,y,t) = 1$ if and only if $x = y$;
(IFM-4) $M(x,y,t) = M(y,x,t)$;
(IFM-5) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(IFM-6) $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous;
(IFM-7) $N(x,y,t) < 1$;
(IFM-8) $N(x,y,t) = 0$ if and only if $x = y$;
(IFM-9) $N(x,y,t) = N(y,x,t)$;
(IFM-10) $N(x,y,t) \Diamond N(y,z,s) \geq N(x,z,t+s)$;
(IFM-11) $N(x,y,.) : (0,\infty) \to [0,1)$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

REMARK 2.1 (12). *Every fuzzy metric space* $(X, M, *)$ *is an IFM-space of the form* $(X, M, 1-M, *, \Diamond)$ *such that t-norm* $*$ *and t-conorm* $\Diamond$ *are associated, i.e.* $x \Diamond y = 1 - ((1-x) * (1-y))$ *for any* $x, y \in [0,1]$. *But the converse is not true.*

Example. [Induced intuitionistic fuzzy metric (12)] Let (X, d) be a metric space. Denote $a * b = ab$ and $a \Diamond b = \min\{1, a+b\}$ for all $a, b \in [0,1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}, \quad N_d(x,y,t) = \frac{d(x,y)}{t + d(x,y)}$$

Then (M_d, N_d) is an intuitionistic fuzzy metric on X. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric. Note that (M_d, N_d) is also an intuitionistic fuzzy metric X even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \Diamond b = \max\{a, b\}$.

DEFINITION 2.4 (12). *Let* $(X, M, N, *, \Diamond)$ *be an IFM-space. Then*
(a) a sequence $\{x_n\}$ in X is said to be convergent x in X if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x, t) > 1 - \epsilon$ and $N(x_n, x, t) < \epsilon$ for all $n \geq n_0$.
(b) a sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ and $N(x_n, x_m, t) < \epsilon$ for all $n, m \geq n_0$.
(c) An intuitionistic fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Throughout this paper, $(X, M, N, *, \Diamond)$ will denote the IFM-space with the following conditions: for all $x, y \in X$ and $t > 0$,

(IFM-12) $\lim_{t\to\infty} M(x, y, t) = 1$;

(IFM-13) $\lim_{t\to\infty} N(x, y, t) = 0$.

PROPOSITION 2.1. *Let $(X, M, N, *, \Diamond)$ be an IFM-space and $\{x_n\}, \{y_n\}$ be sequences in X such that $x_n \to x$, $y_n \to y$ as $n \to \infty$. If t is a continuous point of M and N then $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ and $\lim_{n\to\infty} N(x_n, y_n, t) = N(x, y, t)$.*

Proof. Let $\{x_n\}, \{y_n\}$ be sequences in X such that $x_n \to x$, $y_n \to y$ as $n \to \infty$. Fix $\delta > 0$ such that $\delta < t/2$. Then there is $n_0 \in \mathbb{N}$ such that

$$M(x_n, y_n, t) \geq M(x_n, y_n, t-\delta) \geq M(x_n, x, \delta/2) * M(x, y, t-2\delta) * M(y, y_n, \delta/2),$$

$$N(x_n, y_n, t) \leq N(x_n, y_n, t-\delta) \leq N(x_n, x, \delta/2) \Diamond N(x, y, t-2\delta) \Diamond N(y, y_n, \delta/2)$$

and

$$M(x, y, t+2\delta) \geq M(x, y, t+\delta) \geq M(x, x_n, \delta/2) * M(x_n, y_n, t) * M(y_n, y, \delta/2),$$

$$N(x, y, t+2\delta) \leq N(x, y, t+\delta) \leq N(x, x_n, \delta/2) \Diamond N(x_n, y_n, t) \Diamond N(y_n, y, \delta/2)$$

for all $n \geq n_0$. Letting $n \to \infty$ implies

$$\lim_{n\to\infty} M(x_n, y_n, t) \geq 1 * M(x, y, t-2\delta) * 1 = M(x, y, t-2\delta),$$

$$\lim_{n\to\infty} N(x_n, y_n, t) \leq 0 \Diamond N(x, y, t-2\delta) \Diamond 0 = N(x, y, t-2\delta)$$

and

$$M(x, y, t+2\delta) \geq 1 * \lim_{n\to\infty} M(x_n, y_n, t) * 1 = \lim_{n\to\infty} M(x_n, y_n, t),$$

$$N(x, y, t+2\delta) \leq 0 \Diamond \lim_{n\to\infty} N(x_n, y_n, t) \Diamond 0 = \lim_{n\to\infty} N(x_n, y_n, t).$$

So, by continuity of the functions $t \to M(x, y, t)$ and $t \to N(x, y, t)$, we immediately deduce that $M(x, y, t) = \lim_{n\to\infty} M(x_n, y_n, t)$ and $N(x, y, t) = \lim_{n\to\infty} N(x_n, y_n, t)$.

DEFINITION 2.5 (8). *Two self mappings A and S of an IFM-space $(X, M, N, *, \Diamond)$ are called compatible if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ and $\lim_{n\to\infty} N(ASx_n, SAx_n, t) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X.*

DEFINITION 2.6. *Two self mappings A and S of an IFM-space $(X, M, N, *, \Diamond)$ are called pointwise R-weakly commuting if given x in X there exists $R > 0$ such that $M(ASx, SAx, t) \geq M(Ax, Sx, t/R)$ and $N(ASx, SAx, t) \leq N(Ax, Sx, t/R)$.*

REMARK 2.2. *It is obvious that A and S can fail to be pointwise R-weakly commuting if there is some x in X such that $Ax = Sx$ but $ASx \neq SAx$, that is, only if they possess a coincidence point at which they do not commute. This means that a contractive type mapping pair can not possess a common fixed point without being pointwise R-weakly commuting since a common fixed point is also a coincidence point at which the mappings commute and since contractive conditions exclude the possibility of two types of coincidence points, and compatible mappings are necessarily pointwise R-weakly commuting since compatible mappings commute at coincidence points. However, pointwise R-weakly commuting mappings need not to be compatible as shown in the example.*

Example. Let $X = [2, 20)$ with the usual metric d and define $M(x, y, t) = t/t + |x - y|$ and $N(x, y, t) = |x - y| / t + |x - y|$ for all $x, y \in X$ and $t > 0$. Clearly $(X, M, N, *, \diamondsuit)$ is a complete intuitionistic fuzzy metric space where $*$ and $\diamondsuit$ are defined by $a * b = ab$ and $a \diamondsuit b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$. Let A and S be self mappings of X defined as

$$Ax = \begin{cases} 2, & x = 2 \text{ or } x > 5 \\ 8, & 2 < x \leq 5 \end{cases} \quad \text{and } Bx = \begin{cases} 2, & x = 2 \\ 12 + x, & 2 < x \leq 5 \\ x - 3, & x > 5 \end{cases}$$

It can be verified that A and S are pointwise R-weakly commuting mappings but not compatible. Also, neither A nor S is continuous, not even at their coincidence points.

DEFINITION 2.7. *Two self mappings A and S of an IFM-space $(X, M, N, *, \diamondsuit)$ are called reciprocally continuous on X if $\lim_{n\to\infty} ASx_n = Ax$ and $\lim_{n\to\infty} SAx_n = Sx$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X.*

REMARK 2.3. *If A and S are both continuous, then they are obviously reciprocally continuous but the converse is not true. Moreover, in the setting of common fixed point theorems for pointwise R-weakly commuting mappings satisfying contractive conditions, continuity of one of the mappings A or S implies their reciprocal continuity but not conversely.*

THEOREM 2.1 ([2]). *Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of complete fuzzy metric space $(X, M, *)$ such that*

(a) $AX \subset TX$, $BX \subset SX$,

(b) $M(Ax, By, t) \geq M(x, y, ht)$, $0 < h < 1$, $x, y \in X$ and $t > 0$.

Suppose that (A, S) and (B, T) is compatible pairs of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point.

THEOREM 2.2 ([11]). *Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of complete fuzzy metric space $(X, M, *)$ such that*

(a) $AX \subset TX$, $BX \subset SX$,

(b) $M(Ax, By, t) \geq M(x, y, ht)$, $0 < h < 1$, $x, y \in X$ and $t > 0$.

Let (A, S) and (B, T) be compatible mappings. If any of the mappings in compatible pairs (A, S) and (B, T) is continuous then A, B, S and T have a unique common fixed point.

REMARK 1. *In [11], Pant and Jha proved that Theorem 2 is an analogue of the Theorem 1 by obtaining a connection between continuity and reciprocal continuity in fuzzy metric space.*

3 Main Results

In this section, we extend the Theorem 2 to IFM-spaces. If A, B, S and T are self mappings of IFM-space $(X, M, N, *, \diamondsuit)$ in the sequel we shall denote

$$\mathrm{M}(x, y, t) = \min \left\{ \begin{array}{c} M(Sx, Ty, t), M(Ax, Sx, t), M(Bx, Ty, t), \\ [M(Ax, Ty, t) + M(By, Sx, t)]/2 \end{array} \right\}$$

and

$$\mathrm{N}(x, y, t) = \max \left\{ \begin{array}{c} N(Sx, Ty, t), N(Ax, Sx, t), N(Bx, Ty, t), \\ [N(Ax, Ty, t) + N(By, Sx, t)]/2 \end{array} \right\}.$$

We shall need the following Lemma for proof of our main Theorem:

LEMMA 3.1. *Let* (A, S) *and* (B, T) *be pointwise R-weakly commuting pairs of self mappings of complete IFM-space* $(X, M, N, *, \Diamond)$ *such that*

(a) $AX \subset TX$, $BX \subset SX$,

(b) $M(Ax, By, t) \geq \mathrm{M}(x, y, ht)$ and $N(Ax, By, t) \leq \mathrm{N}(x, y, ht)$, $0 < h < 1$, $x, y \in X$ and $t > 0$.

Then the continuity of one of the mappings in compatible pair (A, S) or (B, T) on $(X, M, N, *, \Diamond)$ implies their reciprocal continuity.

Proof. Suppose that A and S are compatible and S is continuous. We claim that A and S are reciprocally continuous. Let $\{x_n\}$ be a sequence such that $Ax_n \to z$ and $Sx_n \to z$ for some z in X as $n \to \infty$. Since S is continuous, we have $SAx_n \to Sz$ and $SSx_n \to Sz$ as $n \to \infty$ and since (A, S) is compatible, we also have $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ and $\lim_{n\to\infty} N(ASx_n, SAx_n, t) = 0$. This implies that $\lim_{n\to\infty} M(ASx_n, Sz, t) = 1$ and $\lim_{n\to\infty} N(ASx_n, Sz, t) = 0$, that is, $ASx_n \to Sz$ as $n \to \infty$. By (a), for each n, there exists y_n in X such that $ASx_n = Ty_n$. Thus we have $SSx_n \to Sz$, $SAx_n \to Sz$, $ASx_n \to Sz$ and $Ty_n \to Sz$ as $n \to \infty$ whenever $ASx_n = Ty_n$.

We claim that $By_n \to Sz$ as $n \to \infty$. If not, then there exists a subsequence $\{By_m\}$ of $\{By_n\}$ such that for given $t > 0$, there exists a number $\varepsilon > 0$ and a positive integer n_0 such that for all $m > n_0$, $M(By_m, Sz, t) \leq \varepsilon$, $N(By_m, Sz, t) \geq 1 - \varepsilon$, $M(ASx_m, By_m, t) \leq \varepsilon$ and $N(ASx_m, By_m, t) \geq 1 - \varepsilon$. Then

$$\begin{aligned} M(ASx_m, By_m, t) &\geq \mathrm{M}(Sx_m, y_m, ht) = \mathrm{M}(By_m, Ty_m, ht) \\ &= M(By_m, ASx_m, ht), \end{aligned}$$

$$\begin{aligned} N(ASx_m, By_m, t) &\leq \mathrm{N}(Sx_m, y_m, ht) = \mathrm{N}(By_m, Ty_m, ht) \\ &= N(By_m, ASx_m, ht) \end{aligned}$$

which is a contraction. Hence $By_n \to Sz$ as $n \to \infty$. Now, the inequalities

$$M(Az, By_n, t) \geq M(z, y_n, ht) \text{ and } N(Az, By_n, t) \leq N(z, y_n, ht)$$

on letting $n \to \infty$, implies

$$M(Az, Sz, t) \geq M(Az, Sz, ht) \text{ and } N(Az, Sz, t) \leq N(Az, Sz, ht).$$

It follows that $Az = Sz$. Thus, $SAx_n \to Sz$ and $ASx_n \to Sz = Az$ as $n \to \infty$. Therefore A and S are reciprocally continuous on X. If the pair (B, T) is assumed to be compatible and T is continuous, the proof is similar.

THEOREM 3.1. *Let* $(X, M, N, *, \Diamond)$ *be an IFM-space and* A, B, S *and* T *be self maps of* X *satisfying*

(a) $AX \subset TX$, $BX \subset SX$,

(b) the pairs (A, S) and (B, T) are pointwise R-weakly commuting mappings

(c) $M(Ax, By, t) \geq \mathrm{M}(x, y, ht)$ and $N(Ax, By, t) \leq \mathrm{N}(x, y, ht)$, $0 \leq h < 1$, $x, y \in X$ and $t > 0$.

Let (A, S) and (B, T) be compatible mappings. If any of the mappings in compatible pairs (A, S) and (B, T) is continuous then A, B, S and T have a unique common fixed point.

Proof. Suppose that (A, S) are compatible and S is continuous. Then, by Lemma 1, A and S are reciprocally continuous. Let x_0 be any point in X. From condition (a), there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1 = y_0$ and $Bx_1 = Sx_2 = y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, ...$ Then using (c), we have

$$M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n-1}, y_{2n}, ht) \text{ and } N(y_{2n}, y_{2n+1}, t) \leq N(y_{2n-1}, y_{2n}, ht),$$

that is,

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, ht) \geq ... \geq M(y_0, y_1, h^n t),$$
$$N(y_n, y_{n+1}, t) \leq N(y_{n-1}, y_n, ht) \leq ... \leq N(y_0, y_1, h^n t).$$

Thus, any positive integer p, we have

$$\begin{aligned} M(y_n, y_{n+p}, t) &\geq M(y_n, y_{n+1}, \frac{t}{p}) * M(y_{n+1}, y_{n+2}, \frac{t}{p}) * \overset{(p)}{\cdots} * M(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &\geq M(y_n, y_{n+1}, \frac{t}{p}) * M(y_n, y_{n+1}, \frac{ht}{p}) * \overset{(p)}{\cdots} * M(y_n, y_{n+1}, \frac{h^{p-1}t}{p}) \\ &\geq M(y_0, y_1, \frac{h^n t}{p}) * M(y_0, y_1, \frac{h^{n+1}t}{p}) * \overset{(p)}{\cdots} * M(y_0, y_1, \frac{h^{n+p-1}t}{p}), \end{aligned}$$

$$\begin{aligned} N(y_n, y_{n+p}, t) &\leq N(y_n, y_{n+1}, \frac{t}{p}) \Diamond N(y_{n+1}, y_{n+2}, \frac{t}{p}) \Diamond \overset{(p)}{\cdots} \Diamond N(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &\leq N(y_n, y_{n+1}, \frac{t}{p}) \Diamond N(y_n, y_{n+1}, \frac{ht}{p}) \Diamond \overset{(p)}{\cdots} \Diamond N(y_n, y_{n+1}, \frac{h^{p-1}t}{p}) \\ &\leq N(y_0, y_1, \frac{h^n t}{p}) \Diamond N(y_0, y_1, \frac{h^{n+1}t}{p}) \Diamond \overset{(p)}{\cdots} \Diamond N(y_0, y_1, \frac{h^{n+p-1}t}{p}) \end{aligned}$$

According to (IFM-12) and (IFM-13), we have

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * \overset{(p)}{\cdots} * 1 = 1$$

and

$$\lim_{n\to\infty} N(y_n, y_{n+p}, t) \leq 0 \Diamond 0 \Diamond \overset{(p)}{\cdots} \Diamond 0 = 0$$

i.e., $\{y_n\}$ is Cauchy sequence. Since X is complete, then there exists a point z in X such that $y_n \to \infty$ as $n \to \infty$. Moreover,

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z.$$

Suppose that A and S are compatible and reciprocally continuous mappings, then $ASx_{2n} \to Az$ and $SAx_{2n} \to Sz$. Since A and S are compatible mappings, $\lim_{n\to\infty} M(ASx_{2n}, SAx_{2n}, t) = 1$ and $\lim_{n\to\infty} N(ASx_{2n}, SAx_{2n}, t) = 0$, that is, $M(Az, Sz, t) = 1$ and $N(Az, Sz, t) = 0$. Hence, $Az = Sz$.

Since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. Using (c), we have

$$\begin{aligned} M(Az, Bw, t) &\geq \text{M}(z, w, ht) = M(Az, Bw, ht), \\ N(Az, Bw, t) &\leq \text{N}(z, w, ht) = N(Az, Bw, ht). \end{aligned}$$

That is, $Az = Bw$. Thus $Az = Sz = Tw = Bw$. Since A and S are pointwise R-weakly commuting mappings, then there exists $R > 0$ such that $M(ASz, SAz, t) \geq M(Az, Sz, t/R) = 1$ and $N(ASz, SAz, t) \leq N(Az, Sz, t/R) = 0$. That is, $ASz = SAz$ and $AAz = ASz = SAz = SSz$. Similarly, since B and T are pointwise R-weakly commuting mappings, we have $BBw = BTw = TBw = TTw$. Using (c), we have

$$M(AAz, Az, t) = M(AAz, Bw, t) \geq \mathrm{M}(Az, w, ht) = M(AAz, Az, ht),$$

$$N(AAz, Az, t) = N(AAz, Bw, t) \leq \mathrm{N}(Az, w, ht) = N(AAz, Az, ht).$$

That is, $M(AAz, Az, t) = 1$ and $N(AAz, Az, t) = 0$. Hence $Az = AAz$ and $Az = AAz = SAz$. Thus, Az is a common fixed point of A and S. Similarly, by using (c), we find that $Bw(= Az)$ is a common fixed point of B and T. Hence, Az is a common fixed point of A, B, S and T.

To show uniqueness, assume $Aw(\neq Az)$ be an other common fixed point of A, B, S and T. Then, by (c), we have

$$M(Az, Aw, t) = M(AAz, AAw, t) \geq \mathrm{M}(Az, Aw, ht) = M(Az, Aw, ht),$$

$$N(Az, Aw, t) = N(AAz, AAw, t) \leq \mathrm{N}(Az, Aw, ht) = N(Az, Aw, ht).$$

That is, $Az = Aw$. Thus, Az is a unique common fixed point of A, B, S and T.

Example. Let $X = \mathbb{R}^+$ with the metric d defined by $d(x, y) = |x - y|$ and define $M(x, y, t) = \frac{t}{t+d(x,y)}, N(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$ for all $x, y \in X, t > 0$. Clearly $(X, M, N, *, \diamond)$ is a complete intuitionistic fuzzy metric space. Let A, B, S and T be self mappings on X defined as

$A0 = 0, Ax = 1$ if $x > 0$

$Bx = 0$ if $x = 0$ or $x > 6, Bx = 2$ if $0 < x \leq 6$

$S0 = 0, Sx = 2$ if $x > 0$

$T0 = 0, Tx = 4$ if $0 < x \leq 6, Tx = x - 6$ if $x > 6$.

Then A, B, S and T satisfy all the conditions of Theorem 3 with $h \in [0, 1)$ and have a unique common fixed point $x = 0$. Clearly A and S are reciprocally continuous compatible mappings. However A and S are not continuous, not even at the common fixed point. The mappings B and T are noncompatible because suppose that $\{x_n\}$ be a sequence defined as

$$x_n = 6 + \frac{1}{n}, n \geq 1,$$

then $Bx_n = 0, Tx_n \to 0, TBx_n = 0$ and $BTx_n = 2$, hence, B and T are noncompatible but pointwise R-weakly commuting since they commute at their coincidence points.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986), 87-96.

[2] P. Balasubramaniam, S. Muralisankar, R.P. Pant, Common fixed points of four mappings in a fuzzy metric space, J. Fuzzy Math. 10 (2) (2002), 379-384.

[3] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395-399.

[4] M. Grabiec, Fixed points in fuzzy metric spces, Fuzzy Sets and Systems 27 (1988), 385-389.

[5] O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975), 326-334.

[6] S. Kutukcu, D. Turkoglu, C. Yildiz, A common fixed point theorem of compatible maps of type (α) in fuzzy metric spaces, J. Concr. Appl. Math. 5(4) (2007), 347-357.
[7] S. Kutukcu, C. Yildiz, D. Turkoglu, Fixed points of contractive mappings in intuitionistic fuzzy metric spaces, J. Comput. Anal. Appl. 9(2) (2007), 181-194.
[8] S. Kutukcu, C. Yildiz, A. Tuna, On common fixed point theorems for noncompatible mappings in intuitionistic fuzzy metric spaces, Far East J. Math. Sci., in press.
[9] R.P. Pant, Common fixed points of four mappings, Bull. Cal. Math. Soc. 90 (1998), 281-286.
[10] R.P. Pant, Common fixed point theorems for contractive maps, J. Math. Anal. Appl. 226 (1998), 251-258.
[11] R.P. Pant, K. Jha, A remark on common fixed points of four mappings in a fuzzy metric space, J. Fuzzy Math. 12 (2004), 433-437.
[12] J.H. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals 22 (2004), 1039-1046.
[13] B.E. Rhoades, Contractive definitions and continuity, Contemporary Math. 72 (1988), 233-245.
[14] R. Saadati, Y.J. Cho, D. O'Regan, H. Adibi, Common fixed point theorems in L-fuzzy metric spaces, Appl. Math. Comput. doi:10.1016/j.amc.2006.04.045
[15] S. Sharma, Common fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems **127** (2002), 345-352.
[16] S. Sharma, B. Deshpande, Discontinuity and weak compatibility in fixed point consideration on non-complete fuzzy metric spaces, J. Fuzzy Math. **11(2)** (2003), 671-686.
[17] S. Sharma, B. Deshpande, Compatible multivalued mappings satisfying an implicit relation, Southeast Asian Bull. Math. 30 (2006), 535-540.
[18] R. Vasuki, Common fixed points for R-weakly commuting maps in fuzzy metric spaces, Indian J. Pure Appl. Math. 30 (1999), 419-423.
[19] L.A. Zadeh, Fuzzy sets, Inform and Control 8 (1965), 338-353.

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CONVERGENCES IN ALGEBRAIC NUANCED STRUCTURES

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ABSTRACT. Order convergence has been studied by G. Georgescu, F. Liguori and G. Martini for MV algebras, while the convergence with a fixed regulator was studied by Š. Černák for lattice-ordered groups and MV-algebras and by the author for perfect MV algebras. In this paper we study the order convergence and the convergence with a fixed regulator for the n-nuanced MV algebras.

1. Introduction

Gr.C. Moisil introduced in 1940 the 3-valued and 4-valued Łukasiewicz algebras ([17]) and in 1941 the n-valued Łukasiewicz algebras ([18]) with the intention of algebrizing Łukasiewicz's n-valued logic. An example of A. Rose from 1956 established that for $n \geq 5$ the Łukasiewicz implication can no longer be defined on a Łukasiewicz algebra. Algebraic models for the n-valued Łukasiewicz logic are the MV_n-algebras introduced by R. Grigolia in 1977 ([15]).

In fact, Moisil invented a distinct logical system and its algebraic counterpart are the n-valued Łukasiewicz-Moisil algebras. The Łukasiewicz logic has the implication as its primary connector, while the Moisil logic is based on the idea of nuance, expressed algebraically by the Chrysippian endomorphisms. Due to Moisil's determination principle, an n-valued sentence is determined by its Boolean nuances, so one could say that Moisil's logic is derived from the classical logic by the idea of nuancing. This tight relationship is algebraically expressed by the fundamental adjunction between the categories of Boolean and Łukasiewicz algebras.

From these remarks a generic problem arises: starting from a given logical system and using the idea of nuance, how can one construct an n-nuanced logical system based on the given one?

As an answer to this problem, in [9] the notion of *n-nuanced* MV algebras was introduced, which extends both MV algebras and n-valued Łukasiewicz-Moisil algebras. Taking as starting point the nuancing construction sketched in [21], the authors put together two approaches to multiple-valued-ness: that of infinitely valued Łukasiewicz logic and that of Moisil's n-nuanced logic.

A variety of papers has been written on the subject of convergence in ordered structures. Order convergence in a lattice-ordered group is studied in [19] and [20], while L-convergence is presented in [1] and convergence in [2]. Š. Černák studied the convergence with a fixed regulator for abelian ℓ-groups in [7] and for Archimedean ℓ-groups in [5]. Order convergence in MV-algebras is presented in [13], α-convergence was investigated in [16] and various kinds of Cauchy completions of MV-algebras are studied in [3]. Using the Mundici functor, Š. Černák [6] extended the convergence with fixed

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regulator from abelian ℓ-groups to MV-algebras. For the class of perfect MV-algebras, order convergence has been presented in [11] and the convergence with a fixed regulator was treated by the author in [4], based on the Di Nola-Lettieri functors $\mathcal{D}$ and Δ. Order convergence on Łukasiewicz-Moisil algebras was studied in [12]. In the case of residuated lattices the similarity convergence have been developed by G.Georgescu and A.Popescu in [14].
In this paper we introduce the concepts of order convergence and convergence with fixed regulator in n-nuanced MV algebras and we study their properties. We prove that the o-limit is unique in any n-nuanced MV algebra, while the v-limit is unique for locally Archimedean n-nuanced MV-algebra.

2. n-nuanced MV algebras

We present the basic definitions and properties of n-nuanced MV algebras.
For the rest of this section we denote $J = \{1, \cdots, n-1\}$ with $n \in \mathbb{N}$, $n \geq 2$.

Definition 2.1. ([9]) A *generalized De Morgan algebra* is a structure $(L, \oplus, \odot, N, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that the following conditions are satisfied:
(GM_1) $(L, \oplus, 0)$, $(L, \odot, 1)$ are commutative monoids;
(GM_2) $N(x \oplus y) = Nx \odot Ny$ and $NNx = x$ for all $x, y \in L$.

Remarks 2.2. If L is a generalizated De Morgan algebra, then:
(GM_3) $N(x \odot y) = Nx \oplus Ny$ for all $x, y \in A$;
(GM_4) $N1 = 0$ and $N0 = 1$.

Proof. (GM_3): For all $x, y \in L$ we have:

$$Nx \oplus Ny = NN(Nx \oplus Ny) = N(NNx \odot NNy) = N(x \odot y);$$

(GM_4): By (GM_3), for $x = N0$ and $y = 1$ we get:

$$NN0 = NN0 \oplus N1, \text{ so } \mathrm{N1} = 0;$$

Simlarly, by (GM_2), for $x = N1$ and $y = 0$ we get:

$$NN1 = NN1 \odot N0, \text{ so } \mathrm{N0} = 1.$$

□

Examples of generalized De Morgan algebra are supplied by the next proposition.

Proposition 2.3. *Any MV algebra $(A, \oplus, \odot, ^-, 0, 1)$ is a generalized De Morgan algebra.*

Proof. We check the axioms $(GM_1)-(GM_2)$ in the definition of a generalized De Morgan algebra:
1) $(A, \oplus, 0)$ is a commutative monoid by the axioms in $(MV_1)-(MV_3)$ in the definition of an MV algebra;
2) $(A, \odot, 1)$ is a commutative monoid (by the properties of an MV algebra);
3) $Nx \odot Ny = N(NNx \oplus NNy) = N(x \oplus y)$. □

Let's consider the structure L of the form $(L, \oplus, \odot, N, \varphi_1, \cdots, \varphi_{n-1}, 0, 1)$ where $(L, \oplus, \odot, N, 0, 1)$, is a generalized De Morgan algebra and $\varphi_1, \cdots, \varphi_{n-1}$ are unary operations on L. For this structure we consider the following axioms:
(nMV_0) $\varphi_i x \oplus (N\varphi_i x \odot \varphi_i y) = \varphi_i y \oplus (N\varphi_i y \odot \varphi_i x)$ for $i \in J$;

(nMV_1) $\varphi_i(x \oplus y) = \varphi_i x \oplus \varphi_i y$,
$\varphi_i(x \odot y) = \varphi_i x \odot \varphi_i y$,
$\varphi_i(0) = 0$,
$\varphi_i(1) = 1$ for $i \in J$;
(nMV_2) $\varphi_i x \oplus N\varphi_i x = 1$, $\varphi_i x \odot N\varphi_i x = 0$ for $i \in J$;
(nMV_3) $\varphi_i \circ \varphi_j = \varphi_j$ for $i, j \in J$;
(nMV_4) $\varphi_i \circ N = N \circ \varphi_{n-i}$ for $i \in J$;
(nMV_5) If $\varphi_i x = \varphi_i y$ for $i \in J$, then $x = y$ (*Moisil's determination principle*).

Define $M(L) = \{x \in L \mid \varphi_i x = x \text{ for all i} \in \text{J}\}$ called the *MV-center* of L.

Proposition 2.4. *([9]) Let L be an above defined structure. Then:*
(1) If L satisfies (nMV_3), then: for each $x \in L$, $x \in M(L)$ iff [there exists $i \in J$ such that $x = \varphi_i(x)$] iff [there exists $i \in J$ and $y \in L$ such that $x = \varphi_i(y)$] ;
(2) If L satisfies the axioms $(nMV_0)-(nMV_3)$, then $M(L)$ with the operations $\oplus, \odot, N, 0, 1$ induced by L is an MV algebra.

Remark 2.5. On $M(L)$ we have the order $\leq$ defined in an MV algebra.
We can extend this order to L as follows:

$$x \leq y \text{ iff, for each i} \in \text{J} , \ \varphi_{\text{i}}\text{x} \leq \varphi_{\text{i}}\text{y}.$$

Due to the determination principle, the above defined relation is an order. Because of (nMV_3), it is an extension of the order on $M(L)$.

Lemma 2.6. *([9]) Let L be a structure as above. Then:*
(1) 0 is the least and 1 is the greatest element in L w.r.t $\leq$;
(2) for each $x, y \in L$, $x \leq y$ iff $Ny \leq Nx$;
(3) for each $x, x', y, y' \in L$, if $x \leq x'$ and $y \leq y'$, then $x \oplus y \leq x' \oplus y'$ and $x \odot y \leq x' \odot y'$.

Corollary 2.7. *Let L be an $NMVA_n$ and $x, y \in L$. Then:*

$$x \oplus y \leq x, y \leq x \odot y.$$

Definition 2.8. ([9]) An *n-nuanced MV algebra* ($NMVA_n$ for short) is a structure $(L, \oplus, \odot, N, \varphi_1, \cdots, \varphi_{n-1}, 0, 1)$ such that $(L, \oplus, \odot, N, 0, 1)$ is a generalizated De Morgan algebra and $\varphi_1, \cdots, \varphi_n$ satisfy the axioms $(nMV_0) - (nMV_5)$ and the axiom:
(nMV_6) $\varphi_1 x \leq \varphi_2 x \leq \cdots \leq \varphi_{n-1} x$.

Remarks 2.9. (1) For $n = 2$, because of the determination principle, φ_1 is the identity, so an $NMVA_n$ can be identified with an MV algebra;
(2) If $(L, \oplus, \odot, N, 0, 1)$ is a De Morgan algebra, then $(L, \oplus, \odot, N, \varphi_1, \cdots, \varphi_{n-1}, 0, 1)$ is an LM_n-algebra.

Example 2.10. ([9]) Let $(A, \oplus, \odot, ^-, 0, 1)$ be an MV algebra and $T(A)_n = \{(x_1, \cdots, x_{n-1})\} \in A^{n-1} \mid x_1 \leq \cdots \leq x_{n-1}\}$. We denote by 0_1 and 1_1 the usual constant vectors. Then, A^{n-1} is an MV algebra with component-wise operations induced from A and $T(A)_n$ is closed under the operations $0_1, 1_1, \oplus, \odot$. Let's define $N, \varphi_1, \cdots, \varphi_{n-1}$ by:

$$N(x_1, \cdots, x_{n-1}) = (x_{n-1}^-, \cdots, x_1^-)$$

$$\varphi_i(x_1, \cdots, x_{n-1}) = (x_i, \cdots, x_i), \text{ for i} \in \text{J}.$$

Then, $(T(A)_n, \oplus, \odot, N, \varphi_1, \cdots, \varphi_{n-1}, 0_1, 1_1)$ is an $NMVA_n$.

Lemma 2.11. *([9]) If L is an $NMVA_n$, then for any $x \in L$:*

$$\varphi_1 x \leq x \leq \varphi_{n-1} x.$$

Lemma 2.12. *([9]) Let L be an $NMVA_n$ and $x, y \in L$. If $x \oplus y = 1$ and $x \odot y = 0$, then $x, y \in M(L)$ and $y = Nx$.*

Lemma 2.13. *([9]) $M(L) = \{x \in L \mid x \oplus Nx = 1 \text{ and } \mathrm{x} \odot \mathrm{Nx} = 0\}$.*

Proposition 2.14. *Let L be an $NMVA_n$ and $x, y \in L$. Then:*
(1) $x \oplus \varphi_{n-1} Nx = 1$;
(2) $x \odot \varphi_1 Nx = 0$.

Proof. We apply the axiom (nMV_2):
(1) $\varphi_1 x \oplus N\varphi_1 x = 1$ iff $\varphi_1 x \oplus \varphi_{n-1} Nx = 1$.
Taking into consideration that $\varphi_1 x \leq x$, we get:
$1 = \varphi_1 x \oplus \varphi_{n-1} Nx \leq x \oplus \varphi_{n-1} Nx$, so $x \oplus \varphi_{n-1} Nx = 1$.
(2) $\varphi_{n-1} x \odot N\varphi_{n-1} x = 0$ iff $\varphi_{n-1} x \odot \varphi_1 Nx = 0$.
Applying the fact that $x \leq \varphi_{n-1} x$, we get:
$x \odot \varphi_1 Nx \leq \varphi_{n-1} x \odot \varphi_1 Nx = 0$, so $x \odot \varphi_1 Nx = 0$. $\square$

Let L be an $NMVA_n$ and $x, y \in L$, $(x_k)_{k \in K} \subseteq L$. By $x \vee y$ and $\bigvee_{k \in K} x_k$ (respectively $x \wedge y$ and $\bigwedge_{k \in K} x_k$) we denote suprema(respectively infima) in L, whenever they exist.

Proposition 2.15. *([9]) Let L be an $NMVA_n$, $a, b \in M(L)$ and $x \in L$ such that $a \vee b$ and $a \wedge b$ exist. Then:*
(1) $(a \odot x) \vee (b \odot x) = (a \vee b) \odot x$;
(2) $(a \oplus x) \wedge (b \oplus x) = (a \wedge b) \oplus x$.

Remark 2.16. In [10] it was proved that n-valued Łukasiewicz-Moisil algebras are connected with Boolean algebras through an adjunction which allows the transfer of many properties from Boolean case. For any arbitrary $NMVA_n$ L we consider the function $\psi_L : L \to T(M(L))$ defined by

$$\psi_L(x) = (\varphi_1 x, \varphi_2 x, \cdots, \varphi_{n-1} x) \text{ for any } \mathrm{x} \in \mathrm{L}.$$

One can easily check that ψ_L is an injective $NMVA_n$-morphism.
If A is an MV algebra, then the constant vectors are the only elements of $M(T(A)_n)$.
For any MV algebra A, we consider the function $\varphi_A : M(T(A)_n) \to A$ defined by

$$\varphi_A(x, x, \cdots, x) = x \text{ for all } \mathrm{x} \in \mathrm{A}.$$

By the determination principle, it follows that φ_A is an MV-isomorphism.

Let L be an $NMVA_n$ and $d : M(L) \times M(L) \to M(L)$ the distance in the MV algebra $M(L)$, that is

$$d(x, y) = (x \odot Ny) \oplus (y \odot Nx) \text{ for } \mathrm{x}, \mathrm{y} \in \mathrm{M(L)}.$$

We define $\rho : L \times L \to L$ by $\rho(x, y) = \bigvee_{i \in \{1, \cdots, n-1\}} d(\varphi_i x, \varphi_i y)$.

Proposition 2.17. *Let L be an $NMVA_n$ and $x, y \in L$. Then, $\rho(\varphi_i x, \varphi_i y) = d(\varphi_i x, \varphi_i y)$ for all $i \in J$.*

Proof. For all $i \in J$, we have:

$$\rho(\varphi_i x, \varphi_i y) = \bigvee_{j \in J} d(\varphi_j \varphi_i x, \varphi_j \varphi_i y) = \bigvee_{j \in J} d(\varphi_i x, \varphi_i y) =$$

$$= d(\varphi_i x, \varphi_i y).$$

□

Proposition 2.18. *([9]) Let L be an $NMVA_n$. For all $x, y, z, u, v \in L$ we have:*
(1) $\rho(x, y) = 0$ iff x = y;
(2) $\rho(x, 0) = \varphi_{n-1}x$;
(3) $\rho(x, 1) = N\varphi_1 x$;
(4) $\rho(Nx, Ny) = \rho(x, y)$;
(5) $\rho(x, y) = \rho(y, x)$;
(6) $\rho(x, z) \leq \rho(x, y) \oplus \rho(y, z)$;
(7) $\rho(x \oplus u, y \oplus v) \leq \rho(x, y) \oplus \rho(u, v)$;
(8) $\rho(x \odot u, y \odot v) \leq \rho(x, y) \oplus \rho(u, v)$.

Definition 2.19. On an $NMVA_n$ L, we define the *implication*:

$$x \to y = \varphi_{n-1}Nx \oplus y \text{ for all } x,y \in L.$$

Remark 2.20. For $n = 2$, φ_1 is the identity and the $NMVA_n$ becomes an MV algebra. Thus, $x \to y = Nx \oplus y$ is the implication in an MV algebra.

Proposition 2.21. *Let L be an $NMVA_n$ and $x, y \in L$. Then, the following properties hold:*
(1) $1 \to x = x$ *and* $x \to x = 1$;
(2) *If* $x \leq y$, *then* $x \to y = 1$;
(3) *If* $x \leq y$, *then* $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$;
(4) $x \to (y \to x) = 1$;
(5) $x \to (y \to z) = (x \odot y) \to z$;
(6) $x \leq y$ *iff* $\varphi_i x \to \varphi_i y = 1$ *for all* $i \in J$;
(7) $x \to \varphi_1 x = 1$;
(8) $\varphi_i x \to \varphi_i y \leq \varphi_i(x \to y)$ *for all* $i \in J$;
(9) $x \to (y \to z) = y \to (x \to z)$;
(10) $x \to y \leq x \odot z \to y \oplus z$.

Proof. (1) $1 \to x = \varphi_{n-1}N1 \oplus x = \varphi_{n-1}0 \oplus x = 0 \oplus x = x$;
$x \to x = \varphi_{n-1}Nx \oplus x = N\varphi_1 x \oplus x = 1$.
(According to Proposition 2.14 we have $\varphi_{n-1}Nx \oplus x = 1$, so $N\varphi_1 x \oplus x = 1$).
(2) If $x \leq y$, then $\varphi_i x \leq \varphi_i y$ for all $i \in J$.
Hence, in the MV algebra $M(L)$ we have $N\varphi_i x \oplus \varphi_i y = 1$ for all $i \in J$.
In particular $N\varphi_1 x \oplus \varphi_1 y = 1$, that is $\varphi_{n-1}Nx \oplus \varphi_1 y = 1$.
Since $\varphi_1 y \leq y$, we get $1 = \varphi_{n-1}Nx \oplus \varphi_1 y \leq \varphi_{n-1}Nx \oplus y$, so $\varphi_{n-1}Nx \oplus y = 1$.
Thus, $x \to y = 1$.
(3) $x \leq y \Rightarrow \varphi_{n-1}Nz \oplus x \leq \varphi_{n-1}Nz \oplus y$, that is $z \to x \leq z \to y$.
Similarly, $x \leq y \Rightarrow Ny \leq Nx \Rightarrow \varphi_{n-1}Ny \oplus z \leq \varphi_{n-1}Nx \oplus z$, that is $y \to z \leq x \to z$.
(4) For all $i \in J$ we have:

$$\varphi_i x = 0 \oplus \varphi_i x \leq \varphi_{n-1}Ny \oplus \varphi_i x = \varphi_i \circ \varphi_{n-1}Ny \oplus \varphi_i x =$$

$$= \varphi_i(\varphi_{n-1}Ny \oplus x) = \varphi_i(y \to x).$$

From $\varphi_i x \leq \varphi_i(y \to x)$ for all $i \in J$, we get $x \leq y \to x$.
Applying (2), we obtain $x \to (y \to x) = 1$.
(5) $x \to (y \to z) = \varphi_{n-1}Nx \oplus (y \to z) = \varphi_{n-1}Nx \oplus \varphi_{n-1}y \oplus z = \varphi_{n-1}N(x \odot y) \oplus z =$

$(x \odot y) \to x)$.
(6) We have:

$$\begin{gathered} x \le y \text{ iff } \varphi_{\rm i}{\rm x} \le \varphi_{\rm i}{\rm y} \text{ for all i} \in \text{J iff} \\ N\varphi_i x \oplus \varphi_i y = 1 \text{ for all i} \in \text{J in M(L) iff} \\ \varphi_{n-i} N x \oplus \varphi_i y = 1 \text{ for all i} \in \text{J iff} \\ \varphi_{n-1} \circ \varphi_{n-i} N x \oplus \varphi_i y = 1 \text{ for all i} \in \text{J iff} \\ \varphi_{n-1} N \varphi_i x \oplus \varphi_i y = 1 \text{ for all i} \in \text{J iff} \\ \varphi_i x \to \varphi_i y = 1 \text{ for all i} \in \text{J.} \end{gathered}$$

(7) $x \to \varphi_1 x = \varphi_{n-1} N x \oplus \varphi_1 x = N\varphi_1 x \oplus \varphi_1 x = 1$.
(8) We have $\varphi_i x \to \varphi_i y = \varphi_{n-1} N \varphi_i x \oplus \varphi_i y$ for all $i \in J$.
But, $\varphi_{n-1} N \varphi_i x = N\varphi_1 \circ \varphi_i x = N\varphi_i x \le N\varphi_1 x = \varphi_{n-1} N x = \varphi_i \circ \varphi_{n-1} N x$.
(we applied the fact that, for all $i \in J$ $\varphi_1 x \le \varphi_i x$, implies $N\varphi_i x \le N\varphi_1 x$). Hence,

$$\varphi_i x \to \varphi_i y \le \varphi_i \circ \varphi_{n-1} N x \oplus \varphi_i y = \varphi_i(\varphi_{n-1} N x \oplus y) = \varphi_i(x \to y),$$

for all $i \in J$.
(9) We have:

$$\begin{aligned} x \to (y \to z) &= \varphi_{n-1} N x \oplus (y \to z) = \varphi_{n-1} N x \oplus \varphi_{n-1} N y \oplus z = \\ &= \varphi_{n-1} N y \oplus \varphi_{n-1} N x \oplus z = \varphi_{n-1} N y \oplus (x \to z) = y \to (x \to z). \end{aligned}$$

(10) We have:

$$\begin{gathered} x \odot z \to y \oplus z = \varphi_{n-1} N(x \odot z) \oplus (y \oplus z) = \varphi_{n-1}(Nx \oplus Ny) \oplus y \oplus z = \\ = \varphi_{n-1} N x \oplus \varphi_{n-1} N z \oplus y \oplus z = \varphi_{n-1} N x \oplus y \oplus \varphi_{n-1} N z \oplus z = \\ = (x \to y) \oplus (z \to z) = (x \to y) \oplus 1 \ge x \to y. \end{gathered}$$

□

Similarly as in the above results we can establish others properties of the implication on an $NMVA_n$.

Definition 2.22. ([9]) An *ideal* of an $NMVA_n$ L is a nonempty subset I of L such that, for each $x, y \in L$ the following conditions hold:
$(nmv - I_1)$ $x, y \in I$ implies $x \oplus y \in I$;
$(nmv - I_2)$ $x \le y$ and $y \in I$ implies $x \in I$;
$(nmv - I_3)$ $x \in I$ implies $\varphi_{n-1} x \in I$.
An ideal I of L is *proper* if it is different from L.
An ideal I of L is *maximal* if it is maximal in the lattice of ideals of L. A proper ideal I of L is *prime* if for all $x, y \in L$ such that $\varphi_{n-1} x \wedge \varphi_{n-1} y \in I$, then $x \in I$ or $y \in I$.

Remarks 2.23. Let L be an $NMVA_n$ and I an ideal of L.
(1) If $x \in I$, then $\varphi_i x \in I$ for each $i \in J$;
(2) For each $x \in L$, since $x \le \varphi_{n-1} x$, it follows that $x \in I$ iff $\varphi_{n-1} x \in I$.

Let L be an $NMVA_n$. For any ideal I of L we consider the binary relation " $\equiv_I$ " defined by $x \equiv_I y$ iff $\rho(\mathrm{x}, \mathrm{y}) \in \mathrm{I}$.

Proposition 2.24. ([9]) *Let L be an $NMVA_n$ and I an ideal of L. Then:*
(1) *" $\equiv_I$ " is a congruence on L;*
(2) *If $\equiv$ is a congruence on L, then $I_{\equiv} = \{x \in L \mid x \equiv 0\}$ is an ideal of L;*
(3) *The mappings $I \mapsto \equiv_I$ and $I \mapsto I_{\equiv}$ establish a latticial isomorphism between the lattice of ideals of L and the lattice of congruences on L.*

We denote be $Rad(L)$ the intersection of all maximal ideals of L.
$Rad(L)$ is called the *radical* of L.
We denote $Rad(L)^N = \{Nx \mid x \in Rad(L)\}$.

Proposition 2.25. *([9]) Let I be a proper ideal of L. Then:*
(1) I is prime (maximal) iff $I \cap M(L)$ is prime (maximal) MV-ideal of $M(L)$;
(2) $Rad(L) = \varphi_{n-1}^{-1}(Rad(M(L)))$ and $Rad(M(L)) = Rad(L) \cap M(L)$.

Definition 2.26. An $NMVA_n$ L is called *locally Archimedean* whenever $x, y \in Rad(L)$ are such that $mx \leq y$ for all $m \in \mathbb{N}$ it follows that $x = 0$.

3. Convergences in n-nuanced MV algebras

Based on the theory presented in the first section we will introduce two concepts of convergence in an n-nuanced MV algebra.

3.1. Order convergence.

Using the distance function ρ the concept of order convergence in an $NMVA_n$ is introduced in the same way as in the case of MV algebras.

Let $(x_m)_m$ be a sequence in an $NMVA_n$. If $(x_m)_m$ is increasing we denote $(x_m)_m \uparrow$.
Similarly, if $(x_m)_m$ is decreasing we denote $(x_m)_m \downarrow$.
If $(x_m)_m$ is increasing, $\vee_m x_m$ exists and $\vee_m x_m = x$, we denote $(x_m)_m \uparrow x$.
Similarly, if $(x_m)_m$ is decreasing, $\wedge_m x_m$ exists and $\wedge_m x_m = x$, we denote $(x_m)_m \downarrow x$.

Lemma 3.1. *If $(c_m)_m$ and $(d_m)_m$ are two sequences in an $NMVA_n$ L such that $(c_m)_m \downarrow 0$ and $(d_m)_m \downarrow 0$, then $(c_m \oplus d_m)_m \downarrow 0$.*

Proof. The sequence $(c_m)_m$ is decreasing iff $c_{n+p} \leq c_n$ for all $n, p \in \mathbb{N}$ iff $\varphi_i c_{n+p} \leq \varphi_i c_n$ for all $i \in \{1, \cdots, n-1\}$ and $n, p \in \mathbb{N}$. Since $(c_m)_m \downarrow 0$, it follows that $(\varphi_i c_m)_m \downarrow \varphi_i 0 = 0$ for all $i \in \{1, \cdots, n-1\}$.
Similarly, $(\varphi_i d_m)_m \downarrow \varphi_i 0 = 0$ for all $i \in \{1, \cdots, n-1\}$.
Ii follows that $(\varphi_i c_m \oplus \varphi_i d_m)_m \downarrow= 0$ for all $i \in \{1, \cdots, n-1\}$ in the MV algebra $M(L)$.
Hence, $(\varphi_i(c_m \oplus d_m))_m \downarrow 0 = \varphi_i 0$ for all $i \in \{1, \cdots, n-1\}$.
Thus, $(c_m \oplus d_m)_m \downarrow 0$ in L. ☐

Definition 3.2. Let $(x_m)_m$ be a sequence in an $NMVA_n$ L. Then, $(x_m)_m$ *order converges* to $x \in L$ (denoted $x_m \rightarrow_o^n x$) if there exists a sequence $(c_m)_m$ in L such that $(c_m)_m \downarrow 0$ and $\rho(x_m, x) \leq c_m$ for all $m \in \mathbb{N}$.

Proposition 3.3. *Let $(x_m)_m$ in an $NMVA_n$ L. If $x_m \rightarrow_o^n x_1$ and $x_m \rightarrow_o^n x_2$, then $x_1 = x_2$.*

Proof. By the definition of o-convergence there exist two sequences $(c_m)_m$, $(d_m)_m$ in L such that $(c_m)_m \downarrow 0$, $(d_m)_m \downarrow 0$, $\rho(x_m, x_1) \leq c_m$ and $\rho(x_m, x_2) \leq d_m$ for all $m \in \mathbb{N}$. We have: $\rho(x_1, x_2) \leq \rho(x_1, x_m) \oplus \rho(x_m, x_2) \leq c_m \oplus d_m$ for all $m \in \mathbb{N}$.
Since $(c_m \oplus d_m)_m \downarrow 0$, we get $x_1 = x_2$. ☐

Proposition 3.4. *Let $(x_m)_m, (y_m)_m$ be two sequences in an $NMVA_n$ L and $x, y \in L$ such that $x_m \rightarrow_o^n x$ and $y_m \rightarrow_o^n y$. Then, the following hold:*
(1) $Nx_m \rightarrow_o^n Nx$;

(2) $x_m \oplus y_m \rightarrow_o^n x \oplus y$;
(3) $x_m \odot y_m \rightarrow_o^n x \odot y$;
(4) $\varphi_i x_m \rightarrow_o^n \varphi_i x$ *for all* $i \in \{1, \cdots, n-1\}$;
(5) $(x_m \rightarrow y_m) \rightarrow_o^n (x \rightarrow y)$.

Proof. By the definition of the o-convergence there exist two sequences $(c_m)_m$, $(d_m)_m$ in L such that $(c_m)_m \downarrow 0$, $(d_m)_m \downarrow 0$, $\rho(x_m, x_1) \leq c_m$ and $\rho(x_m, x_2) \leq d_m$ for all $m \in \mathbb{N}$. Applying the properties of the distance function ρ we have:
(1) $\rho(Nx_m, Nx) = \rho(x_m, x)$, so $Nx_m \rightarrow_o^n Nx$;
(2) $\rho(x_m \oplus y_m) \leq c_m \oplus d_m$. Taking into consideration that $(c_m \oplus d_m)_m \downarrow 0$, we get $x_m \oplus y_m \rightarrow_o^n x \oplus y$;
(3) Similarly, $\rho(x_m \odot y_m) \leq c_m \oplus d_m$, so $x_m \odot y_m \rightarrow_o^n x \odot y$;
(4) Since $\varphi_i x_m, \varphi_i x \in M(L)$, for all $i \in \{1, \cdots, n-1\}$ we have

$$\rho(\varphi_i x_m, \varphi_i x) = d(\varphi_i x_m, \varphi_i x) \leq \bigvee_{i \in \{1, \cdots, n-1\}} d(\varphi_i x_m, \varphi_i x) = \rho(x_m, x).$$

It follows that $\rho(\varphi_i x_m, \varphi_i x) \leq \rho(x_m, x) \leq c_m$ for all $m \in \mathbb{N}$.
Thus, $\varphi_i x_m \rightarrow_o^n \varphi_i x$ for all $i \in \{1, \cdots, n-1\}$
(5) It follows by the definition of the implication and from $(1)-(4)$. □

Proposition 3.5. *Let L be an $NMVA_n$ L and $(x_m)_m$ a sequence in L such that $(x_m)_m \subseteq Rad(L)$ and $x_m \rightarrow_o^n x$. If in the definition of the o-convergence we have $(c_m)_m \subseteq L$, then $x \in Rad(L)$.*

Proof. We apply the definition of the o-convergence, the properties of the distance ρ and the axioms of an ideal of L.
Because $x_m \rightarrow_o^n x$ it follows that there exists a sequences $(c_m)_m$ in L such that $(c_m)_m \downarrow 0$ and $\rho(x_m, x_2) \leq c_m$ for all $m \in \mathbb{N}$. By Lemma 2.11 and by the properties of the distance function ρ we have:

$$x \leq \varphi_{n-1} x = \rho(x, 0) \leq \rho(x, x_m) \oplus \rho(x_m, 0) = \rho(x_m, x) \oplus \varphi_{n-1} x_m \leq c_m \oplus \varphi_{n-1} x_m.$$

Since $Rad(L)$ is an ideal of L and $x_m \in Rad(L)$, by $(nmv - I_3)$ it follows that $\varphi_{n-1} x_m \in Rad(L)$. By $(nmv - I_1)$ we get $c_m \oplus \varphi_{n-1} x_m \in Rad(L)$ and by $(nmv - I_2)$ it follows that $x \in Rad(L)$. □

The o-Cauchy completion of L can be constructed in a similar way as in the case of order Cauchy completion of a MV algebra (see [3]).

3.2. **Convergence with fixed regulator.**

Definition 3.6. Let L be an $NMVA_n$ and $0 < v \in L$. The sequence $(x_m)_m$ in L *v-converges* to an element $x \in L$ (or x is a *v-limit* of $(x_m)_m$), denoted $x_m \rightarrow_v^n x$ if for every $p \in \mathbb{N}$ there exists $m_0 \in \mathbb{N}$ such that $p\rho(x_m, x) \leq v$ for all $m \in \mathbb{N}$, $m \geq m_0$.

Proposition 3.7. *Let $(x_m)_m, (y_m)_m$ be two sequences in an $NMVA_n$ L and $x, y \in L$ such that $x_m \rightarrow_v^n x$ and $y_m \rightarrow_v^n y$. Then, the following hold:*
(1) $Nx_m \rightarrow_v^n Nx$;
(2) $x_m \oplus y_m \rightarrow_v^n x \oplus y$;
(3) $x_m \odot y_m \rightarrow_v^n x \odot y$;

(4) $\varphi_i x_m \to_v^n \varphi_i x$ *for all* $i \in \{1, \cdots, n-1\}$;
(5) $(x_m \to y_m) \to_v^n (x \to y)$.

Proof. We will apply the properties of the distance ρ defined on an $NMVA_n$.
(1) $p\rho(Nx_m, Nx) = p\rho(x_m, x) \leq v$, so $Nx_m \to_v^n Nx$;
(2) $2p\rho(x_m \oplus y_m, x \oplus y) \leq 2p\rho(x_m, x) \oplus \rho(y_m, y) \leq v \oplus v = 2v$ for all $m \in \mathbb{N}$, $m \geq m_0$. It follows that $p\rho(x_m \oplus y_m, x \oplus y) \leq v$, so $x_m \oplus y_m \to_v^n x \oplus y$.
(3) $2p\rho(x_m \odot y_m, x \odot y) \leq 2p\rho(x_m, x) \oplus \rho(y_m, y) \leq v \oplus v = 2v$ for all $m \in \mathbb{N}$, $m \geq m_0$. It follows that $p\rho(x_m \odot y_m, x \odot y) \leq v$, so $x_m \odot y_m \to_v^n x \odot y$.
(4) Since $\varphi_i x_m, \varphi_i x \in M(L)$, for all $i \in \{1, \cdots, n-1\}$ we have

$$\rho(\varphi_i x_m, \varphi_i x) = d(\varphi_i x_m, \varphi_i x) \leq \bigvee_{i \in \{1, \cdots, n-1\}} d(\varphi_i x_m, \varphi_i x) = \rho(x_m, x).$$

It follows that $p\rho(\varphi_i x_m, \varphi_i x) \leq p\rho(x_m, x) \leq v$.
Thus, $\varphi_i x_m \to_v^n \varphi_i x$ for all $i \in \{1, \cdots, n-1\}$.
(5) It follows by the definition of the implication and from $(1) - (4)$. □

Proposition 3.8. *In any* $NMVA_n$ L *the following hold:*
(1) *If* $(x_m)_m \subseteq Rad(L)$, $0 < v \in Rad(L)$ *and* $x_m \to_v^n x$, *then* $x \in Rad(L)$;
(2) *If* $(x_m)_m \subseteq Rad(L)^N$, $v \in Rad(L)^N$, $v < 1$ *and* $x_m \to_{v^N}^n x$, *then* $x \in Rad(L)^N$.

Proof. We apply the definition of the v-convergence, the properties of the distance ρ and the axioms of an ideal of L.
(1) Because $x_m \to_v^n x$ it follows that for each $p \in \mathbb{N}$ there exists $m_0 \in \mathbb{N}$ such that $p\rho(x_m, x) \leq v$ for all $m \in \mathbb{N}$, $m \geq m_0$. By Lemma 2.11 and by the properties of the distance function ρ we have:

$$x \leq \varphi_{n-1} x = \rho(x, 0) \leq \rho(x, x_m) \oplus \rho(x_m, 0) = \rho(x_m, x) \oplus \varphi_{n-1} x_m \leq v \oplus \varphi_{n-1} x_m.$$

Since $Rad(L)$ is an ideal of L and $x_m \in Rad(L)$, by $(nmv - I_3)$ it follows that $\varphi_{n-1} x_m \in Rad(L)$. By $(nmv - I_1)$ we get $v \oplus \varphi_{n-1} x_m \in Rad(L)$ and by $(nmv - I_2)$ it follows that $x \in Rad(L)$.
(2) We have $(Nx_m)_m \subseteq Rad(L)$ and $0 < Nv \in Rad(L)$ and apply (1). □

Theorem 3.9. *If* L *is a locally Archimedean* $NMVA_n$ *and* $0 < v \in Rad(L)$, *then every sequence* $(x_m)_m$ *has a unique* v*-limit.*

Proof. Suppose $x_m \to_v^n x_1$ and $x_m \to_v^n x_2$. According to Proposition 3.8 we have $x_1, x_2 \in Rad(L)$ and

$$p\rho(x_1, x_2) \leq p\rho(x_1, x_m) \oplus \rho(x_m, x_2) \leq 2v$$

for all $p \in \mathbb{N}$. Because L is locally Archimedean we get $\rho(x_1, x_2) = 0$, so $x_1 = x_2$. □

Proposition 3.10. *Let* L *be a locally Archimedean* $NMVA_n$ *and* $0 < v \in Rad(L)$. *If* $(x_m)_m, (y_m)_m \subseteq L$ *such that* $x_m \to_v^n x$ *and* $y_m \to_v^n y$ *and* $x_m \leq y_m$ *for any* $m \in \mathbb{N}$, *then* $x \leq y$.

Proof. We have $x_m \leq y_m$ iff $\varphi_i x_m \leq \varphi_i y_m$ for all $i \in \{1, \cdots, n-1\}$.
Because $\varphi_i x_m, \varphi_i y_m \in M(L)$ and $M(L)$ is an MV algebra, the last inequality is equivalent with $N\varphi_i x_m \oplus \varphi_i y_m = 1$ for all $i \in \{1, \cdots, n-1\}$.
We get $N\varphi_i x_m \oplus \varphi_i y_m \to_v^n 1$ for all $i \in \{1, \cdots, n-1\}$.
On the other hand, applying Proposition 3.7 we get $N\varphi_i x_m \oplus \varphi_i y_m \to_v^n N\varphi_i x \oplus \varphi_i y$.
By Theorem 3.9 we get $N\varphi_i x \oplus \varphi_i y = 1$, so $\varphi_i x \leq \varphi_i y$ for all $i \in \{1, \cdots, n-1\}$. Thus, $x \leq y$. □

The v-Cauchy completion of L can be constructed in a similar way as in the case of v-Cauchy completion of an MV algebra (see [6]).

References

[1] R.N. Ball, *Convergence and Cauchy structures on lattice-ordered groups*, Transactions of the American Mathematical Society 259 (1980), 357-392.

[2] R.N. Ball, *The structure of α-completion of lattice ordered groups*, Houston Journal of Mathematics 15 (1989), 481-515.

[3] R.N. Ball, G. Georgescu, I. Leuştean, *Cauchy Completion of MV-algebras*, Algebra Universalis, 47 (2002), 367-407.

[4] L. Ciungu, *Convergence with a fixed regulator in perfect MV-algebras*, Demonstratio Mathematica, to appear.

[5] Š. Černák, *Convergence with a fixed regulator in Archimedean lattice ordered groups*, Mathematical Slovaca, Vol. 56, no. 2 (2006), p. 167-180.

[6] Š. Černák, *Convergence with a fixed regulator in MV-algebras*, submitted.

[7] Š. Černák, J. Lihová, *Convergence with a regulator in lattice ordered groups*, Tatra Mountatins Mathematical Publications, Vol. 30, no. 0 (2005), p. 21-34.

[8] G. Georgescu, *Bosbach states on fuzzy structures*, Soft Computing 8(2004), 217-230.

[9] G. Georgescu, A. Popescu, *A common generalization for MV algebras and Łukasiewicz-Moisil algebras*, Arch.Math.Log 45(8)(2006), 947-981.

[10] G. Georgescu, C. Vraciu, *On the characterization of centered Łukasiewicz algebras*, J. Algebras 16(1970), 486-495.

[11] G. Georgescu, I. Leuştean, *Convergence in perfect MV-algebras*, Journal of Mathematical Analysis and Applications, Vol. 228, No. 1 (1998), p. 96-111 (16).

[12] G. Georgescu, I. Leuştean, A. Popescu, *Order convergence and distance on Łukasiewicz-Moisil algebras*, Journal of Multiple-Valued Logic and Soft Computing, Vol. 12, No. 1-2(2006), 33-69.

[13] G. Georgescu, F. Liguori, G. Martini, *Convergence in MV-algebras*, Soft Computing, 4(1997), 41-52.

[14] G. Georgescu, A. Popescu, *Similarity convergence in residuated structures*, Logic Journal of IGPL, (2005), 13(4), 389 - 413.

[15] R. S. Grigolia, *Algebraic analysis of Łukasiewicz-Tarski logical system*, R. Wojcicki, G. Malinkowski, editors, Selected Papers on Łukasiewicz Sentential Calculi, Osolineum, Wroclav, 1977, 81-92.

[16] I. Leuştean, *α-convergence and complete distributivity in MV-algebras*, Journal of Multiple-Valued Logic and Soft Computing Vol.12, 3-4(2006), 309-319.

[17] Gr. C. Moisil, *Recherches sur les logiques non-chrysippiennes,* Ann. Sci. Univ. Jassy 26(1940), 431-466.

[18] Gr. C. Moisil, *Notes sur les logiques non-chrysippiennes*, Ann. Sci. Univ. Jassy 27(1941), 86-98.

[19] F. Papangelou, *Some considerations on convergence in abelian lattice-groups*, Pacific Journal of Mathematics, Vol. 15, No.4(1965).

[20] F. Papangelou, *Order Convergence and Topological Completion of Commutative Lattice-Groups*, Mathematische Annalen, Vol. 155, No. 4(1965).

[21] A. Popescu, *Tones of truth*, submitted.

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On Wavelet Packet Frames

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Abstract: The essential problem in signal analysis is to find a numerically stable algorithm for reconstruction of a signal from its atomic decomposition [2]. This leads to the notion of frames [3, 7] which is a main ingredient in the analysis and synthesis of signals. In this paper, we have obtained the frame bounds for wavelet packet frames which are more general than that of wavelet frames.

Keywords: Frame, trace class operator and density.

AMS Subject Classifications: 41A58, 42C15.

1. Introduction and Preliminaries:

Introduced by Duffin and Schaeffer [6] in the context of non-harmonic Fourier series, the theory of frames has been developed for Gabor and Wavelets by many authors, see especially the papers by Daubechies [3], Heil and Walnut [7], Christensen [1], Sun and Zhou [9], Shang and Zhou [8] and Yang and Zhou [10].

A system of elements $\{f_n\}_{n\in\Lambda}$ in a Hilbert space H is called a frame for H if there exists two +ve numbers A and B such that for any $f \in H$,

$$A\ \|f\|^2 \ \leq\ \sum_{n\in\Lambda} |\langle f, f_n\rangle|^2 \ \leq\ B\ \|f\|^2.$$

The numbers A and B are called frame bounds. If $A = B$, the frame is said to be tight. The frame is called exact if it ceases to be a frame whenever any single element is deleted from the frame.

The continuous wavelet transformation of a L^2-function f with respect to the wavelet ψ, which satisfies admissibility condition, is defined as:

$$(T^{wav} f)(a, b) \ =\ |a|^{-1/2} \int_{-\infty}^{\infty} f(t)\ \overline{\psi\left(\frac{t-b}{a}\right)} dt, \quad a, b \in \mathbb{R}, a \neq 0.$$

The term wavelet denotes a family of functions of the form $\psi_{a,b} \ =\ |a|^{-1/2}\ \psi\left(\frac{t-b}{a}\right)$, obtained from a single function ψ by the operation of dilation and translation.

Wavelet Packets : We have the following sequence of functions due to Wickerhauser [11]. For $l = 0, 1, 2, ...,$

$$\psi_{2l}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} a_k \psi_l(2x - k) \ \text{ and } \ \psi_{2l+1}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} b_k \psi_l(2x - k), \qquad (*)$$

where $a = \{a_k\}$ is the filter such that $\sum_{n\in\mathbb{Z}} a_{n-2k}a_{n-2l} = \delta_{kl}$, $\sum_{n\in\mathbb{Z}} a_n = \sqrt{2}$ and $b_k = (-1)^k a_{1-k}$. For $l = 0$ in $(*)$, we get

$$\psi_0(x) = \psi_0(2x) + \psi_0(2x-1), \quad \psi_1(x) = \psi_0(2x) - \psi_0(2x-1),$$

where ψ_0 is a scaling function and may be taken as a characteristic function. If we increase l, we get the following

$$\psi_2(x) = \psi_1(2x) + \psi_1(2x-1), \quad \psi_3(x) = \psi_1(2x) - \psi_1(2x-1)$$

$$\psi_4(x) = \psi_1(4x) + \psi_1(4x-1) + \psi_1(4x-2) + \psi_1(4x-3)$$

and so on.

Here ψ_l's have a fixed scale but different frequencies. They are Walsh functions in [0,1[. The functions $\psi_l(t-k)$, for integers k, l with $l \geq 0$, form an orthonormal basis of $L^2(\mathbb{R})$.

Theorem 1.1. For every partition P of the non negative integers into the sets of the form $I_{lj} = \{2^j l, \ldots.., 2^j(l+1) - 1\}$, the collection of functions $\psi_{l;j,k} = 2^{j/2}\psi_l(2^j x - k)$, $I_{lj} \in P, k \in \mathbb{Z}$, is an orthonormal basis of $L^2(\mathbb{R})$.

We have used the inner product of functions $f, g \in L^2(\mathbb{R})$ as $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$, the Fourier transform of $f \in L^2(\mathbb{R})$ as $\hat{f}(w) = \int_{-\infty}^{\infty} e^{-iwx} f(x)\, dx$ and the relationship between functions and their Fourier transform as $2\pi\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$,. For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier transform $\hat{f}$ of f is in $L^2(\mathbb{R})$ and satisfies the Parseval identity $\|\hat{f}\|_2^2 = 2\pi\|f\|_2^2$.

2. Main Results:

Theorem 2.1. If $\psi_{l;j,k} = 2^{j/2}\psi_l(2^j x - k)$, $j, k \in \mathbb{Z}, l = 1, 2, \ldots., k$, constitute a wavelet packet frame for $L^2(\mathbb{R})$ with frame bounds A, B, then

$$A\ ln2 \leq \int_0^\infty \xi^{-1}|\hat{\psi}_l(\xi)|^2 d\xi \leq\ B\ ln2.$$

Proof. We use the Ad^+ to denote the set of all functions whose Fourier transform is continuous and compactly supported in the positive half real line, and vanishes in some neighborhood of zero. We use P^+ to denote the right half-plane, i.e., $P^+ = \{(a, b) \in \mathbb{R}^2 : a > 0, b \in \mathbb{R}\}$. Let $w(s) = e^{-\lambda^2\pi s^2}$, where λ is a positive parameter. We define the weight function,

$$c(a,b) = \begin{cases} w(\frac{|b|}{a}), & 1 \leq a \leq 2 \\ \\ 0, & \text{otherwise} , \end{cases}$$

and define
$$C = \int_0^\infty \frac{da}{a^2} \int_{-\infty}^{\infty} db \langle ., h^{a,b}\rangle h^{a,b} w\left(\frac{|b|}{a}\right),$$
where, $h^{a,b} = |a|^{-1/2}\ h\left(\frac{.-b}{a}\right)$.
Since $\{\psi_{l;j,k} = 2^{j/2}\psi_l(2^j x - k)\}_{j,k\in\mathbb{Z},\ l=1,2,\ldots,k}$, constitutes a wavelet packet frame for $L^2(\mathbb{R})$ with frame bounds A, B, we have

$$A\ \|h\|^2 \ \le\ \sum_l \sum_{j,k} |\langle \psi_{l;j,k}, h^{a,b}\rangle|^2 \ \le\ B\ \|h\|^2,\ \text{for } (a,b) \in P^+. \tag{1}$$

Multiplying equation (1) by weight function c(a,b) and integrating over P^+ , we have

$$A\ TrC \le\ \sum_l \sum_{j,k} \langle C\psi_{l;j,k}, \psi_{l;j,k}\rangle \ \le\ B\ TrC \tag{2}$$

where,

$$\begin{aligned} TrC &= \textstyle\int_{p^+} \frac{dadb}{a^2} c(a,b)\|h\|^2 \\ &= \textstyle\int_1^2 \frac{da}{a^2} \int_{-\infty}^{\infty} db\ w\left(\frac{|b|}{a}\right) \|h\|^2 \\ &= \textstyle\int_1^2 \frac{da}{a} \int_{-\infty}^{\infty} ds\ w(s)\|h\|^2 \\ &= ln2\ 2\left(\int_0^\infty ds\ w(s)\|h\|^2\right). \end{aligned}$$

For weight function w we have chosen, $\int_0^\infty dtw(t) = \frac{1}{2}$.
Hence,
$$TrC\ = ln2\ \|h\|^2. \tag{3}$$

The middle term of equation (2), becomes

$$\textstyle\sum_l \sum_{j,k} \langle C\psi_{l;j,k}, \psi_{l;j,k}\rangle \ =\ \sum_l \sum_{j,k} \int_1^2 \frac{da}{a^2} \int_{-\infty}^{\infty} db\ w\left(\frac{|b|}{a}\right) |\langle \psi_{l;j,k}, h^{a,b}\rangle|^2. \tag{4}$$

Now,
$$\begin{aligned} \langle \psi_{l;j,k}, h^{a,b}\rangle &= \textstyle\int \psi_{l;j,k}(x)\ \overline{h^{a,b}(x)} dx \\ &= \textstyle\int 2^{j/2}\psi_l(2^j x - k) a^{-1/2}\ \overline{h\left(\frac{x-b}{a}\right)} dx \\ &= 2^{j/2}a^{-1/2} \textstyle\int \psi_l(y)\ \overline{h\left(\frac{y+k-b2^j}{2^j a}\right)} 2^{-j} dy \\ &= 2^{-j/2}a^{-1/2} \textstyle\int \psi_l(y)\ \overline{h\left(\frac{y-(b2^j-k)}{2^j a}\right)} dy \\ &= \langle \psi_l, h^{2^j a, b2^j - k}\rangle. \end{aligned}$$

Now equation(4) becomes,

$$\sum_l \sum_{j,k} \langle C\psi_{l;j,k}, \psi_{l;j,k}\rangle = \sum_l \sum_{j,k} \int_1^2 \frac{da}{a^2} \int_{-\infty}^{\infty} dbw\left(\frac{|b|}{a}\right) |\langle \psi_l, h^{2^j a, b2^j - k}\rangle|^2.$$

By changing variables $2^j a = a'$, $2^j b = b'$, we have

$$\begin{aligned}
\sum_l \sum_{j,k} \langle C\psi_{l;j,k}, \psi_{l;j,k}\rangle &= \sum_l \sum_{j,k} \int_{2^j}^{2^{j+1}} \frac{da'}{a'^2} \int_{-\infty}^{\infty} db' w\left(\frac{|b'|}{a'}\right) |\langle \psi_l, h^{a', b'-k}\rangle|^2 \\
&= \sum_l \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \frac{da}{a^2} \int_{-\infty}^{\infty} db \sum_{k\in\mathbb{Z}} w\left(\frac{b+k}{a}\right) |\langle \psi_l, h^{a,b}\rangle|^2 \\
&= \sum_l \sum_{j\in\mathbb{Z}} \int_0^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} db |\langle \psi_l, h^{a,b}\rangle|^2 \sum_{k\in\mathbb{Z}} w\left(\frac{b+k}{a}\right) \qquad (5)
\end{aligned}$$

The function $w(s) = e^{-\lambda^2 \pi s^2}$ has only one local maximum and is monotonically decreasing as $|s|$ increases.

The Lemma 2.2 of Daubechies [3], shows that for such function w and $\alpha\beta \in \mathbb{R}$, $\beta > 0$,

$$\int_{-\infty}^{\infty} dtw(t) - \beta w_{max} \leq \beta \sum_{k\in\mathbb{Z}} w(\alpha + k\beta) \leq \int_{-\infty}^{\infty} dtw(t) + \beta w_{max},$$

or, for particular w,

$$\sum_{k\in\mathbb{Z}} w\left(\frac{|b+k|}{a}\right) = a + \rho(a,b)$$

with $\rho(a,b) \leq w(0) = \lambda$.
Thus we have,

$$\begin{aligned}
\sum_l \sum_{j,k} \langle C\psi_{l;j,k}, \psi_{l;j,k}\rangle &= \int_0^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} db |\langle \psi_l, h^{a,b}\rangle|^2 (a + \rho(a,b)) \\
&= \int_0^{\infty} \frac{da}{a} \int_{-\infty}^{\infty} db |\langle \psi_l, h^{a,b}\rangle|^2 + R \qquad (6)
\end{aligned}$$

where,

$$\begin{aligned}
|R| &= \int_0^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} db |\langle \psi_l, h^{a,b}\rangle|^2 \rho(a,b) \\
&\leq \lambda C_h \|\psi_l\|^2. \qquad (7)
\end{aligned}$$

$$\begin{aligned}
\int_0^{\infty} \frac{da}{a} \int_{-\infty}^{\infty} db |\langle \psi_l, h^{a,b}\rangle|^2 &= \int_0^{\infty} \frac{da}{a} \int_{-\infty}^{\infty} db \left|\frac{1}{2\pi} \langle \hat{\psi}_l, \hat{h}^{a,b}\rangle\right|^2 \\
&= \int_0^{\infty} \frac{da}{a} \int_{-\infty}^{\infty} db \frac{1}{4\pi^2} \left| \left[\int_0^{\infty} \hat{\psi}_l(\xi) a^{1/2} \hat{h}(a\xi) e^{ib\xi} d\xi \right] \right|^2 \\
&= \frac{1}{2\pi} \int_0^{\infty} da' |\hat{h}(a')|^2 \int_0^{\infty} d\xi \xi^{-1} |\hat{\psi}_l(\xi)|^2 \\
&= \frac{1}{2\pi} \|\hat{h}\|_2^2 \int_0^{\infty} d\xi \xi^{-1} |\hat{\psi}_l(\xi)|^2
\end{aligned}$$

$$= \quad \|h\|_2^2 \int_0^\infty d\xi \xi^{-1} |\hat{\psi}_l(\xi)|^2. \tag{8}$$

Thus equation (6) becomes,

$$\sum_l \sum_{j,k} \langle C\psi_{l;j,k}, \psi_{l;j,k}\rangle = \|h\|_2^2 \int_0^\infty d\xi \xi^{-1} |\hat{\psi}_l(\xi)|^2 + R. \tag{9}$$

Substituting the values of equations (3),(9) and R in equation (2), we get

$$A\|h\|_2^2 \; ln2 \leq \|h\|_2^2 \int_0^\infty d\xi \xi^{-1} |\hat{\psi}_l(\xi)|^2 + \lambda C_h \|\psi_l\| \leq B\|h\|_2^2 \; ln2.$$

Taking λ tends to zero, we get the conclusion immediately.

Theorem 2.2. Let $\psi_l \in L^2(\mathbb{R})$ be admissible for each l. If the system $\psi_{l;j,k} = 2^{j/2}\psi_l(2^j x - k)$, $j,k \in \mathbb{Z}, l = 1,2,....,k$, constitute a wavelet packet frame for $L^2(\mathbb{R})$ with frame bounds A, B, then

$$A \; \delta \leq \sum_l \sum_{j \in \mathbb{Z}} |\hat{\psi}_l(2^j\xi)|^2 \leq \; B \bigtriangleup .$$

where, $\delta \leq k \leq \bigtriangleup < \infty$.

Proof. Taking the weight function,

$$c_{\lambda,\epsilon}(a,b) := \begin{cases} w_\lambda(\frac{|b|}{a}), & e^{-\epsilon} \leq a \leq e^\epsilon \\ 0, & \text{otherwise} \; , \end{cases}$$

and

$$A \; TrC_{\lambda,\epsilon,h} \leq \; \sum_l \sum_{j,k} \langle C_{\lambda,\epsilon,h}\psi_{l;j,k}, \psi_{l;j,k}\rangle \; \leq \; B \; TrC_{\lambda,\epsilon,h} \tag{10}$$

and using the similar technique as in Theorem 2.1., we have

$$TrC_{\lambda,\epsilon,h} \; = \; 2\|h\|^2 \epsilon \tag{11}$$

and

$$\sum_l \sum_{j,k} \langle C_{\lambda,\epsilon,h}\psi_{l;j,k}, \psi_{l;j,k}\rangle \; = \; \sum_l \sum_{j \in \mathbb{Z}} \int_{2^j e^{-\epsilon}}^{2^j e^\epsilon} \frac{da}{a^2} \int_{-\infty}^\infty db |\langle \psi_l, h^{a,b}\rangle|^2 \sum_{k \in \mathbb{Z}} w_\lambda \left(\frac{b+k}{a}\right). \tag{12}$$

By Lemma 2.3 [10], we have the following estimates,

$$\frac{a}{\bigtriangleup} - \lambda \leq \sum_{k \in \mathbb{Z}} w_\lambda \left(\frac{|b+k|}{a}\right) \leq \frac{a}{\delta} + \frac{2\lambda\bigtriangleup}{\delta}. \tag{13}$$

Combining equations (10)-(13), and letting $\lambda \to 0$, we have

$$A\|h\|^2 \leq \frac{1}{\delta} I_\epsilon \qquad \text{and} \qquad B\|h\|^2 \geq \frac{1}{\triangle} I_\epsilon, \tag{14}$$

where,

$$\begin{aligned} I_\epsilon &= \tfrac{1}{2\epsilon}\sum_l \sum_{j\in\mathbb{Z}} \int_{2^j e^{-\epsilon}}^{2^j e^\epsilon} \tfrac{da}{a} \int_{-\infty}^{\infty} db |\langle \psi_l, h^{a,b}\rangle|^2 \\ &= \tfrac{1}{4\pi^2}\tfrac{1}{2\epsilon}\sum_l \sum_{j\in\mathbb{Z}} \int_{2^j e^{-\epsilon}}^{2^j e^\epsilon} \tfrac{da}{a} \int_{-\infty}^{\infty} db \left[\int_0^\infty |\hat{\psi}_l(\xi)|^2 a |\hat{h}(a\xi)|^2 d\xi e^{ib\xi}\right] \\ &= \tfrac{1}{2\pi}\sum_l \int_0^\infty d\xi |\hat{\psi}_l(\xi)|^2 \sum_{j\in\mathbb{Z}} \tfrac{1}{2\epsilon}\int_{2^j e^{-\epsilon}}^{2^j e^\epsilon} da |\hat{h}(a\xi)|^2 \\ &= \tfrac{1}{2\pi}\sum_l \int_0^\infty d\xi |\hat{\psi}_l(\xi)|^2 H_\epsilon(\xi). \end{aligned} \tag{15}$$

Since $h \in Ad^+$, we can assume that $supp\hat{h} \subseteq [x_h, X_h]$ and $\hat{h}(\xi) \leq M$, where x_h and X_h $(x_h < X_h)$ are two positive numbers.
Here,

$$\begin{aligned} H_\epsilon(\xi) &= \sum_{j\in\mathbb{Z}} \tfrac{1}{2\epsilon}\int_{2^j e^{-\epsilon}}^{2^j e^\epsilon} da |\hat{h}(a\xi)|^2 \\ &= \sum_{j\in\mathbb{Z}} \tfrac{1}{\xi 2\epsilon}\int_{\xi 2^j e^{-\epsilon}}^{\xi 2^j e^\epsilon} da' |\hat{h}(a')|^2 \\ &= \sum_{j\in\mathbb{Z}} \tfrac{1}{\xi 2\epsilon}\int_{\xi 2^j e^{-\epsilon}}^{\xi 2^j e^\epsilon} da |\hat{h}(a)|^2 \\ &\leq \sum_{j\in\mathbb{Z}} \tfrac{M}{\xi 2\epsilon}\int_{\xi 2^j e^{-\epsilon}}^{\xi 2^j e^\epsilon} da \chi_{[x_h, X_h]}(a) \\ &\leq \sum_{\xi X_h^{-1} e^{-\epsilon} < 2^j < \xi x_h^{-1} e^\epsilon} \tfrac{M}{\xi 2\epsilon}\int_{\xi 2^j e^{-\epsilon}}^{\xi 2^j e^\epsilon} da \\ &\leq \sum_{\xi X_h^{-1} < 2^j < \xi x_h^{-1}} \tfrac{M}{\xi 2\epsilon} \xi 2^j (e^{-\epsilon} - e^\epsilon) \\ &\leq \tfrac{2M}{\xi/2X_h < 2^j < 2\xi/x_h} \tfrac{1}{2^j} \\ &= 2M\rho_{1/2X_h, 2/x_h}(\xi). \end{aligned} \tag{16}$$

By Lemma 2.1 [10], the function $|\hat{\psi}_l(\xi)|^2\rho_{1/2X_h,2/x_h}(\xi)$ is integrable over $(0,\infty)$. So by the use of dominated convergence theorem, from eqn.(15) we can deduce that,

$$\begin{aligned}
lim_{\epsilon\to 0} I_\epsilon &= \tfrac{1}{2\pi}\sum_l \int_0^\infty d\xi |\hat{\psi}_l(\xi)|^2 H_\epsilon(\xi) \\
&= \tfrac{1}{2\pi}\sum_l \int_0^\infty d\xi |\hat{\psi}_l(\xi)|^2 lim_{\epsilon\to 0}\tfrac{1}{2\epsilon}\sum_{j\in\mathbb{Z}}\int_{\xi 2^j e^{-\epsilon}}^{\xi 2^j e^{\epsilon}} da|\hat{h}(a)|^2 \\
&= \tfrac{1}{2\pi}\sum_l \int_0^\infty d\xi |\hat{\psi}_l(\xi)|^2 \xi^{-1} lim_{\epsilon\to 0}\sum_{j\in\mathbb{Z}}\tfrac{1}{2\epsilon}\int_{log\xi - log2^j-\epsilon}^{log\xi-log2^j+\epsilon} du\ e^u|\hat{h}(e^u)|^2 \\
&= \tfrac{1}{2\pi}\sum_l \int_0^\infty d\xi |\hat{\psi}_l(\xi)|^2 \xi^{-1}\sum_{j\in\mathbb{Z}}\tfrac{\xi}{2^j}\left|\hat{h}\left(\tfrac{\xi}{2^j}\right)\right|^2 \\
&= \tfrac{1}{2\pi}\sum_l \int_0^\infty d\xi |\hat{h}(\xi)|^2 \sum_{j\in\mathbb{Z}}|\hat{\psi}_l(2^j\xi)|^2 \\
&= \|h\|^2\sum_l\sum_{j\in\mathbb{Z}}|\hat{\psi}_l(2^j\xi)|^2. \qquad (17)
\end{aligned}$$

Combining equations (14), (17) and Lemma 2.2 [10], the result follows.

References

[1] Christensen, I.; *Frames, Riesz bases and discrete Gabor/wavelet expansions*, Bull. Amer. Math. Soc., 38, no.3, pp. 273-291, 2001.

[2] Coifman, R.R. and Rochberg, R.; *Representation for holomorphic and harmonic functions in L_p*, Asterisque 77, pp. 11-66, 1980.

[3] Daubechies, I.; *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. theory, 36, no.5, pp. 961-1005, 1990.

[4] Daubechies, I.; *Ten lectures on wavelets*, SIAM, Philadelphia 1992.

[5] Debnath, L.; *Wavelet transforms and their Applications*, Birkhauser, 2002.

[6] Duffin, R.J. and Schaeffer, A.C.; *A class of non harmonic Fourier series*, Trans. Amer. Math. Soc., 72, pp. 341-366. 1952.

[7] Heil, C. and Walnut, D.; *Continuous and discrete wavelet transforms*, SIAM Review, 31, pp. 628-666, 1989.

[8] Shang, Z. and Zhou, X.; *Dual generators for weighted irregular wavelet frames and reconstruction error*, Appl. Comp. Harm. Anal., Article in press.

[9] Sun, W. and Zhou, X.; *Irregular wavelet/Gabor frames*, Appl. Comp. Harm. Anal., 13, pp. 63-76, 2002.

[10] Yang, S. and Zhou, X.; *A note on irregular wavelet frames*, Appl. Math. Comp., Article in press.

[11] Wickerhauser, M. V.; *Acoustic signal compersion with wavelet packets*, In C.K. Chui (editor), pp. 679-700, 1991.

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ADAPTIVE UPDATE LIFTING SCHEME FOR IMAGE CODING

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ABSTRACT

In this paper, we present the application of adaptive wavelet in lossy image compression. The construction of this adaptive wavelet is realised by the use of lifting scheme. In our application the lifting scheme is composed by an adaptive update lifting step and a fixed prediction lifting step. Finally, experiments with The EBCOT (Embedded Block Coding with Optimized Truncations) algorithm applied on synthetic and real images are reported.

Keywords: Adaptive wavelet, update lifting scheme, EBCOT, lossy Image compression,

I. INTRODUCTION

Wavelet transforms have received significant attention in deferent fields, such as mathematics, digital signal and image processing, because of their ability to represent and analyze data [1][2][3][4].

The wavelet transform, defined by Yves Meyer and J. Lemarié, offers good localization in both, space and frequency domains, and can be implemented by fast algorithms.

Since the discrete wavelet transform (DWT) was presented by Mallat, many researchers on signal analysis and image compression have reached fruitful results due to its well time-frequency decomposition.

Recently, a new wavelet construction called lifting scheme, has been developed by Wim Sweldens and Ingrid Daubechies [5][6]. This method has gained increasing interest in scientific community, due to its reduced computational complexity by first factoring a classical wavelet filter into lifting steps.

Based on the lifting scheme work, a number of adaptive wavelet transforms have been proposed. D. Taubman [7] has developed adaptive wavelet transforms to modify the prediction step by using the properties of the image. Since this predictor takes into account the fact that the discontinuities in images tend to occur along continuous curves, this adaptation of the predictor makes the transform non-linear. Calypoole et al [9] used local orientation information at edge boundaries, to define a prediction operator.

[10] have proposed a prediction operator based on the local properties of the image. In [11] they described an adaptive polyphase structure based on the reduction of the variance. In [12] they first calculate the optimal predictors, by minimizing the prediction error variance, and then they apply these optimal predictor filters with adaptive update filters.

Our work consists in applying the update adaptive wavelet filter presented in [13], [14], [15] and [16], to lossy image compression.

This paper is organized as follows. Section II represent an overview of the construction of lifting schemes. In section III, we introduce the construction of the adaptive update lifting. Sections IV and V describe the two-dimensional adaptive update lifting scheme and its application in image coding respectively. Simulation results and concluding remarks are presented in sections VI and VII, respectively.

II. THE LIFTING SCHEME

A canonical lifting structure consists of three stages, as shown in fig.1:

1- Split: divide the original signal x into two disjoint parts:

x_o: the set of odd samples of x
x_e: the set of even samples of x

2- Predict: apply a predictor P on x_e to predict x_o

$$d=x_o-P(x_e)$$

d: prediction error (the detail of the signal x)

3- Update: apply an updater U on d

$$S=x_e+U(d)$$

The inverse transform is easily obtained since every stage above is invertible.

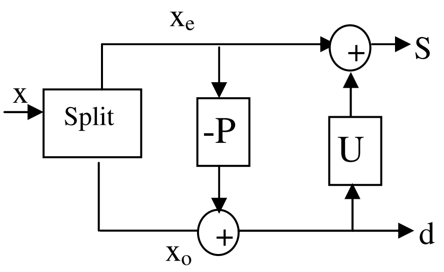

Fig.1. The lifting scheme

III. ADAPTIVE UPDATE LIFTING SCHEME

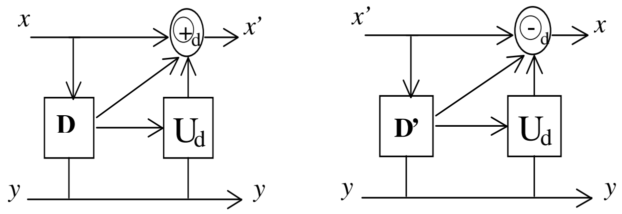

Fig.2. Adaptive Update Lifting scheme

The input signal X is decomposed into x,y.

y containing more than one channel;

$y=\{y_{b1},y_{b2},\ldots,y_{bM}\}$. (1)

The update step is adaptive while the prediction step is fixed.

The update value $x'(n)$ is given by

$x'(n)=x(n)\oplus_{dn}U_{dn}(y)(n)$ (2)

U_{dn} : is the update operator for the decision d_n

D: is the decision map wich uses inputs (from all bands) and its output dn is a function representing N-valued decision

$d_n=D(x,y)(n) \in\{0,1,\ldots\ldots,\text{N-1}\}$ (3)

$$x\oplus_d u=\alpha_0(x+u) \qquad \alpha_j\neq 0$$

The update filter is

$$U_d(y)(n)=\sum_{j=1}^{J}\lambda_{d,j}y_j(n) \quad \text{where } y_j(n)=y_{bj}(n+l_j)$$

$$j=1,\ldots,J \text{ and } l_j\in L,$$

where L is a window in Z^2 centered around the origin.

$\lambda_{d,j}$ depend on the decision d at location n

The update equation used at analysis

$x'(n)=\alpha_{d_n}x(n)+\sum\beta_{d_n,j}y_j(n)$ (4)

$$\beta_{d_n,j}=\alpha_d\lambda_{d_n,j}$$

where we assume that $\alpha_{dn}+\sum_{j=1}^{J}\beta_{d,j}=1$ and $d=0,\ldots\ldots,N-1$ with $\alpha_d\neq 0$ for all d.

The decision map depends only on the gradient vector $v(n)\in R^J$, with components $v_i(n)$ given by

$v_j(n)=x(n)-y_j(n)=x(n)-y(n+j),$ (5)

where $j=1,\ldots.,J$.

The gradient vector at synthesis $v'(n)$ is related to $v(n)$ by means of the linear relation

$v_j'(n)=x'(n)-y_j(n)$ (6)

$$v'(n)=A_d v(n)$$

where $A_d=I-ub_d^T$

I is $J\times J$ is the identity matrix,

and $u=(1,..,1)^T$, $b_d=(\beta_{d,1},\ldots.,\beta_{d,J})^T$

To have perfect reconstruction (PR) we need to satisfy the decision conservation condition $D(x,y)(n)=D'(x',y)(n)$.

The lifting scheme can choose between several update linear filters.

The conservation of the signal and image discontinuities is a potential characteristic of the method. even at low resolutions.

IV. TWO-DIMENSIONAL ADAPTIVE UPDATE LIFTING SCHEME D

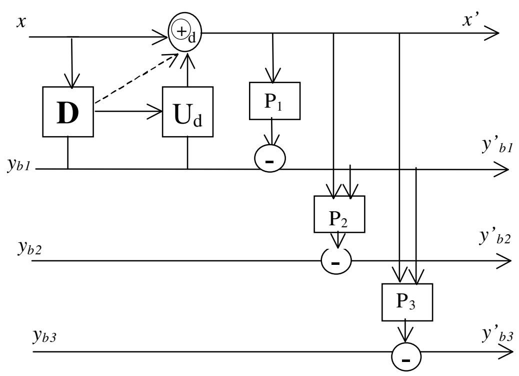

Fig.3. 2D wavelet decomposition with adaptive update lifting step and three prediction steps

The input images are x, y_{b1}, y_{b2} and y_{b3}

$$x(m,n) = x_0(2m,2n)$$
$$y_{b1}(m,n) = x_0(2m,2n+1)$$
$$y_{b2}(m,n)=x_0(2m+1,2n)$$
$$y_{b3}(m,n)=x_0(2m+1,2n+1)$$

where x_0 is the original image

x' is the approximation band and y'_{b1}, y'_{b2}, y'_{b3} are the horizontal, vertical, diagonal detail bands.

We consider a 3×3 portion of the image centered around the center pixel $x(m,n)$ as shown in fig.4. The decision d_n and the output of the update filter depend only on the values associated with this portion.

$y_7(n)$	$y_4(n)$	$y_8(n)$
$y_3(n)$	$x(n)$	$y_1(n)$
$y_6(n)$	$y_2(n)$	$y_5(n)$

Fig.4. Sample image segment centred around $x(n)=x_0(2m,2n)$

$y_1(n)=y_{b1}(n)$, $y_2(n)=y_{b2}(n)$ $y_{13}(n)=y_{b1}(m,n-1)$

$y_4(n)=y_{b2}(m-1,n)$ $y_5(n)=y_{b3}(n)$, *etc...*

We give two examples. After the update step, we compute the detail images with a symmetric prediction scheme:

$$y'_{b_1}=y_{b_1}-(x'(n+1,m)+x'(n))/2$$

$$y'_{b_2}=y_{b_2}-(x'(n,m+1)+x'(n))/2$$

$$y'_{b_3}=y_{b_3}-(x'(n+1,m+1)+x'(n))/2-y'_{b_1}(n)-y_{b_2}(n)$$

where P is taken as a weighted gradient seminorm.

$$P(v)=\left|a^T v\right|=\left|\sum_j a_j v_j\right| \qquad (7)$$

Example 1

$$N=2 \qquad d_n=D(x,y)(n) \in\{0,1\}$$

$$D(x,y)(n)=[P(v(n)>T)]$$

$$[P]=\begin{cases}1 & \text{if } P(v(n))>T \text{ is true}\\ 0 & \text{if } P(v(n))>T \text{ is false}\end{cases}$$

We use a = (1,1,1,1,-1/2,-1/2,-1/2,-1/2). In order to verify the Perfect Reconstruction condition we choose λ_0=1/4 and $\lambda_1=0$.

For the input image Zelda shown at the top left of fig. 5, we take T=25. The decision map is shown in the top right. The approximation and horizontal detail (after two levels of decomposition) in the non adaptive case (d=0) are shown in the middle row. The adaptive results are presented in the bottom row.

Example 2

We now present a way of constructing the dicision map by comparing different seminorms.

We have four decision regions, described by the following conditions:

$$p_0(v)\le P_1(v) \text{ and } p_2(v)\le T_0 \quad\Leftrightarrow\quad d=0$$

$$p_0(v)\le P_1(v) \text{ and } p_2(v)>T_0 \quad\Leftrightarrow\quad d=1$$

$$p_0(v)>P_1(v) \text{ and } p_3(v)\le T_0 \quad\Leftrightarrow\quad d=2$$

$$p_0(v)>P_1(v) \text{ and } p_3(v)>T_0 \quad\Leftrightarrow\quad d=3$$

where $p_0(v)=\left|v_1+v_3\right|$ $p_1(v)=\left|v_{21}+v_{43}\right|$

$$p_2(v)=\left|v_1+\frac{1}{2}v_2+v_3+\frac{1}{2}v_4\right| \qquad p_3(v)=\left|\frac{1}{2}v_1+v_2+\frac{1}{2}v_3+v_4\right|$$

We choose $b_0=\frac{1}{4}a_2$ $b_2=\frac{1}{4}a_3$ and $b_1=b_3=0$, which implies that no update filtering is performed.

The thresholds T and T_0 are chosen heuristically, with their values depending on the test images and the seminorm values.

Fig 5 Wavelet decomposition (level 2). Top: input image (left) and decision map (right). Middle: approximation (left), horizontal detail (right) images in adaptive mode. Bottom: approximation (left), horizontal detail (right) images in non-adaptive mode

V. APPLICATIONS IN IMAGE COMPRESSION

The adapted image coding method is the EBCOT (Embedded Block Coding with Optimized Truncations) presented in [17] and [18]. Three levels of decomposition were selected. The adaptive update lifting scheme was applied to the test images shown in fig. 6. At the left of the figure a synthetic image, in the right, the image lena, and in th down the bird and the montage images .

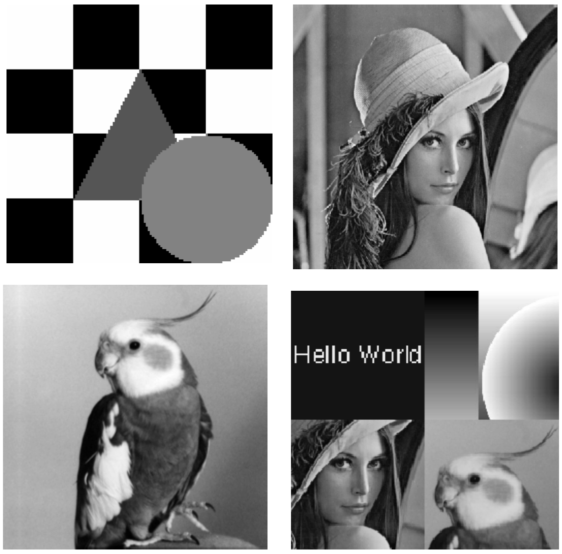

Fig.6 test images : in the top: synthetic image(left) and the image lenna in right, in the down: image bird (left) and the image montage in the right

VI. RESULTS

VI.A. Synthetic Data

Fig. 7 shows the rate distortion curves of the adaptive case against non adaptive of example 1 and example 2 applied to a synthetic image. Here the differences are considerably large in the adaptive example than its non-adaptive one. And we notice that in the adaptive algorithm the example1 has better PSNR performance than example 2.

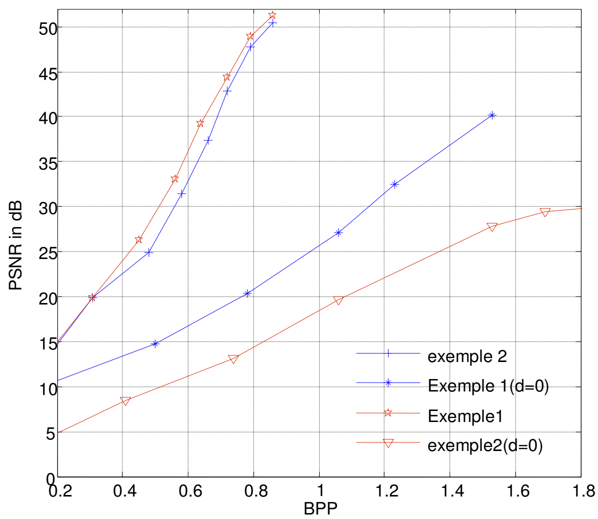

Fig.7. Rate distortion curve for the synthetic image of fig. 6. This test image was designed to compare example 1 with example 2 (adaptive case and non adaptive case).

We have compared our performance to the Daubechies (9, 7) wavelet because it is very popular in image compression. The PSNR curve demonstrates that, for this synthetic test image, the adaptive algorithm of example 1 has better performance than both, the adaptive algorithm of example 2 and Daubechies (9, 7).

VI.B. Real Data

The proposed coding methods were applied to the standard monochromatic (8 b/pixel) image Lena of size 512×512. Fig. 8 shows the results. The performance of the popular Daubechies (9,7) transform is shown for reference. Although our adaptive algorithm does not match the PSNR performance of the Daubechies (9,7) transform, example1 goes closer to Daubechies (9,7) in some area; this is due to the non linearity characteristic. In Fig 9 example2 done better results than example1 and The PSNR results of adaptive example are than the Daubechies (9,7). To the contrary, in fig10, example 1 gives better results over a considerable area, on the image montage.

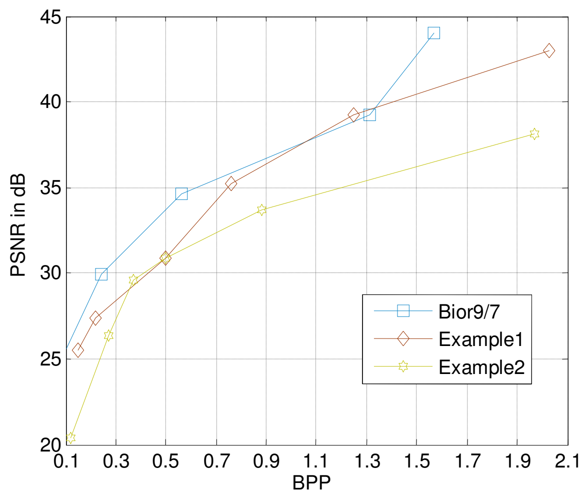

Fig. 8. Comparison of adaptive algorithm example 1 and adaptive algorithm example2 with Bior 9/7 on the Lena image

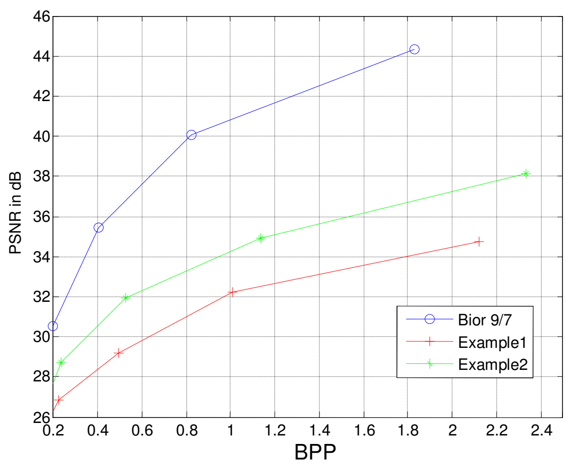

Fig.9 PSNR curve for the image bird: comparison of adaptive filters and Bior 9/7

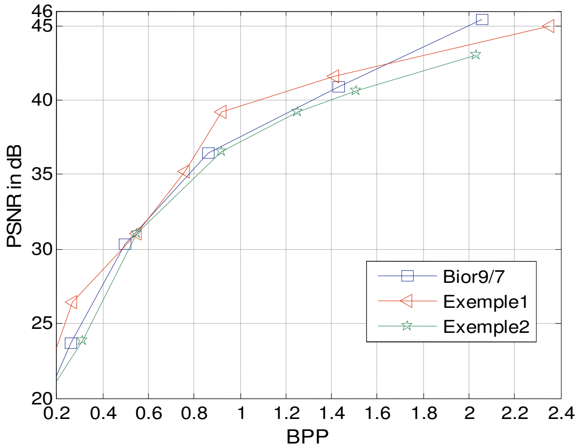

Fig. 10 PSNR curve for the montage image: comparaison of adaptive filters and Bior 9/7

VII. CONCLUSION

The lifting scheme is a new framework in wavelet research. It develops the concept of wavelet. And further, extends the wavelet research. It could be used to implement existing wavelets and to construct completely new wavelets, and allows us to combine, nonlinear and adaptive properties.

Within this scheme, we used an algorithm that switches between various linear update filters in lossy image compression. This adaptive lifting scheme using seminorms, have produced good results in synthetic images, and promising results on certain test images.

Future work aims to develop an application of adaptive wavelet transforms that map integers to integers based on the adaptive update lifting scheme, in lossy image compression..

VIII. REFERENCES

[1] L.Szu Y.Sheng "Wavelet processing and optics " , IEEE 84(5) , 1996 , pp 720-732 . Imaging , October 1999, Vol.7(4).
[2] M.Borland "Wavelet in image communication". Elsevier 1994.
[3] Charles Hsu et al. "Wavelet brassboards for live video via ratio", Journal of Electronic Imaging, October 1999, Vol 7(4)
[4] O.Strome et al "Study of wavelet decompositions for image/video compression by software codecs", IEEE 1997, pp 125-132
[5] W. Sweldens "The lifting scheme: A construction of second generation wavelets" SIAM J. Math. Anal., Vol.29, No.2, pp.511-546, 1997.
[6] I. Daubechies, W. Sweldens "Factoring wavelet into lifting steps", J. Fourier Anal. Appl. Vol.4, Nr.3, pp.247-269, 1998.
[7] D. Taubman "Adaptive non-separable lifting transforms for image compression", in Proc. Int. Conf. Image Processing, Kobe, Japan, Oct.1999.
[8] A. Gouze, M. Antonini, M. Barland "Quincunx lifting scheme for lossy image compression" IEEE 2000, pp 665-668
[9] A. Gouze, M. Antonini, M. Barland, Benoit Macq "Design of signal –adapted multidimensional lifting scheme for lossy coding" IEEE Trans. on Image Processing, Vol.13, No.12, pp 1589-1603, December 2004
[10] R.L. Caypoole, G.M. Davis, W. Sweldens, and R.Gboranuk "Nonlinear wavelet transforms for image coding via lifting" IEEE Trans. on Image Processing, Vol.12, pp 1149-1459, Dec.2003.
[11] Ömer Nezih Gerek,, A.Enisçetin "Adaptive polyphase decomposition structure for image compression" IEEE Trans. on Image Processing, Vol.9, No.10, October 2000.
[12] N.V. Boulgouris, Dimitrios Tzavaras and Michael Gerassiomos Strintzis "Lossless image compression based on optimal prediction, adaptive lifting and conditional arithmetic coding", IEEE Transaction on Image Processing, Vol.10, No.1, January 2001, pp1-14.
[13] G. Piella, H.J.A.M. Heijmans "Adaptive lifting scheme with perfect reconstruction", IEEE Trans. on Signal Processing, Vol.50, No7, July 2002, pp1620-1630.
[14] G. Piella, B. Pesquet-Popescu, H.J.A.M. Heijmans "Adaptive update lifting with a decision rule based on derivative filters", IEEE Signal Processing Letters, Vol.9, No.10, October 2002.
[15] G.Piella, B. Pesquet-Popescu, H.J.A.M. Heijmans "Content-adaptive multiresolution anaalysise" in Advanced Concepts for Intelligent Vision Systems", Brussels, Belgiun, Aug.31-Sept.3, 2004, pp289-296.
[16] G.Piella.; G.Pau.; B.Pesquet-Popescu "Adaptive lifting schemes combining seminorms for lossless image compression" IEEE International Conference on Image processing Volume 1, Issue , 11-14 Sept. 2005 pp: 753-759
[17] D. Taubman "High performances scalable image compression with EBCOT", IEEE Trans. on Image Processing, Vol.9, No.7, July 2000.
[18] Majid Rabbani, R. Joshi "An overview of JPEG2000 still image compression standard", Signal Processing, Image Communication 17(2002), 3-48.

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Market Stochastic bounds with elliptical distributions

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MARKET STOCHASTIC BOUNDS WITH ELLIPTICAL DISTRIBUTIONS

Abstract: This paper explores and analyzes the implications and the advantages of safety first analysis in portfolio choice problems. In particular, we describe the market bounds in order to examine the evolution of investor's choices in relation with the market trend. Thus, first we prove the uniqueness in distribution of market bounds when limited short sales are allowed. Then we analytically determine the link between stochastic choices and market bounds when the returns are elliptical distributed and no short sales are allowed.
Key words: Stochastic bounds, efficient frontier, stochastic dominance, elliptical distributions.

1 Introduction

The aim of this paper is to provide a comprehensive presentation of the measurement and applications of the market stochastic bounds in order to understand and forecast the investors' choices changes. In particular, the paper proposes some new tools to manage return portfolios considering the evolution of the optimal choices related to the market trend.

The paper describes some management tools to forecast the behavior of future choices. In particular, we study the evolution of investors' optimal choices depending on the market stochastic bounds when no short sales are allowed. The distribution functions of market stochastic bounds are the envelopes of the portfolio distribution functions and the upper stochastic bound represents optimal choices of all admissible investors. When the returns are elliptically distributed, we can analytically describe how choices change in relation with the upper market stochastic bound even if the process requires to rewrite the Markowitz' efficient frontier in terms of the mean (see Markowitz [11], [12]). So, first we describe an alternative algorithm to Markowitz' one in order to rewrite recursively the efficient frontier when no short sales are allowed. The algorithm describes analytically the evolution of optimal choices as a function of the upper stochastic bound that is generated by the optimal choices of the efficient frontier.

In the next section, 2, we define the stochastic bounds of the market. The third section introduces the analysis of market trend when returns are elliptically distributed. All the proofs of the results are described in the appendix.

2 Stochastic Bounds

In this section we describe the stochastic bounds of all admissible choices. In particular, we show that the stochastic bound distributions represent the envelopes of all portfolio distribution functions and, for this reason, they are strictly related to stochastic ordering theory. Recall that X first stochastically dominates

Y (X FSD Y), if and only if $E(\phi(X)) \geq E(\phi(Y))$ for every non-decreasing function ϕ such that the two expectations exist, i.e., if and only if $F_X(t) \leq F_Y(t)$ for every real t. More generally, we say that X dominates Y in the sense of the α ($\alpha \geq 1$) stochastic dominance order ($X \underset{\alpha}{\geq} Y$) iff $F_X^{(\alpha)}(t) \leq F_Y^{(\alpha)}(t)$ for every real t, where

$$F_X^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-y)^{\alpha-1} dF_X(y) = \frac{E\left((t-X)_+^{\alpha-1}\right)}{\Gamma(\alpha)}$$

(see Fishburn [9]).

Suppose we have a frictionless market without arbitrage opportunities where all investors act as price takers. Suppose that all investors have the same temporal horizon at a fixed date in the future. Given n risky securities, we indicate with r_i the rate of return of the i^{th} security and $Z_i = 1 + r_i$ the respective gross return. The random vector of the gross returns $Z = [Z_1, ..., Z_n]'$ is defined on the Polish probability space $(\Omega, \Im, P)$. In a market with limited short selling opportunities every admissible vector of portfolio weights $x = [x_1, ..., x_n]'$ belongs to the compact set

$$T = \left\{ y \in \Re^n / \sum_i y_i = 1 \ ; \ g_i \leq y_i \leq h_i \right\},$$

where $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_n)$ are fixed vectors belonging to $\Re^n$. Under these assumptions the portfolios are stochastically bounded, i.e., there exist two random variables Y_U and Y_L such that for every admissible vector of portfolio weights $x = [x_1, ..., x_n]'$ belonging to the set T we have Y_U FSD $x'Z$ FSD Y_L. As we can observe in the following theorem we can express analytically the distribution functions of the optimal bounds which are given by

$$F_U(\lambda) = \inf_{x \in T} \mathbb{P}(x'Z \leq \lambda) \quad \text{and} \quad F_L(\lambda) = Q(\lambda^+) = \lim_{\lambda \to \lambda^+} Q(\lambda) = \sup_{x \in T} \mathbb{P}(x'Z \leq \lambda).$$

and they are related to the vectors of portfolio weights

$$x_U(\lambda) \in \arg\left(\inf_{x \in T} \mathbb{P}(x'Z \leq \lambda)\right) \quad \text{and} \quad x_L(\lambda) \in \arg\left(\sup_{x \in T} \mathbb{P}(x'Z \leq \lambda)\right)$$

Theorem 1
Assume limited short sales are allowed in the market. Then F_U is the "smallest" cumulative distribution (in FSD sense) which FSD dominates all portfolios, and it has support $[a, b]$ where $a = \sup_{x \in T} c(x)$, $b = \sup_{x \in T} d(x)$, and

$$c(x) = \sup\left\{c \in \overline{\Re} \mid \mathbb{P}(x'Z \leq c) = 0\right\}$$
$$d(x) = \inf\left\{d \in \overline{\Re} \mid \mathbb{P}(x'Z > d) = 0\right\}.$$

Similarly, F_L is the "greatest" cumulative distribution (in FSD sense) which is FSD dominated by all portfolios, and it has support $[\widetilde{a}, \widetilde{b}]$ where $\widetilde{a} = \inf_{x\in T} c(x)$, and $\widetilde{b} = \inf_{x\in T} d(x)$. Moreover, when a riskless gross return z_0 is allowed, then $a \geq z_0$ and $\widetilde{b} \leq z_0$.

Note that two cases are possible:

i) there exists a portfolio $x'Z$ that FSD dominates all the others (or is FSD dominated by all the others), then it should have F_U as distribution (it should have F_L as distribution);

ii) in the market it does not exist a portfolio $x'Z$ that FSD dominates (or is FSD dominated by) all the others.

Next, we will focus our attention on case ii), that is the most common one. When only limited short sales are allowed, we call F_U the *stochastically dominating distribution* of all the admissible portfolios and F_L the *stochastically dominated distribution*. Let Y_U and Y_L be two random variables (unique in distribution) defined on $(\Omega, \Im, P)$, with respectively the stochastically dominating distribution F_U and the stochastically dominated distribution F_L. We call them *stochastic bounds of all admissible portfolios.* In particular we call Y_U *preferential market growth* since it represents the maximum factor of growth of future wealth. In some sense, Y_U is the maximum price that we can give at the future risky wealth for a unity of wealth invested today. By definition Y_U FSD $x'Z$ FSD Y_L for every portfolio weight x belonging to the bounded set of admissible choices T. Clearly, the same considerations can be easily extended to a given category of investors. As a matter of fact, when limited short selling opportunities are allowed, we can consider the non-dominated choices in the sense of a given $\alpha-$stochastic order (see Fishburn [9]) belonging to $\overline{T}^{\alpha}$, closure of the set

$$T^{\alpha} = \left\{ x_{(\alpha)}(t) \in T \Big/ t \in R,\, x_{(\alpha)}(t) \in \arg\left(\inf_{x\in T} E\left((t - x'Z)_{+}^{\alpha-1} \right) \right) \right\}.$$

Note that $\overline{T}^{\alpha}$ is a compact subspace of T. Thus, we can easily prove the following theorem.

Theorem 2

Assume limited short sales are allowed in the market. Then

$$F_{U,(\alpha)}(\lambda) = \inf_{y\in\overline{T}^{\alpha}} \mathbb{P}(y'Z \leq \lambda)$$

is the "smallest" distribution (in FSD sense) which FSD dominates all portfolios in $\overline{T}^{\alpha}$. Similarly,

$$F_{L,(\alpha)}(\lambda) = Q_{(\alpha)}(\lambda^{+}), \text{ where } Q_{(\alpha)}(\lambda) = \sup_{x\in\overline{T}^{\alpha}} \mathbb{P}(x'Z \leq \lambda),$$

is the "greatest" cumulative distribution (in FSD sense) which is FSD dominated by all portfolios in $\overline{T}^{\alpha}$.

As for the stochastic bounds of all admissible portfolios, we can define on the probability space $(\Omega, \Im, P)$ two random variables $Y_{U,(\alpha)}$ and $Y_{L,(\alpha)}$ (unique in distribution) with respectively the stochastically dominating distribution $F_{U,(\alpha)}$ and the stochastically dominated distribution $F_{L,(\alpha)}$.

3 Market trend analysis with elliptically distributed returns

In this and in the following sections we will consider portfolio choice among n risky assets with returns $Z = [Z_1, ..., Z_n]'$ when no short sales are allowed, i.e. portfolio weights

$$x \in S = \left\{ y \in R^n : y_i \geq 0; \sum_{i=1}^{n} y_i = 1 \right\}.$$

Even if this hypothesis can be relaxed allowing limited short sales. We also assume that it does not exist a portfolio $x'Z$ in the market that FSD dominates (or is FSD dominated by) all the others, and we examine stochastic bounds for all admissible portfolios. First of all we discuss the uniqueness, for every λ belonging to $[a, b]$, of the portfolio weight

$$x_U(\lambda) \in \arg \left(\inf_{x \in S} \mathbb{P}(x'Z \leq \lambda) \right)$$

when limited short sales are allowed. Actually, the uniqueness is not always satisfied in presence of riskless assets. As a matter of fact, we can easily find a counter example. Let us consider a scalar and translation invariant family of unbounded distribution functions $\Xi = \{F_X\}$ (i.e., such that if $F_X \in \Xi$ then also $F_{\alpha X}$ and F_{X+t} belong to Ξ for any real t and any $\alpha > 0$) that is uniquely determined by the first k moments, then the following statement holds.

Corollary 1
Let the portfolios of risky gross returns $x'Z$ belong to a scalar and translation invariant family of unbounded distribution functions absolutely continuous and uniquely determined by the the mean μ and the first k central moments σ^2, $\mu_3, ..., \mu_k$. Suppose a riskless portfolio is allowed with gross return z_0 and suppose there exists a risky portfolio with mean greater than z_0. Then $x_U(\lambda)$ with $\lambda = z_0$ is not unique.

The above Corollary can be further extended to more general scalar and translation invariant families of distribution functions (see Ortobelli [14]). On the other hand, we can guarantee the uniqueness when the risk-free asset is not allowed and the risky returns are elliptically distributed (see Ortobelli and Rachev [15]). For this reason in all the following considerations we assume that risk-free assets are not allowed.

Since the stochastic bounds are the market limits, then the market trend is implicitly described by them and it has sense to study the evolution of investor's choices in relation with the market trend. Assume now that $x_U(\lambda) =$

$\arg\left(\inf_{x\in S}\mathbb{P}(x'Z \le \lambda)\right)$ is a measurable vectorial function. Then, by the definition of the stochastically dominating distribution F_U it follows that

$$F_{Y_U}(\lambda) = F_U(\lambda) = \mathbb{P}(x'_U(\lambda)Z \le \lambda)\ \forall\lambda \in \Re.$$

Next, we say that it is satisfied the *based target* axiom when we assume non satiable investors minimize the probability that their returns are lower than a given value of the preferential market growth. The based-target axiom can be justified even in terms of the classic von Neumann–Morgenstern approach. As a matter of fact, Castagnoli and LiCalzi [4] and Bordley and LiCalzi [3] have shown that the based-target approach is a generalization of the von Neumann–Morgenstern (vNM) one and even the vNM investors minimize the probability of being under a given target. Assuming that the based-target axiom holds, optimal choices are well represented by the random vectorial surjective function X_U defined on $(\Omega, \Im, P)$ by

$$X_U(\omega) = x_U(Y_U(\omega)) = \arg\left(\inf_{x\in S}\mathbb{P}(x'Z \le Y_U(\omega))\right) \text{ for every } \omega \in \Omega.$$

We call the random vectorial function X_U *dominating portfolio of all admissible choices*. The dominating portfolio is a random indicator of the investors' optimal choices, since it represents all the "safety first" choices (see Roy [19], Tesler [21]). Moreover, we can distinguish different market indicators that can be interpreted by an economic point of view:

***a*)** The vector of the expected values of the dominating portfolio

$$x_D = E(X_U) = \int_{\widetilde{\Omega}} x_U \circ Y_U dP = \int_{[a,b]} x_U(\lambda)dF_U(\lambda),$$

represents the portfolio weights where the investors' preferences collapse in average. In addition, when the set of all optimal safety first portfolios

$$U = \left\{x : x = \arg\left(\inf_{x\in S}\mathbb{P}(x'Z \le \lambda)\right), \lambda \in \Re\right\}$$

is convex, then x_D belongs to the portfolio weight efficient frontier U, since it is a convex combination of optimal portfolios.

***b*)** The average of the upper stochastic bound

$$\widetilde{\lambda} = E(Y_U) = \int_{\widetilde{\Omega}} Y_U dP = \int_{[a,b]} \lambda dF_U(\lambda),$$

is another interesting indicator of the market growth, since Y_U is the first random variable which dominates all portfolios. Then $\widetilde{\lambda}$ can be considered as the average of the maximum future price of risky wealth for a unity of wealth invested today.

Clearly, it could be important to understand how optimal allocations vary among the n risky assets under preferential market growth changes. Since we have no idea of the behavior of X as function of the upper bound, we can think to use the least square estimator (LSE) to get polynomial approximation of any component $X_{U,i}$

$$X_{U,i} = a_{U,i}^{(0)} + a_{U,i}^{(1)}(Y_U - \mathrm{E}(Y_U)) + a_{U,i}^{(2)}(Y_U - \mathrm{E}(Y_U))^2 + \\ ... + a_{U,i}^{(s)}(Y_U - \mathrm{E}(Y_U))^s + \varepsilon_i, \quad i = 1, \dots, n,$$

While this is only an approximation, when the distributions are elliptical we can also give an analytical expression for the dominating portfolio X_U.

It is well-known that asset returns are not normally distributed, since the excess kurtosis found in many empirical investigations led them to reject the normality assumption and to propose alternative distributional assumption for asset returns (see Mandelbrot [10] and Fama [8]). There exist many elliptical distribution families which present heavier tails than Gaussian distribution, for example stable sub-Gaussian distributions and t-student families (see, among others, Rachev and Mittnik [18]). This is the main reason why we suppose that the vector Z of risky returns admits a joint unbounded elliptical distribution (see Chamberlain [5] and Owen and Rabinovitch [17]). Thus every risky portfolio return $x'Z$ is an unbounded random variable with cumulative distribution:

$$\mathbb{P}(x'Z \leq \lambda) = \int_{-\infty}^{\lambda} \frac{1}{C_0 (x'Qx)^{1/2}} g\left(\frac{(t - x'\mu)^2}{x'Qx}\right) dt, \tag{1}$$

where g is a non-negative integrable function, $\mu = E(Z)$ is the mean of the vector Z, Q is the positive definite dispersion matrix of the primary asset returns, and

$$C_0 := \int_{-\infty}^{+\infty} \frac{1}{(x'Qx)^{1/2}} g\left(\frac{(t - x'\mu)^2}{x'Qx}\right) dt.$$

Under this distributional assumption, the vectorial application

$$x_U(\lambda) = \arg\left(\inf_{x \in S} \mathbb{P}(x'Z \leq \lambda)\right) = \arg\left(\inf_{x \in S} \frac{\lambda - x'\mu}{(x'Qx)^{1/2}}\right)$$

is a measurable vectorial function. Clearly, in order to understand the relation between X_U and Y_U we need to find the function $x_U(\lambda)$ because $X_U = x_U(Y_U)$. In some special cases we can easily represent the frontier U as we evince by the following proposition.

Proposition 1
Assume the primary gross returns to be unbounded random variables jointly elliptically distributed with non singular dispersion matrix and with distribution function (1). Suppose that there are risky assets traded in a frictionless economy where only limited short sales are allowed. Thus, if all the components of

portfolio weights on the Markowitz-Tobin efficient frontier are positive and the return portfolio with global maximum mean $x_*'Z$ admits also global maximum dispersion, then

$$X_U = \begin{cases} \frac{Y_U Q^{-1}e - Q^{-1}\mu}{Y_U C - B} & \text{if } Y_U \leq \frac{E(x_*'Z)B - A}{E(x_*'Z)C - B} \\ x_* & \text{if } Y_U > \frac{E(x_*'Z)B - A}{E(x_*'Z)C - B} \end{cases} \tag{2}$$

and

$$F_{Y_U}(\lambda) = \begin{cases} \int\limits_{(-\infty,\lambda)} \frac{\lambda C - B}{C_0(\lambda^2 C - 2\lambda B + A)^{1/2}} g\left(\frac{\left(t\lambda^2 C - -tB - \lambda B + A\right)^2}{(\lambda^2 C - 2\lambda B + A)}\right) dt, \\ \qquad \text{if } \lambda < \frac{E(x_*'Z)B - A}{E(x_*'Z)C - B} \\ \int\limits_{-\infty}^{\lambda} \frac{1}{C_0(x_*'Qx_*)^{1/2}} g\left(\frac{(t - x_*'\mu)^2}{x_*'Qx_*}\right) dt \quad \text{if } \lambda \geq \frac{E(x_*'Z)B - A}{E(x_*'Z)C - B}, \end{cases}$$

where $A = \mu' Q^{-1}\mu$, $B = e'Q^{-1}\mu$; $C = e'Q^{-1}e$ and $x_* = e_i$ *(i.e., the vector with* 1 *in the i-th component and 0 in the other components assuming that the i-th component corresponds to the gross return with maximum mean and maximum dispersion).*

Note that under the assumptions of Proposition 1 the choices of non-satiable investors are the same as those of non-satiable risk averse investors (see Bawa [1]), thus $Y_U = Y_{U,(2)}$ and $X_U = X_{U,(2)}$. Generally, Markowitz- Tobin efficient frontier presents switching points on the components of optimal portfolios, thus the analysis of the dependence between X_U and Y_U is more complex. Traditional wisdom says that each switching point corresponds to a kink, while Dybvig [6] has shown that there is a kink at a risky portfolio only if that portfolio is a portfolio of securities of equal expected return. Thus, if all securities have a different expected return, a kink on the frontier can occur only if a single security is held at that point.

Markowitz [11], [12] has proposed an algorithm to determine recursively the efficient frontier. The Markowitz-Tobin efficient frontier was not expressed by Markowitz in terms of the mean and of a dispersion measure as we do in the next subsection. Moreover, the following extension permits us to describe the efficient frontier using, for example, the stable sub-Gaussian dispersion measure instead of the variance (see Ortobelli et al. [16]), i.e., using a more realistic approximation of return distributions.

3.1 A general analytical formulation for $X_U(Y_U)$ and F_U

Let us assume that the vector of returns z is ordered in the sense of the mean, i.e. $E(Z_1) < E(Z_2) < ... < E(Z_n)$. Therefore, we assume that the securities have different expected returns in order to limit the presence of kinks on the efficient frontier (see Markowitz [11], [12], Dybvig [6], [7], Vörös [22], [23]). Markowitz has shown that when:

- there are not redundant assets and variance covariance matrix Q is definite positive;

- there is one portfolio for each mean-variance combination;
then the mean-variance efficient frontier obtained solving the optimization problem

$$\begin{array}{c} \min_x x'Qx \\ subject \quad to \\ x_i \geq 0; \;\; x'e = 1, \quad x'\mu = m, \end{array} \tag{3}$$

for different values of the mean m, is described by some parabolic arcs where the variance is an increasing function of the mean. The same analysis holds in a mean-dispersion plane when returns are elliptically distributed and matrix Q is a definite positive dispersion matrix (see Ortobelli and Rachev [15]).

In particular, let us denote with $i_k \in I_k = \{i \in I = \{1, 2, ..., n\} \mid x_i = 0\}$ the null components of the optimal portfolio x for the k-th parabolic arc. Then the optimal solutions of k-th parabolic arc are univocally determined by the optimization problem

$$\begin{array}{c} \min_x \frac{1}{2} \left(x^{(k)}\right)' Q^{(k)} x^{(k)} \\ subject \quad to \\ \left(x^{(k)}\right)' e^{(k)} = 1, \quad \left(x^{(k)}\right)' \mu^{(k)} = m, \end{array} \tag{4}$$

for some opportune values of the mean m, where $Q^{(k)}$ is the matrix obtained by Q eliminating every i_k-th row and every i_k-th column $\forall i_k \in I_k$, while $e^{(k)}$, $x^{(k)}$ and $\mu^{(k)}$ are the vectors obtained eliminating every i_k-th element by $e = [1, ..., 1]'$, $x = [x_1, ..., x_n]'$ and $\mu = E(Z)$. Problem (4) can be analytically described using the Merton's derivation [13], thus the portfolio weights of the k-th arc are given, for $m \in (m_k, m_{k+1}]$, by

$$\begin{aligned} x^{(k)} = & \frac{\left(C^{(k)} \left(Q^{(k)}\right)^{-1} \mu^{(k)} - B^{(k)} \left(Q^{(k)}\right)^{-1} e^{(k)}\right) m}{A^{(k)} C^{(k)} - \left(B^{(k)}\right)^2} + \\ & + \frac{A^{(k)} \left(Q^{(k)}\right)^{-1} e^{(k)} - B^{(k)} \left(Q^{(k)}\right)^{-1} \mu^{(k)}}{A^{(k)} C^{(k)} - \left(B^{(k)}\right)^2}, \end{aligned} \tag{5}$$

where

$$\begin{array}{c} A^{(k)} = \left(\mu^{(k)}\right)' \left(Q^{(k)}\right)^{-1} \mu^{(k)}; \; B^{(k)} = \left(e^{(k)}\right)' \left(Q^{(k)}\right)^{-1} \mu^{(k)} \\ \text{and} \;\; C^{(k)} = \left(e^{(k)}\right)' \left(Q^{(k)}\right)^{-1} e^{(k)}, \end{array}$$

while m_k is the mean in a switching point i.e., it is the mean of a portfolio where the components are changing (see the Appendix on how to get the means m_k and the sets I_k). Such $x^{(k)}$ represents the portfolio weights of positive portfolio components, and we denote with $\widetilde{x}^{(k)}$ the previous portfolio valued in $m = m_k$. Since we assume that all securities have different mean, then portfolio $\widetilde{x}^{(k)}$ represents a kink in the mean-dispersion frontier only if a single security is held at that point. Here k varies from 1 to $\overline{k}$, the value m_1 is the mean of the global minimum dispersion portfolio $x^{\min}$, and $m_{\overline{k}} = E(Z_n)$ is mean of the return with the greatest mean. Moreover, if the asset return with the greatest mean has also the greatest dispersion, then the efficient frontier of non satiable investors is equal to the frontier of non-satiable risk averse investors (see Bawa [1]). Otherwise, we have to consider further optimal portfolios in order to obtain the optimal choices of all non-satiable investors. For this last case, recall that Bawa [2] has shown that the optimal portfolios for non-satiable investors is equivalent to

$$U = \left\{ x : x = \arg\left(\inf_{x \in S} \mathbb{P}\left(x'Z \leq \lambda\right) \right), \lambda \in \Re \right\}$$

when returns' distributions are elliptical, and these portfolios are given by:
a) the Markowitz-Tobin efficient frontier;
b) some parabolic arcs of maximum dispersion portfolios obtained as the linear combination of couple of risky returns with dispersion greater than σ_n (dispersion associated to the maximum mean return).

Thus, in order to describe the frontier U we can use Bawa's recursive algorithm. In particular, in each additional arc of the non-satiable efficient frontier we obtain that the components of portfolio are all null except for two assets (see Bawa [1]). At the extremes of these arcs we have portfolio composed by an unique security that creates a kink. If we distinguish with $i_p \in J_p = \{i \in I = \{1, 2, ..., n\} \mid x_i = 0\}$ the null components of the optimal portfolio x for the p-th parabolic arc, then the optimal solutions of the p-th additional parabolic arc are $x_i = 0\ \forall i_p \in J_p$ and, for $i_p \notin J_p$,

$$x^{(p)} = \frac{\left(C^{(p)} \left(Q^{(p)}\right)^{-1} \mu^{(p)} - B^{(p)} \left(Q^{(p)}\right)^{-1} e^{(p)} \right) m}{A^{(p)} C^{(p)} - \left(B^{(p)}\right)^2} + $$
$$+ \frac{A^{(p)} \left(Q^{(p)}\right)^{-1} e^{(p)} - B^{(p)} \left(Q^{(p)}\right)^{-1} \mu^{(p)}}{A^{(p)} C^{(p)} - \left(B^{(p)}\right)^2},$$

if $m \in (m_{p+1}, m_p]$, where $Q^{(p)}$ is the $2{\times}2$ matrix obtained by Q eliminating every i_p-th row and every i_p-th column, $\forall i_p \in J_p$; $e^{(p)}$, $x^{(p)}$ and $\mu^{(p)}$ are the vectors obtained eliminating every i_p-th element from $e = [1, ..., 1]'$, $x = [x_1, ..., x_n]'$ and $\mu = E(Z)$, respectively, while $A^{(p)} = \left(\mu^{(p)}\right)' \left(Q^{(p)}\right)^{-1} \mu^{(p)}$, $B^{(p)} = \left(e^{(p)}\right)' \left(Q^{(p)}\right)^{-1} \mu^{(p)}$ and $C^{(p)} = \left(e^{(p)}\right)' \left(Q^{(p)}\right)^{-1} e^{(p)}$. We denote with $\widetilde{x}_{m_p}$ the portfolio composed by an unique security with mean m_p. The procedure concludes with the asset return having the greatest dispersion (that we assume is represented by portfolio x_*). Note that in order to determine m_{p+1} we can apply Bawa's algorithm. This can be summarized in the following theorem, where the dominating portfolio X_U is expressed as function of Y_U. In the statement the means m_k and the sets I_k and $I_{\min}$ (set of the null components of the global minimum dispersion portfolio $x^{\min} = [x_1^{\min}, ..., x_n^{\min}]$) are assumed to be known, but a procedure to get them is described in the proof of this theorem given in the Appendix.

Theorem 3

Assume that the primary gross returns are unbounded random variables jointly elliptically distributed with non singular dispersion matrix. Suppose that there are risky assets with different expected returns $E(Z_1) < E(Z_2) < ... < E(Z_n)$ *traded in a frictionless economy where no short sales are allowed. Thus, we have the following analytical expression for the distribution function of* F_U*:*

$$F_U(\lambda) = \begin{cases} \int\limits_{-\infty}^{\lambda} \frac{1}{C_0((x^{\min})'Qx^{\min})^{1/2}}\ g\left(\frac{(t-(x^{\min})'\mu)^2}{(x^{\min})'Qx^{\min}}\right)dt, \\ \qquad \text{if } I_{\min} \neq I_1 \text{ and } \lambda \leq \frac{m_1 B^{(1)}-A^{(1)}}{m_1 C^{(1)}-B^{(1)}}; \\ \int\limits_{-\infty}^{\lambda} \frac{\lambda C^{(1)}-B^{(1)}}{C_0\left(\lambda^2 C^{(1)}-2\lambda B^{(1)}+A^{(1)}\right)^{1/2}}\ g\left(\frac{\left(t\lambda^2 C^{(1)}-tB^{(1)}-\lambda B^{(1)}+A^{(1)}\right)^2}{\left(\lambda^2 C^{(1)}-2\lambda B^{(1)}+A^{(1)}\right)}\right)dt, \\ \qquad \text{if } I_{\min} = I_1 \text{ and } \lambda \leq \frac{m_2 B^{(1)}-A^{(1)}}{m_2 C^{(1)}-B^{(1)}}; \\ \int\limits_{-\infty}^{\lambda} \frac{1}{C_0((\widetilde{x}^{(k)})'Q\widetilde{x}^{(k)})^{1/2}}\ g\left(\frac{(t-(\widetilde{x}^{(k)})'\mu)^2}{(\widetilde{x}^{(k)})'Q\widetilde{x}^{(k)}}\right)dt, \\ \qquad \text{if } \frac{m_k B^{(k-1)}-A^{(k-1)}}{m_k C^{(k-1)}-B^{(k-1)}} < \lambda \leq \frac{m_k B^{(k)}-A^{(k)}}{m_k C^{(k)}-B^{(k)}}; \\ \int\limits_{-\infty}^{\lambda} \frac{\lambda C^{(k)}-B^{(k)}}{C_0\left(\lambda^2 C^{(k)}-2\lambda B^{(k)}+A^{(k)}\right)^{1/2}}\ g\left(\frac{\left(t\lambda^2 C^{(k)}-tB^{(k)}-\lambda B^{(k)}+A^{(k)}\right)^2}{\left(\lambda^2 C^{(k)}-2\lambda B^{(k)}+A^{(k)}\right)}\right)dt, \\ \qquad \text{if } \frac{m_k B^{(k)}-A^{(k)}}{m_k C^{(k)}-B^{(k)}} < \lambda \leq \frac{m_{k+1} B^{(k)}-A^{(k)}}{m_{k+1} C^{(k)}-B^{(k)}}; \\ \int\limits_{-\infty}^{\lambda} \frac{1}{C_0(x_*'Qx_*)^{1/2}}\ g\left(\frac{(t-x_*'\mu)^2}{x_*'Qx_*}\right)dt, \qquad \text{if } \lambda \geq \frac{m^* B^{(p^*)}-A^{(p^*)}}{m^* C^{(p^*)}-B^{(p^*)}}. \end{cases}$$

Moreover, we can obtain the following analytical formulations for the vector X_U *as function of* Y_U*.*

◇ *If* $I_{\min} \neq I_1$ *then*

- $X_{U,i}(Y_U) = 0, \forall i \in I_{\min}$ *and* $X_{U,i}(Y_U) = x_i^{\min}\ \forall i \notin I_{\min}$, *for* $Y_U \leq \frac{m_1 B^{(1)}-A^{(1)}}{m_1 C^{(1)}-B^{(1)}}$;
- $X_{U,i}(Y_U) = 0, \forall i \in I_1$ *and* $X_{U,i}(Y_U) = \frac{Y_U\left(Q^{(1)}\right)^{-1}_{(i)}e^{(1)}-\left(Q^{(1)}\right)^{-1}_{(i)}\mu^{(1)}}{Y_U C^{(1)}-B^{(1)}}\ \forall i \notin I_1$, *for* $\frac{m_1 B^{(1)}-A^{(1)}}{m_k C^{(1)}-B^{(1)}} < Y_U \leq \frac{m_2 B^{(1)}-A^{(1)}}{m_2 C^{(1)}-B^{(1)}}$,

◇ *If* $I_{\min} = I_1$ *then*

- $X_{U,i}(Y_U) = 0, \forall i \in I_1$ *and* $X_{U,i}(Y_U) = \frac{Y_U\left(Q^{(1)}\right)^{-1}_{(i)}e^{(1)}-\left(Q^{(1)}\right)^{-1}_{(i)}\mu^{(1)}}{Y_U C^{(1)}-B^{(1)}}\ \forall i \notin I_1$, *for* $Y_U \leq \frac{m_2 B^{(1)}-A^{(1)}}{m_2 C^{(1)}-B^{(1)}}$.

Then for $k = 2, ..., \overline{k}-1$ $(I_{\min} = I_1$ *or* $I_{\min} \neq I_1)$

- $X_{U,i}(Y_U) = 0, \forall i \in I_k$ *and* $X_{U,i}(Y_U) = \widetilde{x}_i^{(k)}, \forall i \in I - I_k$, *for* $\frac{m_k B^{(k-1)}-A^{(k-1)}}{m_k C^{(k-1)}-B^{(k-1)}} < Y_U \leq \frac{m_{k+1} B^{(k)}-A^{(k)}}{m_{k+1} C^{(k)}-B^{(k)}}$;
- $X_{U,i}(Y_U) = 0, \forall i \in I_k$ *and* $X_{U,i}(Y_U) = \frac{Y_U\left(Q^{(k)}\right)^{-1}_{(i)}e^{(k)}-\left(Q^{(k)}\right)^{-1}_{(i)}\mu^{(k)}}{Y_U C^{(k)}-B^{(k)}}, \forall i \in I - I_k$, *for* $\frac{m_k B^{(k)}-A^{(k)}}{m_k C^{(k)}-B^{(k)}} < Y_U \leq \frac{m_{k+1} B^{(k)}-A^{(k)}}{m_{k+1} C^{(k)}-B^{(k)}}$.

◇ *When the portfolio with the greatest mean is also the portfolio with the greatest dispersion, then the last step is given by*:

- $X_U(Y_U) = [0, 0,..,0, 1]'$, *for* $Y_U > \frac{E(Z_n)B^{(\overline{k})}-A^{(\overline{k})}}{E(Z_n)C^{(\overline{k})}-B^{(\overline{k})}}$

Otherwise, we have maximum dispersion arcs and:

- $X_{U,j}(Y_U) = 0, \forall j \in J_p$ *and* $X_{U,j}(Y_U) = \frac{Y_U\left(Q^{(p)}\right)^{-1}_{(j)}e^{(p)}-\left(Q^{(p)}\right)^{-1}_{(j)}\mu^{(p)}}{Y_U C^{(p)}-B^{(p)}}\ \forall j \notin J_p$, *for* $\frac{m_p B^{(p)}-A^{(p)}}{m_p C^{(p)}-B^{(p)}} < Y_U \leq \frac{m_{p+1} B^{(p)}-A^{(p)}}{m_{p+1} C^{(p)}-B^{(p)}}$;

- $X_U(Y_U) = \widetilde{x}_{m_p}$ *for* $\frac{m_p B^{(p-1)}-A^{(p-1)}}{m_p C^{(p-1)}-B^{(p-1)}} < Y_U \leq \frac{m_p B^{(p)}-A^{(p)}}{m_p C^{(p)}-B^{(p)}}$

and the last step is given by:

- $X_{U,j}(Y_U) = 0$, $\forall j \neq j^*$ *and* $X_{U,j^*}(Y_U) = 1$, *for* $Y_U > \frac{m^* B^{(\overline{t})}-A^{(\overline{t})}}{m^* C^{(\overline{t})}-B^{(\overline{t})}}$,

where m^* *is the mean of the primary gross return with the greatest dispersion.*

Note that with Theorem 3 we implicitly obtain an analytical formulation of the vector $X_{U,(2)}$ as a function of $Y_{U,(2)}$, since $Y_{U,(2)} = Y_U$ for $Y_U < \frac{E(Z_n)B^{(\overline{k})}-A^{(\overline{k})}}{E(Z_n)C^{(\overline{k})}-B^{(\overline{k})}}$. Thus,

$$X_{U,(2)}(Y_{U,(2)}) = X_U(Y_U) \quad \text{for} \quad Y_{U,(2)} < \frac{E(Z_n)B^{(\overline{k})} - A^{(\overline{k})}}{E(Z_n)C^{(\overline{k})} - B^{(\overline{k})}}$$

and

$$X_{U,(2)}(Y_{U,(2)}) = [0, 0, .., 0, 1]' \quad \text{if} \quad Y_{U,(2)} \geq \frac{E(Z_n)B^{(\overline{k})} - A^{(\overline{k})}}{E(Z_n)C^{(\overline{k})} - B^{(\overline{k})}}.$$

Example. Suppose we have three risky assets which are Gaussian distributed with vector of means μ and variance covariance matrix Q given by

$$\mu = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 1.1 & 2 \\ 0 & 2 & 4.1 \end{pmatrix}.$$

This example does not satisfy the condition of Propostion 1, since the components of portfolio weights on the Markowitz-Tobin efficient frontier are not all positive. As observed by Dybvig [6], under these assumptions the mean variance frontier with no short sales consists of two arcs: the first arc given by convex combinations of assets 1 and 2 and the second one by convex combinations of assets 2 and 3 (being $m_2 = 3$ the frontier between these two arcs). Solving the optimization problem (3) one can get the portfolio with global minimum variance, which is $x^{\min} = [0.9167, 0.0833, 0]$, having mean $m_1 = 1.1667$ (and variance equal to 0.091667). Since for this case it holds $I_0 = I_1 = \{3\}$, $I_2 = \{1\}$, and since $A^{(1)} = 18.1819$, $B^{(1)} = 12.7273$, $C^{(1)} = 10.9091$,and $A^{(2)} = 12.7451$, $B^{(2)} = 5.2941$, $C^{(2)} = 2.3529$,(as one can easily verify), we get

$$\frac{m_2 B^{(1)} - A^{(1)}}{m_2 C^{(1)} - B^{(1)}} = 1, \ \frac{m_2 B^{(2)} - A^{(2)}}{m_2 C^{(2)} - B^{(2)}} = 1.7778 \text{ and } \frac{m_3 B^{(2)} - A^{(2)}}{m_3 C^{(2)} - B^{(2)}} = 2.0476.$$

Thus we can subdivide the support of Y_U in the sets $S_1 = (-\infty, 1]$, $\widetilde{S}_1 = (1, 1.7778]$, $S_2 = (1.7778, 2.0476]$ and $\widetilde{S}_2 = (2.0476, +\infty)$, obtaining the following expression for $X_U(Y_U)$:

$$X_U(Y_U) = \begin{cases} \begin{pmatrix} \frac{10Y_U-10}{10{,}9091Y_U-12{,}7273} \\ \frac{0{,}9091Y_U-2{,}7273}{10{,}9091Y_U-12{,}7273} \\ 0 \end{pmatrix} & \text{if } Y_U \in S_1, \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \text{if } Y_U \in \widetilde{S}_1, \\ \begin{pmatrix} 0 \\ \frac{4{,}1176Y_U-8.4314}{2.3529Y_U-5.2941} \\ \frac{-1{,}7647Y_U+3.1373}{2.3529Y_U-5.2941} \end{pmatrix} & \text{if } Y_U \in S_2, \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \text{if } Y_U \in \widetilde{S}_2. \end{cases}$$

■

Once we determined the mean m_k for k=1,..., $\overline{k}$, we have determined a closed form solution for the efficient frontier, and the algorithm described in the Appendix can be furthermore extended to the most general case analyzed by Markowitz. In addition, we can easily determine the portfolio on the risky efficient frontier that maximizes the Sharpe ratio $\frac{E(x'Z)-z_0}{\sqrt{x'Qx}}$ when no short sales are allowed (see Sharpe [20] and the references therein) and there are risky assets with return mean greater than z_0. As a matter of fact, when unlimited short sales are allowed then the portfolio weights that maximize the Sharpe ratio are given by $\widetilde{x} = \frac{Q^{-1}(\mu-z_0e)}{B-z_0C}$ with mean $m = \frac{A-z_0B}{B-z_0C}$ and dispersion $\sigma = \frac{\sqrt{A-2Bz_0+Cz_0^2}}{B-Cz_0}$. Thus the portfolio that maximizes the Sharpe ratio when no short sales are allowed should be either a tangent portfolio or a portfolio corresponding to a kink. For it, recall that the "tangent" portfolio obtained with the assets belonging to $I - I_k$ is given by

$$v_{(k)} = \frac{\left(Q^{(k)}\right)^{-1}\left(\mu^{(k)} - z_0 e^{(k)}\right)}{B^{(k)} - z_0 C^{(k)}} \quad (k = 1, ..., \overline{k}),$$

and it presents the Sharpe ratio

$$\frac{E(v'_{(k)}Z) - z_0}{\sqrt{v'_{(k)}Qv_{(k)}}} = \sqrt{A^{(k)} - 2B^{(k)}z_0 + C^{(k)}z_0^2}.$$

This portfolio is on the k-th arc if $E(v'_{(k)}Z) = \frac{A^{(k)}-z_0B^{(k)}}{B^{(k)}-z_0C^{(k)}} \in (m_k, m_{k+1})$, otherwise $v_{(k)}$ presents negative components. Let SR_k denote the maximum Sharpe ratio on the k-th arc, then

$$SR_k = \begin{cases} \sqrt{A^{(k)} - 2B^{(k)}z_0 + C^{(k)}z_0^2} & \text{if } \frac{A^{(k)}-z_0B^{(k)}}{B^{(k)}-z_0C^{(k)}} \in (m_k, m_{k+1}) \\ \max\left(\frac{m_k-z_0}{\sigma_k}, \frac{m_{k+1}-z_0}{\sigma_{k+1}}\right) & \text{otherwise} \end{cases},$$

where σ_k is dispersion of the optimal portfolio with mean m_k. Then the maximum Sharpe ratio is given by $\max_{1<k\leq\overline{k}}(SR_k)$ and the optimal portfolio weights that maximize the Sharpe ratio should be the corresponding arguments.

4 Appendix

Proof of Theorem 1. In order to prove that F_U and F_L are not decreasing, let us consider $\lambda_1 \leq \lambda_2$. Thus

$$F_U(\lambda_1) = \mathbb{P}(x'_U(\lambda_1)Z \leq \lambda_1) \leq \mathbb{P}(x'_U(\lambda_2)Z \leq \lambda_1) \leq \mathbb{P}(x'_U(\lambda_2)Z \leq \lambda_2) = F_U(\lambda_2),$$

where the first inequality is a consequence of the definition of $x_U(\lambda_1)$, while the second inequality follows from the properties of every distribution function (which is a non decreasing function). We see that F_L is right continuous by definition, and that F_L is not decreasing if and only if $Q(\lambda)$ is not decreasing. Since,

$$Q(\lambda_1) = \mathbb{P}(x'_L(\lambda_1)Z \leq \lambda_1) \leq \mathbb{P}(x'_L(\lambda_1)Z \leq \lambda_2) \leq \mathbb{P}(x'_L(\lambda_2)Z \leq \lambda_2) = Q(\lambda_2),$$

we get that F_L is a not decreasing function. In addiction, F_U is right continuous since if $\lambda_n \searrow \lambda_0$, we have that

$$\begin{aligned} F_U(\lambda_0) &\leq \lim_{n\to+\infty} F_U(\lambda_n) = \lim_{n\to+\infty} \mathbb{P}(x'_U(\lambda_n)Z \leq \lambda_n) \leq \\ &\leq \lim_{n\to+\infty} \mathbb{P}(x'_U(\lambda_0)Z \leq \lambda_n) = F_U(\lambda_0), \end{aligned}$$

where the first inequality is a consequence of the fact that F_U is a non decreasing function, the second inequality is due to the definition of $x_U(\lambda_n)$, while the final equality follows from the properties of every distribution function (which is a right continuous function). Hence, $\lim_{n\to+\infty} F_U(\lambda_n) = F_U(\lambda_0)$.

From definition of a (or $\widetilde{a}$), for every $\lambda < a$, we have $F_U(\lambda) = 0$, and for every $\lambda > a$ (or $\widetilde{a}$), it is $F_U(\lambda) > 0$ (or $F_L(\lambda) > 0$). Moreover, by right continuous property it is also $F_U(a) \geq 0$ (or $F_L(\widetilde{a}) \geq 0$). Similarly, from definition of b (or $\widetilde{b}$), for every $\lambda < b$ (or $\widetilde{b}$) we have $F_U(\lambda) < 1$ (or $F_L(\lambda) < 1$), and, from definition of $\widetilde{b}$ for every $\lambda > \widetilde{b}$, $F_L(\lambda) = 1$. Considering the random portfolios $x'(b)Z$, $y'(\widetilde{b})Z$, $x'_U(a)Z$ and $x'_L(\widetilde{a})Z$, then

$$\lim_{\lambda\to+\infty} \mathbb{P}(x'_L(\widetilde{b})Z \leq \lambda) = \lim_{\lambda\to+\infty} \mathbb{P}(x'_U(b)Z \leq \lambda) = 1$$

and

$$0 = \lim_{\lambda\to-\infty} \mathbb{P}(x'_L(\widetilde{a})Z \leq \lambda) = \lim_{\lambda\to-\infty} \mathbb{P}(x'_U(a)Z \leq \lambda).$$

Thus, F_U and F_L are distribution functions. Finally, for every portfolio weight vector x we get

$$F_U(\lambda) \leq \mathbb{P}(x'Z \leq \lambda) \leq F_L(\lambda)$$

for every λ and the inequality is strict for some λ. If we consider portfolio weights that includes a weight for the riskless gross return z_0, we have $\mathbb{P}(z_0 = z_0) = 1$ and for every $\lambda < z_0$, $\mathbb{P}(z_0 \leq \lambda) = 0$. Then in this case $F_U(\lambda) = 0$, which implies that $a \geq z_0$. Similarly, for every $\lambda > z_0$, $\mathbb{P}(z_0 \leq \lambda) = 1$ and $F_L(\lambda) = 1$, which implies that $\widetilde{b} \leq z_0$. ■

Proof of Theorem 2. Similar to the proof of Theorem 1. ■

Proof of Corollary 1. Let us call $f(t, \mu(x'Z), \sigma^2(x'Z), \mu_3(x'Z), ..., \mu_k(x'Z))$ the density of the portfolio $x'Z$ valued in the point t. Then

$$\mathbb{P}((1-x'e)z_0 + x'Z \leq \lambda) =$$

$$= \int_{-\infty}^{\lambda} f(t, (1-x'e)z_0 + x'\mathrm{E}(Z), \sigma^2(x'Z), \mu_3(x'Z), ..., \mu_k(x'Z))dt$$

$$= \int_{-\infty}^{\frac{\lambda-(1-x'e)z_0-\mu(x'Z)}{\sigma(x'Z)}} f(u, 0, 1, \tfrac{\mu_3(x'Z)}{\sigma^3(x'Z)}, ..., \tfrac{\mu_k(x'Z)}{\sigma^k(x'Z)})du$$

since the parameters $\frac{\mu_i(x'Z)}{\sigma^i(x'Z)}$ are scalar and translation invariant. Thus, when there exists a portfolio with mean $\mu(x'Z) = (1-x'e)z_0 + x'E(Z)$ greater than λ, then all the portfolios that minimize $\mathbb{P}((1-x'e)z_0 + x'Z \leq \lambda)$ also maximize $\frac{(1-x'e)z_0+x'E(Z)-\lambda}{\sqrt{x'Qx}}$ for some given ratios $\frac{\mu_i(x'Z)}{\sigma^i(x'Z)}$. Let us consider now a portfolio weights $\widetilde{x}$, with components in the risky assets $\widetilde{x}_i \geq 0; \sum_{i=1}^n \widetilde{x}_i \leq 1$ that minimizes the probability $\mathbb{P}((1-x'e)z_0 + x'Z \leq z_0)$, and assume it has mean $\widetilde{\mu}$, and the other central moments $\widetilde{\sigma}^2, \widetilde{\mu}_3, ..., \widetilde{\mu}_k$. This portfolio maximizes $\frac{(1-x'e)z_0+x'E(Z)-\lambda}{\sqrt{x'Qx}}$ with $\lambda = z_0$ for the fixed ratios $\frac{\widetilde{\mu}_3(x'Z)}{\widetilde{\sigma}^2(x'Z)}$. However, all the portfolios $(1-\alpha\widetilde{x}'e)z_0 + \alpha\widetilde{x}'Z$, (with $\alpha \in (0, \frac{1}{\sum_{i=1}^n \widetilde{x}_i})$) have portfolio weights belonging to

$$S = \left\{(x, 1-x'e) \in R^{n+1} : x \in R^n; 1-x'e \geq 0, x_i \geq 0\right\}$$

have the same ratios $\frac{\widetilde{\mu}_3(x'Z)}{\widetilde{\sigma}^2(x'Z)}$ and maximize the Sharpe ratio $\frac{x'(\mathrm{E}(Z)-ez_0)}{\sqrt{x'Qx}}$ (see Sharpe [20]). Hence they belong to $\arg\inf \mathbb{P}((1-x'e)z_0 + x'Z \leq z_0)$. ■

Proof of Proposition 1. In any arc of the efficient frontier we can use the analytical form deducted by the case in which short sales are allowed (see Bawa [2] and Ortobelli and Rachev [15]). Since the efficient frontier does not present kinks, we know that

$$x_U(\lambda) = \begin{cases} \frac{\lambda Q^{-1}e - Q^{-1}\mu}{\lambda C - B} & \text{if } \lambda < \frac{E(x_*'Z)B-A}{E(x_*'Z)C-B} \\ x_* & \text{if } \lambda \geq \frac{E(x_*'Z)B-A}{E(x_*'Z)C-B}, \end{cases}$$

while formulation of

$$F_U(\lambda) = \int_{-\infty}^{\lambda} \frac{1}{C_0(x_U'(\lambda)Qx_U(\lambda))^{1/2}} g\left(\frac{(t - x_U'(\lambda)\mu)^2}{x_U'(\lambda)Qx_U(\lambda)}\right) dt$$

holds by the substitution of the mean and dispersion formulas

$$E(x'_U(\lambda)z) = \frac{\lambda B - A}{\lambda C - B} \quad \text{and} \quad \sigma^2(\lambda) = x'_U(\lambda)Qx_U(\lambda) = \frac{\lambda^2 C - 2\lambda B + A}{(\lambda C - B)^2}.$$

Thus, we can easily deduce the desired results. ■

Proof of Theorem 3. In order to prove Theorem 3 we first need to explain how one can compute recursively the values m_{k+1}. For it, assume that m_k is known (see below on how to compute m_1). The composition of the optimal portfolio with mean m_{k+1} could change:
1. because some strictly positive components of the optimal portfolio weights become null;
2. because some components of the optimal portfolio weights contribute to minimize the dispersion becoming strictly positive.

As observed by Markowitz [11], [12] one can prosecute recalling that the limits of each arc are determined by the constrains:

a) $x_i \geq 0, i = 1, ..., n;$

$\frac{\partial L}{\partial x_i} = 0$ if $x_i > 0$ (i.e., for $i \in I - I_k$) and $\frac{\partial L}{\partial x_i} \geq 0$ if $x_i = 0$ (i.e., for $i \in I_k$), where

$$L = \frac{1}{2}x'Qx - \lambda_1 (x'e - 1) - \lambda_2 (x'\mu - m).$$

Since the efficient portfolios could change their components that are strictly positive, we want to know what components (among those strictly positive, i.e., $i \in I - I_k$) become null when the mean is m_{k+1}. Thus from the constraints a) we have that

$$m_{k+1} \leq m^-_{k+1} = \min_{i \in I - I_k} \left\{ t_i > m_k \mid t_i = \frac{B^{(k)} \left(Q^{(k)}\right)^{-1}_{(i)} \mu^{(k)} - A^{(k)} \left(Q^{(k)}\right)^{-1}_{(i)} e^{(k)}}{C^{(k)} \left(Q^{(k)}\right)^{-1}_{(i)} \mu^{(k)} - B^{(k)} \left(Q^{(k)}\right)^{-1}_{(i)} e^{(k)}} \right\},$$

where $\left(Q^{(k)}\right)^{-1}_{(i)}$ is the i-*th* row of the matrix $\left(Q^{(k)}\right)^{-1}$ and t_i is the mean derived by equating to zero the portfolio weight x_i as expressed in the Merton's formulation (5).
In order to get those new components of the optimal portfolio weights that contribute to minimize the dispersion becoming strictly positive, we consider the constraint b). In fact:
- from the condition $\frac{\partial L}{\partial x_i} = 0$ $\forall i \in I - I_k$, we can calculate the Lagrangian coefficients

$$\lambda_1 = \frac{A^{(k)} - mB^{(k)}}{A^{(k)}C^{(k)} - \left(B^{(k)}\right)^2} \quad \text{and} \quad \lambda_2 = \frac{mC^{(k)} - B^{(k)}}{A^{(k)}C^{(k)} - \left(B^{(k)}\right)^2}$$

(this derivation follows because the expected returns are different for different assets, see Vörös [22] and Dybvig [6]);

- the conditions $\frac{\partial L}{\partial x_i} \geq 0$, $\forall i \in I_k$, can be used to determine if an additional asset has to be considered in optimal efficient frontier. As a matter of fact, we can consider the partial derivatives $\frac{\partial L}{\partial x_i}$, $\forall i \in I_k$, evaluated on an optimal portfolio with mean belonging to $[m_k, m_{k+1}]$. It holds

$$\frac{\partial L}{\partial x_i} = Q_{(i)}x - \lambda_1 - \lambda_2 E(Z_i) = \sum_{p \in I - I_k} \sigma_{pi} x_p - \frac{A^{(k)} - mB^{(k)}}{A^{(k)}C^{(k)} - \left(B^{(k)}\right)^2} - \\ - \frac{mC^{(k)} - B^{(k)}}{A^{(k)}C^{(k)} - \left(B^{(k)}\right)^2} E(Z_i),$$

where $x_p = \frac{\left(C^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}\mu^{(k)} - B^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}e^{(k)}\right)m}{A^{(k)}C^{(k)} - \left(B^{(k)}\right)^2} + \frac{A^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}e^{(k)} - B^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}\mu^{(k)}}{A^{(k)}C^{(k)} - \left(B^{(k)}\right)^2}$ is determined by the Merton's formulation (5) letting $x_i = 0$ $\forall i \in I_k$. If the i-*th* asset, $i \in I_k$, enters in the composition of the optimal portfolio when the portfolio mean is given by $m = m_{k+1}$, then it should be $\frac{\partial L}{\partial x_i} = 0$. Thus we can get the mean t_i of this optimal portfolio by equating to zero $\frac{\partial L}{\partial x_i}$, $\forall i \in I_k$, and observing that

$$m_{k+1} \leq m^+_{k+1} = \min_{i \in I_k} \Bigg\{ t_i > m_k \ / \\ t_i = \frac{A^{(k)} - B^{(k)}E(z_i) + \sum_{p \in I - I_k} \sigma_{pi}\left(B^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}\mu^{(k)} - A^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}e^{(k)}\right)}{B^{(k)} - C^{(k)}E(z_i) + \sum_{p \in I - I_k} \sigma_{pi}\left(C^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}\mu^{(k)} - B^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}e^{(k)}\right)} \Bigg\}$$

Now we can set

$$m_{k+1} = \min\left\{m^-_{k+1}, m^+_{k+1}\right\},$$

observing that if $m_{k+1} = m^-_{k+1} < m^+_{k+1}$ then $I_{k+1} = I_k \cup I^-_k$, where

$$I^-_k = \arg\left(\min_{i \in I - I_k}\left\{t_i > m_k \mid t_i = \frac{B^{(k)}\left(Q^{(k)}\right)^{-1}_{(i)}\mu^{(k)} - A^{(k)}\left(Q^{(k)}\right)^{-1}_{(i)}e^{(k)}}{C^{(k)}\left(Q^{(k)}\right)^{-1}_{(i)}\mu^{(k)} - B^{(k)}\left(Q^{(k)}\right)^{-1}_{(i)}e^{(k)}}\right\}\right),$$

while, if $m_{k+1} = m^+_{k+1} < m^-_{k+1}$, then $I_{k+1} = I_k - I^+_k$ where

$$I^+_k = \arg\Bigg(\min_{i \in I_k}\Bigg\{t_i > m_k \ / \\ t_i = \frac{A^{(k)} - B^{(k)}E(z_i) + \sum_{p \in I - I_k} \sigma_{pi}\left(B^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}\mu^{(k)} - A^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}e^{(k)}\right)}{B^{(k)} - C^{(k)}E(z_i) + \sum_{p \in I - I_k} \sigma_{pi}\left(C^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}\mu^{(k)} - B^{(k)}\left(Q^{(k)}\right)^{-1}_{(p)}e^{(k)}\right)}\Bigg\}\Bigg).$$

In the particular case that $m_{k+1} = m^+_{k+1} = m^-_{k+1}$ then we consider $\widetilde{I}_{k+1} = I_k - I^+_k$ and we define

$$\widetilde{m}^-_{k+1} = \min_{i \in I - \widetilde{I}_{k+1}}\left\{t_i \geq m_{k+1} \mid t_i = \frac{\widetilde{B^{(k)}}\left(\widetilde{Q^{(k)}}\right)^{-1}_{(i)}\widetilde{\mu^{(k)}} - \widetilde{A^{(k)}}\left(\widetilde{Q^{(k)}}\right)^{-1}_{(i)}\widetilde{e^{(k)}}}{\widetilde{C^{(k)}}\left(\widetilde{Q^{(k)}}\right)^{-1}_{(i)}\widetilde{\mu^{(k)}} - \widetilde{B^{(k)}}\left(\widetilde{Q^{(k)}}\right)^{-1}_{(i)}\widetilde{e^{(k)}}}\right\},$$

where $\widetilde{Q^{(k)}}$, $\widetilde{\mu^{(k)}}$, $\widetilde{e^{(k)}}$, are, respectively, the matrix or vectors obtained eliminating every i_k-*th* row and every i_k-*th* column, $\forall i_k \in \widetilde{I}_{k+1}$, in the matrix Q

and in the vectors μ and e, respectively, while the values $\widetilde{A^{(k)}}$, $\widetilde{B^{(k)}}$, $\widetilde{C^{(k)}}$ are defined in the usual way from the matrix $\widetilde{Q^{(k)}}$ and the vectors $\widetilde{\mu^{(k)}}$ and $\widetilde{e^{(k)}}$ (observe that the matrix $\widetilde{Q^{(k)}}$ is dimensionally greater than matrix $Q^{(k)}$, since it considers the aport of the new components in the optimal portfolios). Now, it could be that $\widetilde{m}_{k+1}^{-} > m_{k+1}$; in this case we set $I_{k+1} = \widetilde{I}_{k+1}$. Otherwise we assign $m_{k+1} = \widetilde{m}_{k+1}^{-}$ and $I_{k+1} = \widetilde{I}_{k+1} \cup \widetilde{I}_k^{-}$, where

$$\widetilde{I}_k^{-} = \arg\left(\min_{i \in I - \widetilde{I}_{k+1}} \left\{ t_i \geq m_{k+1} \mid t_i = \frac{\widetilde{B^{(k)}}\left(\widetilde{Q^{(k)}}\right)^{-1}_{(i)}\widetilde{\mu^{(k)}} - \widetilde{A^{(k)}}\left(\widetilde{Q^{(k)}}\right)^{-1}_{(i)}\widetilde{e^{(k)}}}{\widetilde{C^{(k)}}\left(\widetilde{Q^{(k)}}\right)^{-1}_{(i)}\widetilde{\mu^{(k)}} - \widetilde{B^{(k)}}\left(\widetilde{Q^{(k)}}\right)^{-1}_{(i)}\widetilde{e^{(k)}}} \right\}\right).$$

Then the set I_{k+1} is given by

$$I_{k+1} = \begin{cases} I_k \cup I_k^{-} & \text{if } m_{k+1} = m_{k+1}^{-} < m_{k+1}^{+} \\ I_k - I_k^{+} & \text{if } m_{k+1} = m_{k+1}^{+} < m_{k+1}^{-} \\ I_k - I_k^{+} & \text{if } m_{k+1} = m_{k+1}^{+} = m_{k+1}^{-} < \widetilde{m}_{k+1}^{-} \\ \left(I_k - I_k^{+}\right) \cup \widetilde{I}_k^{-} & \text{if } m_{k+1} = m_{k+1}^{+} = m_{k+1}^{-} = \widetilde{m}_{k+1}^{-}\ . \end{cases}$$

Considering that the efficient frontier of non satiable risk averse investors is bounded from below by the global minimum dispersion portfolio, and from above by risky return with the greatest mean, then we can determine the efficient frontier with the following recursive algorithm.

Step 1 First we determine the global minimum dispersion portfolio $x^{\min} = [x_1^{\min}, ..., x_n^{\min}]$ of the efficient frontier, which is the solution of the system

$$\begin{gathered} \min_x x'Qx \\ \textit{subject to} \\ x_i \geq 0;\ \ x'e = 1. \end{gathered}$$

Let $I_{\min} = \left\{ i \subseteq I = \{1, 2, ..., n\} \;\middle|\; x_i^{\min} = 0 \right\}$ denotes the set of the null components of $x^{\min}$, and let $i_{\min}$ denotes the elements of $I_{\min}$. We can easily compute:

– the dispersion matrix $Q^{(\min)}$ obtained eliminating every $i_{\min}$-th row and every $i_{\min}$-th column from Q, for all $i_{\min} \in I_{\min}$;

– the vectors $e^{(\min)}$, $x^{(\min)}$ and $\mu^{(\min)}$, obtained eliminating every $i_{\min}$-th element from $e = [1, ..., 1]'$, $x = [x_1, ..., x_n]'$ and $\mu = E(Z)$;

– the values

$$A^{(\min)} = \left(\mu^{(\min)}\right)' \left(Q^{(\min)}\right)^{-1} \mu^{(\min)},$$
$$B^{(\min)} = \left(e^{(\min)}\right)' \left(Q^{(\min)}\right)^{-1} \mu^{(\min)} \text{ and } C^{(\min)} = \left(e^{(\min)}\right)' \left(Q^{(\min)}\right)^{-1} e^{(\min)};$$

– the mean of the global minimum dispersion portfolio, which is given by $m_1 = \frac{B^{(\min)}}{C^{(\min)}}$.

Since we do not know if $x^{\min}$ is associated or not to a switching point, we have to compute the portfolio weights $\widetilde{y}^{(1)}$ solution of the problem

$$\begin{gathered} \min_x x'Qx \\ \textit{subject to} \\ x_i \geq 0;\ \ x'e = 1,\ \ x'\mu = m_1 + \varepsilon. \end{gathered}$$

for an opportune little $\varepsilon > 0$. As a consequence, we get $I_1 = \left\{ i \in I \left| \widetilde{y}_i^{(1)} = 0 \right. \right\}$. Clearly, the global minimum portfolio is a point where the optimal portfolio weights change its components only if $I_1 \neq I_{\min}$. $\diamond$

Step 2 Once we know I_1, then we should calculate $m_2 = \min\left\{m_2^-, m_2^+\right\}$ as mentioned above. The portfolio weights of the first arc that are given, for $m \in (m_1, m_2]$, by

$$x^{(1)} = \frac{\left(C^{(1)}\left(Q^{(1)}\right)^{-1}\mu^{(1)} - B^{(1)}\left(Q^{(1)}\right)^{-1}e^{(1)}\right)m}{A^{(1)}C^{(1)} - \left(B^{(1)}\right)^2} + \\ + \frac{A^{(1)}\left(Q^{(1)}\right)^{-1}e^{(1)} - B^{(1)}\left(Q^{(1)}\right)^{-1}\mu^{(1)}}{A^{(1)}C^{(1)} - \left(B^{(1)}\right)^2},$$

where $Q^{(1)}$, $\mu^{(1)}$, $e^{(1)}$, are respectively the matrix and vectors obtained eliminating every i_1-*th* row and every i_1-*th* column $\forall i_1 \in I_1$ in the matrix Q and in the respective vectors, while the values $A^{(1)}$, $B^{(1)}$, $C^{(1)}$ are obtained by the matrix $Q^{(1)}$, and vectors $\mu^{(1)}$, $e^{(1)}$. In the case $I_1 \neq I_{\min}$, then $A^{(1)}$, $B^{(1)}$, $C^{(1)}$ are generally different by $A^{(\min)}$, $B^{(\min)}$ and $C^{(\min)}$. The new set of the new null components is given by $I_2 = \begin{cases} I_1 \cup I_1^- & \text{if } m_2 = m_2^- < m_2^+ \\ I_1 - I_1^+ & \text{if } m_2 = m_2^+ < m_2^- \\ I_1 - I_1^+ & \text{if } m_2 = m_2^+ = m_2^- < \widetilde{m}_2^- \\ \left(I_1 - I_1^+\right) \cup \widetilde{I}_1^- & \text{if } m_2 = m_2^+ = m_2^- = \widetilde{m}_2^-. \end{cases}$ $\diamond$

Step 3 We can proceed in the same way in order to find the other arcs of the efficient frontier, until, after a finite number $\overline{k}$ of recursive steps, we arrive at the last portfolio of the Markowitz-Tobin efficient frontier, that corresponds to the asset with the greatest expected return Z_n (since the vector of returns Z is ordered in the sense of the mean, i.e. $E(Z_1) < E(Z_2) < ... < E(Z_n)$). This part of the efficient frontier represents the non-satiable risk averse investors' choices. If the portfolio with the greatest mean is also the portfolio with the greatest dispersion then the efficient choices of non-satiable investors are the same choices of non-satiable risk averse investors, otherwise we need to compute the arcs with maximum dispersion using Bawa's algorithm in order to get optimal choices of non-satiable investors who are not necessarily risk averse. $\diamond$

Once the means m_k and the sets I_k are known, it is possible to get the analytical formulation for $X_U(Y_U)$ and F_U. In fact, when returns are elliptically and unbounded distributed then every portfolio $x_U(\lambda) \in \arg\left(\inf_{y \in S} \mathbb{P}(y'z \leq \lambda)\right)$ is on the non-satiable efficient frontier and, for each mean-dispersion arc, it holds $x^{(k)}(\lambda) = \frac{\lambda\left(Q^{(k)}\right)^{-1}e^{(k)} - \left(Q^{(k)}\right)^{-1}\mu^{(k)}}{\lambda C^{(k)} - B^{(k)}}$ with $m_k(\lambda) = \frac{\lambda B^{(k)} - A^{(k)}}{\lambda C^{(k)} - B^{(k)}}$ (see Ortobelli and Rachev [15]). Thus, λ as function of the mean is given by $\lambda(m) = \frac{mB^{(k)} - A^{(k)}}{mC^{(k)} - B^{(k)}}$. Therefore:

$x_j(\lambda) = 0$, $\forall j \in I_{\min}$ and $x_i(\lambda) = x_i^{\min}$, $\forall i \in I - I_{\min}$, when $\lambda \leq \frac{m_1 B^{(1)} - A^{(1)}}{m_1 C^{(1)} - B^{(1)}}$ (if $I_1 \neq I_{\min}$);

$x_j(\lambda) = 0,\ \forall j \in I_1$ and $x_i(\lambda) = \frac{\lambda\left(Q^{(1)}\right)_{(i)}^{-1} e^{(1)} - \left(Q^{(1)}\right)_{(i)}^{-1} \mu^{(1)}}{\lambda C^{(1)} - B^{(1)}} \forall i \in I - I_1$

for $\frac{m_1 B^{(1)} - A^{(1)}}{m_1 C^{(1)} - B^{(1)}} < \lambda \leq \frac{m_2 B^{(1)} - A^{(1)}}{m_2 C^{(1)} - B^{(1)}}$, (or for $\lambda \leq \frac{m_2 B^{(1)} - A^{(1)}}{m_2 C^{(1)} - B^{(1)}}$, if $I_1 = I_{\min}$)

and, for every $2 \leq k < \overline{k}$,

$x_j(\lambda) = \widetilde{x}_j^{(k)}$ when $\frac{m_k B^{(k-1)} - A^{(k-1)}}{m_k C^{(k-1)} - B^{(k-1)}} < \lambda \leq \frac{m_k B^{(k)} - A^{(k)}}{m_k C^{(k)} - B^{(k)}}$;

$x_j(\lambda) = 0, \forall j \in I_k$ and $x_i(\lambda) = \frac{\lambda\left(Q^{(k)}\right)_{(i)}^{-1} e^{(k)} - \left(Q^{(k)}\right)_{(i)}^{-1} \mu^{(k)}}{\lambda C^{(k)} - B^{(k)}}\ \forall i \in I - I_k$

for $\frac{m_k B^{(k)} - A^{(k)}}{m_k C^{(k)} - B^{(k)}} < \lambda \leq \frac{m_{k+1} B^{(k)} - A^{(k)}}{m_{k+1} C^{(k)} - B^{(k)}}$.

If the portfolio with the greatest mean is also the portfolio with the greatest dispersion, then the last step is given by:

$x_j(\lambda) = 0,\ \forall j < n$ and $x_n(\lambda) = 1$ when $\lambda \geq \frac{m_{\overline{k}} B^{(\overline{k})} - A^{(\overline{k})}}{m_{\overline{k}} C^{(\overline{k})} - B^{(\overline{k})}}$, where $m_{\overline{k}} = E(Z_n)$.

Otherwise we have that in any maximum dispersion arc

$$x_j(\lambda) = 0,\ \forall j \in J_p \text{ and } x_i(\lambda) = \frac{\lambda\left(Q^{(p)}\right)_{(i)}^{-1} e^{(p)} - \left(Q^{(p)}\right)_{(i)}^{-1} \mu^{(p)}}{\lambda C^{(p)} - B^{(p)}}\ \forall i \in I - J_p$$

when $\frac{m_p B^{(p)} - A^{(p)}}{m_p C^{(p)} - B^{(p)}} < \lambda \leq \frac{m_{p+1} B^{(p)} - A^{(p)}}{m_{p+1} C^{(p)} - B^{(p)}}$, and

$$x(\lambda) = \widetilde{x}_{m_p} \ \text{ if } \ \frac{m_p B^{(p-1)} - A^{(p-1)}}{m_p C^{(p-1)} - B^{(p-1)}} < \lambda \leq \frac{m_p B^{(p)} - A^{(p)}}{m_p C^{(p)} - B^{(p)}}.$$

In this last case after p^* steps we arrive to the j^*-*th* risky return with the greatest dispersion. Thus the last step is given by

$$x_j(\lambda) = 0,\ \forall j \neq j^* \text{ and } x_{j^*}(\lambda) = 1$$

when $\lambda \geq \frac{m^* B^{(\overline{t})} - A^{(\overline{t})}}{m^* C^{(\overline{t})} - B^{(\overline{t})}}$, where $m^* = E(Z_{j^*})$. Observe that intervals $\frac{m_p B^{(p)} - A^{(p)}}{m_p C^{(p)} - B^{(p)}} < \lambda \leq \frac{m_{p+1} B^{(p)} - A^{(p)}}{m_{p+1} C^{(p)} - B^{(p)}}$ do never intersect with the others (see Bawa [1] and Ortobelli and Rachev [15]). Thus, from definition of $x_U(\lambda)$, and similarly as we did in the proof of Proposition 1, we get the expression of $X_U(Y_U)$ and of the distribution function of preferential market growth stated in the assertion. ■

References

[1] V. S. Bawa, Admissible portfolio for all individuals, *Journal of Finance,* 31, 1169-1183 (1976).

[2] V. S. Bawa, Safety-first stochastic dominance and optimal portfolio choice, *Journal of Financial and Quantitative Analysis,* 13, 255-271 (1978).

[3] R.Bordley, and M. LiCalzi, Decision analysis using targets instead of utility functions, *Decisions in Economics and Finance,* 23, 53-74 (2000).

[4] E. Castagnoli, and M. LiCalzi, Expected utility without utility, *Theory and Decision*, 41, 281-301 (1996).

[5] G. Chamberlain, A characterization of the distributions that imply mean-variance utility functions, *Journal of Economic Theory*, 29, 185-201 (1983).

[6] P. Dybvig, Short sales restrictions and kinks on the mean variance frontier, *Journal of Finance*, 39, 239-244 (1984).

[7] P. Dybvig, Acknowledgment: Kinks on the mean variance frontier, *Journal of Finance*, 40, 245 (1985).

[8] E. Fama, The behavior of stock market prices, *Journal of Business*, 38, 34-105 (1965).

[9] P.C. Fishburn, Continua of stochastic dominance relations for unbounded probability distributions, *Journal of Mathematical Economics*, 7, 271-285 (1980).

[10] B.B. Mandelbrot, New methods in statistical economics, *Journal of Political Economy*, 71, 421-440 (1963).

[11] H. M. Markowitz, *Portfolio selection; efficient diversification of investment*, Wiley & Sons, New York, 1959.

[12] H. M. Markowitz, *Mean Variance analysis in portfolio choice and capital markets*, Blackwell Ltd., Oxford 1987.

[13] R.C. Merton, An analytical derivation of the efficient portfolio frontier, *Journal of Financial and Quantitative Analysis,* 7, 1851-1872 (1972).

[14] L. S. Ortobelli, The classification of parametric choices under uncertainty: analysis of the portfolio choice problem, *Theory and Decision*, 51, 297-327 (2001).

[15] L. S. Ortobelli, and S.T. Rachev, Safety first analysis and stable paretian approach to portfolio choice theory, *Mathematical and Computer Modelling*, 34, 1037-1072 (2001).

[16] L. S. Ortobelli, A. Biglova, I. Huber; S. Stoyanov, and B. Racheva, Portfolio selection with heavy tailed distributions, *Journal of Concrete and Applicable Mathematics,* 3, 353-376 (2005).

[17] J. Owen, and R. Rabinovitch, On the class of elliptical distributions and their applications to the theory of portfolio choice, *Journal of Finance*, 38, 745-752 (1983).

[18] S.T. Rachev, and S. Mittnik, *Stable paretian models in finance,* Wiley.J. & Sons, Chichester, 2000.

[19] A.D. Roy, Safety-first and the holding of assets, *Econometrica*, 20, 431-449 (1952).

[20] W.F. Sharpe, The Sharpe ratio, *Journal of Portfolio Management*, 45-58, (1994).

[21] L.G. Tesler, Safety first and hedging, *Review of Economic Studies*, 23, 1-16 (1955/6).

[22] J. Vörös, The explicit derivation of the efficient portfolio frontier in the case of degeneracy and general singularity, *European Journal of Operational Research*, 32, 302-310 (1987).

[23] J. Vörös, Portfolio analysis: an analytical derivation of the efficient portfolio frontier, *European Journal of Operational Research*, 23, 294-300 (1986).

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POINTWISE APPROXIMATION BY THE GENERALIZATION OF PICARD AND GAUSS-WEIERSTRASS SINGULAR INTEGRALS

ALI ARAL

Abstract. In a recent generalization of Picard and Gauss - Weierstrass singular integrals, obtained as exponential functions replace by the q-exponential functions, are defined as depending on the q-parameter where $0 < q < 1$ and it is called q-Picard and q-Gauss-Weierstrass singular integrals (see [3]). This adds flexibility on the approximation properties of these operators that is, as depending on choosing the parameter q, the rate of convergence of these operators in L_p-norm and weighted L_p-norm is better than the classical case while retaining their basic properties. In this paper, as a continuation of the previous paper [3], we first study the pointwise approximation properties in the polynomial weighted space by the q-Picard and $q-$Gauss-Weierstrass singular integrals. Then, we deduce Voronovskaya type theorems related to the second q-derivatives of a function exists in any point.

1. Introduction and notations

Recently the author [3] introduced the generalization of the well known Picard and Gauss-Weierstrass singular integrals (their different generalization and basic properties are explained widely in Anastassiou and Gal's book [2]) via the q-exponential functions. Research results show that they possess good convergence and approximation properties on $L_p(\mathbb{R})$ and weighted-$L_{p,m}(\mathbb{R})$. Briefly we recall these constructions by replacing $[\lambda]_q$ with $1/[\lambda]_q$. For a real valued function f, the q-generalization of Picard and Gauss-Weierstrass singular integrals of f are introduced in [3] and defined as

$$P_\lambda(f;q,x) \equiv P_\lambda(f;x) := \frac{[\lambda]_q(1-q)}{2\ln q^{-1}} \int_{-\infty}^{\infty} \frac{f(x+t)}{E_q\left([\lambda]_q(1-q)|t|\right)} dt \tag{1.1}$$

and

$$W_\lambda(f;q,x) \equiv W_\lambda(f;x) := \frac{\sqrt{[\lambda]_q}}{\pi\left(q^{1/2};q\right)_{1/2}} \int_{-\infty}^{\infty} \frac{f(x+t)}{E_q\left([\lambda]_q t^2\right)} dt, \tag{1.2}$$

for $0 < \lambda < \infty$ and $0 < q < 1$.

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Now, we collect some well known definitions and formulas for the q-calculus used throughout the paper. For $q>0$, q-number is

$$[\lambda]_q=\begin{cases}\frac{1-q^\lambda}{1-q}, & q\neq 1\\ \lambda, & q=1\end{cases}$$

for all nonnegative λ. If λ is an integer, i.e. $\lambda=n$ for some n, we write $[n]_q$ and call it $q-$integer. Also, we define a $q-$factorial as

$$[n]_q!=\begin{cases}[n]_q\,[n-1]_q\cdots[1]_q\,, & n=1,\,2,...\\ 1 & n=0.\end{cases}.$$

For integers $0\leq k\leq n$, the q-binomial coefficients are given by

$$\left[\begin{array}{c} n\\ k\end{array}\right]_q=\frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

Furthermore, the q-extension of exponential function e^x is

$$E_q\left(x\right):=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{(q;\,q)_n}x^n=\left(-x;\,q\right)_\infty\,, \tag{1.3}$$

where $(a;\,q)_n=\prod\limits_{k=0}^{n}\left(1-aq^k\right)$ and $(-x;\,q)_\infty=\prod\limits_{k=0}^{\infty}\left(1+xq^k\right)$. More details on these can be found in the monograph [9] by G. Gasper and M. Rahman and [8] by T. Ernst.

Following two integrals will play an important role throughout the paper. For $0<q<1$, the first integral, called the q-extension of Euler integral representation for the gamma function given in [6] and [1] that we use to define the q-Picard singular integral, is

$$c_q\left(x\right)\Gamma_q\left(x\right)=\frac{1-q}{\ln q^{-1}}q^{\frac{x(x-1)}{2}}\int_0^{\infty}\frac{t^{x-1}}{E_q\left(\left(1-q\right)t\right)}dt\,,\quad \Re x>0 \tag{1.4}$$

where $\Gamma_q\left(x\right)$ is the q-gamma function defined by

$$\Gamma_q\left(x\right)=\frac{(q;\,q)_\infty}{(q^x;\,q)_\infty}\left(1-q\right)^{1-x}\,,\;\;0<q<1$$

and $c_q\left(x\right)$ satisfies the following conditions:

(1) $c_q\left(x+1\right)=c_q\left(x\right)$
(2) $c_q\left(n\right)=1,\,n=0,\,1,\,2,...$
(3) $\lim\limits_{q\rightarrow1^-}c_q\left(x\right)=1.$

When $x=n+1$ with n a non-negative integer, we obtain

$$\Gamma_q\left(n+1\right)=[n]_q!. \tag{1.5}$$

The second integral that we use to define the q-Gauss-Weierstrass singular integral and is given in [7] is

$$\int_{-\infty}^{\infty} \frac{t^{2k}}{E_q\left(t^2\right)} dt = \pi \left(q^{1/2};\, q\right)_{1/2} q^{-\frac{k^2}{2}} \left(q^{1/2};\, q\right)_k, \quad k = 0,\, 1,\, 2... \tag{1.6}$$

where

$$(a;\, q)_\alpha = \frac{(a;\, q)_\infty}{(aq^\alpha;\, q)_\infty}$$

for any real number α.

We denote by $L_{p,\, m}(\mathbb{R})$ the linear space of p-absolutely integrable functions on $\mathbb{R}$ with respect to the weight function $\omega(x) = \left(1 + x^{2m}\right)^{-1}$ with the norm

$$\|f\|_{p,\, m} := \begin{cases} \left(\int\limits_{\mathbb{R}} \left|\frac{f(t)}{1+t^{2m}}\right|^p dt\right)^{\frac{1}{p}} < \infty & \text{for } 1 \le p < \infty. \\ \sup\limits_{t\in\mathbb{R}} \frac{|f(t)|}{1+t^{2m}} & \text{for } p = \infty \end{cases}$$

Our aim to present new approximation properties of the operators defined in (1.1) and (1.2). First we deal with the pointwise estimation of the rate of convergence of these approximation processes under the condition that the approximating function satisfy the some smoothness conditions at a fixed point. We note that another generalization of Gauss-Weierstrass singular integral, called λ-Gauss Weierstrass singular integral, in which the kernel depends on nonisotropic distance defined in [4] and [5] , and some pointwise approximation results in L_p and exponential weighted-L_p spaces are investigated. The results show that the pointwise order of convergence does not change depending on nonisotropic distance. However, we show that the pointwise order of convergences in the polynomial weighted space $L_{p,\, m}(\mathbb{R})$ of our operators can be made better depending on the selection of q. Second, we prove that a Voronovskaya type theorems for twice *q-differentiable* functions in $L_{\infty,\, m}(\mathbb{R})$. As a particular case, we obtain the theorems for twice differentiable functions in $L_{\infty,\, m}(\mathbb{R})$. These theorems show that for our operators defined by (1.1) and (1.2) pointwise convergence speed is at least so faster than classical case.

2. Pointwise Approximation

Principal step in this section is the integral inequality given in the following lemma connected with μ-Hardy-Littewood maximal function.

For $g \in L[0,\, \infty)$ (integrable function on $[0,\, \infty)$) and $x \in \mathbb{R}$, we define following μ-Hardy-Littewood maximal function (see [12, pp. 62])

$$M_\mu(g)(x) := \sup_{t>0} \frac{1}{\mu(t)} \int_0^t |g(z-x)|\, dz,$$

where $\mu(t)$ is a nondecreasing function on $(0,\infty)$ such that

$$\mu(t)\rightarrow 0 \text{ as } t\rightarrow 0 \quad \text{and} \quad \mu(t)\rightarrow\infty \text{ as } t\rightarrow\infty \tag{2.1}$$

Lemma 1. *Let $g\in L[0,\ \infty)$ and $\phi\in L_{loc}$. If positive decreasing function ϕ defined on $(0,\infty)$ satisfies the condition*

$$\mu(t)\,\phi(t)\rightarrow 0 \ \textit{as}\ t\rightarrow 0 \ \textit{and}\ t\rightarrow\infty, \tag{2.2}$$

then we have

$$\int_0^\infty |g(x)|\,\phi(x)\,dx \le M_\mu(g)(0)\int_0^\infty \phi(x)\,\mu'(x)\,dx.$$

Proof. Let $U(x)=\int_0^x |g(t)|\,dt$. By the method of integration by parts, we get

$$\begin{aligned}\int_0^\infty |g(x)|\,\phi(x)\,dx &= \int_0^\infty \phi(x)\,dU(x)\\ &= \lim_{x\rightarrow\infty}\phi(x)\,U(x)+\int_0^\infty U(x)\,d(-\phi(x)).\end{aligned}$$

Definition of M_μ shows that

$$U(x)\le M_\mu(g)(0)\,\mu(x).$$

This inequality, the condition (2.2) and the fact that ϕ is a decreasing function imply that

$$\begin{aligned}\int_0^\infty |g(x)|\,\phi(x)\,dx &\le M_\mu(g)(0)\int_0^\infty \mu(x)\,d(-\phi(x))\\ &\le M_\mu(g)(0)\left(\lim_{x\rightarrow\infty}-\mu(x)\,\phi(x)+\int_0^\infty \phi(x)\,\mu'(x)\,dx\right).\end{aligned}$$

Using again (2.2) we have desired result. □

Corollary 1. *Suppose that the assumptions of Lemma 1 is satisfied and $x_0\in\mathbb{R}$. Then we have*

$$\int_0^\infty |g(x-x_0)|\,\phi(x)\,dx \le M_\mu(g)(x_0)\int_0^\infty \phi(x)\,\mu'(x)\,dx.$$

Theorem 1. *Let be $q_\lambda\in(0,\ 1)$ such that $q_\lambda\rightarrow 1$ as $\lambda\rightarrow\infty$. Suppose that for every $\lambda>0$*

$$\frac{\mu(t)}{E_q\left([\lambda]_{q_\lambda}(1-q_\lambda)\,t\right)}\rightarrow 0 \quad \textit{as}\ \ t\rightarrow 0\ \textit{and}\ \ t\rightarrow\infty \tag{2.3}$$

and

$$\Delta_{q_\lambda}^{1}(\lambda)=\frac{[\lambda]_{q_\lambda}(1-q_\lambda)}{\ln q_\lambda^{-1}}\int_{0}^{\infty}\frac{\mu^{'}(t)}{E_q\left([\lambda]_{q_\lambda}(1-q_\lambda)t\right)}dt\to 0,\quad as\ \lambda\to\infty \tag{2.4}$$

where $\mu(t)$ is defined as in (2.1). If $f\in L_{p,\ m}(\mathbb{R})$ with some $1\leq p<\infty$ and positive integer m and also for some $\delta>0$ and fixed point $x_0\in\mathbb{R}$

$$\sup_{0<\tau<\delta}\frac{1}{\mu(\tau)}\int_{|t|<\tau}\frac{|f(t-x_0)+f(t+x_0)-2f(x_0)|}{1+t^{2m}}dt=A_\delta(x_0)<\infty, \tag{2.5}$$

then we have

$$|P_\lambda(f,\ q_\lambda;\ x_0)-f(x_0)|=\mathcal{O}\left(\Delta_{q_\lambda}^{1}(\lambda)\right),\ as\ \lambda\to\infty.$$

Proof. Let x_0 satisfy (2.5) for $f\in L_{p,\ m}(\mathbb{R})$ and choose $\delta>0$. Then

$$\begin{aligned}|P_\lambda(f,\ q_\lambda;\ x_0)-f(x_0)| &\leq \frac{[\lambda]_{q_\lambda}(1-q_\lambda)}{\ln q_\lambda^{-1}}\int_{0}^{\infty}\frac{|f(x_0-t)+f(x_0+t)-2f(x_0)|}{E_q\left([\lambda]_{q_\lambda}(1-q_\lambda)t\right)}dt\\ &= \int_{0}^{\delta}\cdots+\int_{\delta}^{\infty}\cdots\\ &= I_1+I_2.\end{aligned}$$

We estimate, below, $I_1$and I_2.

For I_1, we introduce the function g defined by

$$g(t-x_0)=\begin{cases}\frac{f(x_0-t)+f(x_0+t)-2f(x_0)}{1+t^{2m}} & 0\leq t<\delta\\ 0 & \delta\leq t<\infty\end{cases}.$$

Since $1/E_q\left([\lambda]_{q_\lambda}(1-q_\lambda)t\right)$ is decreasing on $(0,\ \infty)$ then we have from (2.3) and Corollary 1

$$\begin{aligned}I_1 &\leq \frac{[\lambda]_{q_\lambda}(1-q_\lambda)}{\ln q_\lambda^{-1}}\int_{0}^{\infty}|g(t-x_0)|\frac{1+\delta^{2m}}{E_q\left([\lambda]_{q_\lambda}(1-q_\lambda)t\right)}dt\\ &\leq \left(1+\delta^{2m}\right)A_\delta(x_0)\frac{[\lambda]_{q_\lambda}(1-q_\lambda)}{\ln q_\lambda^{-1}}\int_{0}^{\infty}\frac{\mu^{'}(t)}{E_q\left([\lambda]_{q_\lambda}(1-q_\lambda)t\right)}dt\\ &= \mathcal{O}\left(\Delta_{q_\lambda}^{1}(\lambda)\right),\ \text{as}\ \lambda\to\infty. \end{aligned} \tag{2.6}$$

Now we consider I_2. Applying Hölder's inequality with $\frac{1}{p}+\frac{1}{s}=1$, then we have

$$\begin{aligned} I_2 \quad &\leq \quad 2\left|f\left(x_0\right)\right| \frac{[\lambda]_{q_\lambda}\left(1-q_\lambda\right)}{\ln q_\lambda^{-1}} \int\limits_0^{\infty} \chi_\delta\left(t\right) \frac{dt}{E_{q_\lambda}\left([\lambda]_{q_\lambda}\left(1-q_\lambda\right)t\right)} \\ &\quad +2^{2m}\left(1+x_0^{2m}\right)\|f\|_{p,\ m} \frac{[\lambda]_{q_\lambda}\left(1-q_\lambda\right)}{\ln q_\lambda^{-1}} \left\| \chi_\delta \frac{1+t^{2m}}{E_q\left([\lambda]_{q_\lambda}\left(1-q_\lambda\right)t\right)}\right\|_{s,\ m} \end{aligned} \tag{2.7}$$

where χ_δ is the characteristic function of the interval $[\delta,\ \infty)$.

Since $\underset{q\to 1}{lim} E_q\left(\left(1-q\right)t\right)=e^t$ (see [9, pp.9, (1.3.16)]) we apply Lebesgue Dominated Convergence theorem, the first term of above inequality satisfies

$$\begin{aligned} \frac{[\lambda]_{q_\lambda}\left(1-q_\lambda\right)}{\ln q_\lambda^{-1}} \int\limits_0^{\infty} \chi_\delta\left(t\right) \frac{dt}{E_{q_\lambda}\left([\lambda]_{q_\lambda}\left(1-q_\lambda\right)t\right)} &= \frac{\left(1-q_\lambda\right)}{\ln q_\lambda^{-1}} \int\limits_{\delta[\lambda]_{q_\lambda}}^{\infty} \frac{dt}{E_{q_\lambda}\left(\left(1-q_\lambda\right)t\right)} \\ &\to \ 0 \text{ as } \lambda\to\infty. \end{aligned} \tag{2.8}$$

For the second term in (2.7), we get

$$\begin{aligned} &\frac{[\lambda]_{q_\lambda}\left(1-q_\lambda\right)}{\ln q_\lambda^{-1}} \left\| \chi_\delta \frac{1+t^{2m}}{E_{q_\lambda}\left([\lambda]_{q_\lambda}\left(1-q_\lambda\right)t\right)}\right\|_{s,\ m} \\ &= \frac{[\lambda]_{q_\lambda}\left(1-q_\lambda\right)}{\ln q_\lambda^{-1}} \left(\int\limits_\delta^{\infty} \frac{1+t^{2m}}{E_{q_\lambda}\left([\lambda]_{q_\lambda}\left(1-q_\lambda\right)t\right)} \left[\frac{1+t^{2m}}{E_{q_\lambda}\left([\lambda]_{q_\lambda}\left(1-q_\lambda\right)t\right)}\right]^{\frac{s}{p}} \right)^{\frac{1}{s}} \\ &\leq \left[\sup_{t>\delta} \frac{1+t^{2m}}{E_{q_\lambda}\left([\lambda]_{q_\lambda}\left(1-q_\lambda\right)t\right)}\right]^{\frac{1}{p}} \frac{[\lambda]_{q_\lambda}\left(1-q_\lambda\right)}{\ln q_\lambda^{-1}} \left\| \chi_\delta \frac{1+t^{2m}}{E_{q_\lambda}\left([\lambda]_{q_\lambda}\left(1-q_\lambda\right)t\right)}\right\|_1^{\frac{1}{s}}, \end{aligned}$$

where $\|f\|_1=\int_{\mathbb{R}}|f(x)|\,dx$.

As in (2.6) the right hand side of above inequality tends to 0 as $\lambda\to\infty$. Thus we have

$$I_2\to 0 \quad \text{as} \quad \lambda\to\infty. \tag{2.9}$$

From (2.6) and (2.9), the proof of Theorem 1 is completed. □

Similarly we can prove the following theorem for the operator $W_\lambda\left(f,\ \cdot\right)$

Theorem 2. *Let be $q_\lambda\in(0,\ 1)$ such that $q_\lambda\to 1$ as $\lambda\to\infty$. Suppose that for every $\lambda>0$*

$$\frac{\mu\left(t\right)}{E_q\left([\lambda]_{q_\lambda}t^2\right)}\to 0 \qquad \textit{as} \ \ t\to 0 \ \textit{and} \ \ t\to\infty$$

and

$$\Delta_{q_\lambda}^2(\lambda) = \frac{\sqrt{[\lambda]_{q_\lambda}}}{\pi\left(q_\lambda^{1/2};\ q_\lambda\right)_{1/2}} \int_0^\infty \frac{\mu^{'}(t)}{E_{q_\lambda}\left([\lambda]_{q_\lambda} t^2\right)} dt \to 0, \quad as\ \lambda \to \infty$$

where $\mu(t)$ is defined as in (2.1). If $f \in L_{p,\ m}(\mathbb{R})$ with some $1 \le p < \infty$ and positive integer m and also for some $\delta > 0$ and a fixed point $x_0 \in \mathbb{R}$

$$\sup_{0<\tau<\delta} \frac{1}{\mu(\tau)} \int_{|t|<\tau} \frac{|f(t-x_0)+f(t+x_0)-2f(x_0)|}{1+t^{2m}} dt = A_\delta(x_0) < \infty,$$

then we have

$$|W_\lambda(f,\ q_\lambda;\ x_0) - f(x_0)| = \mathcal{O}\left(\Delta_{q_\lambda}^2(\lambda)\right), as\ \lambda \to \infty.$$

Example 1. *When $\mu(t) = t^{\alpha+1}$, $\alpha > 0$ from (2.4) and (1.4) it follows that*

$$\Delta_{q_\lambda}^1(\lambda) = \frac{c_{q_\lambda}(\alpha+1)\,\Gamma_{q_\lambda}(\alpha+1)}{q^{\frac{\alpha(\alpha+1)}{2}}} \frac{(\alpha+1)}{[\lambda]_{q_\lambda}^\alpha}$$

which from the property (3) *tends to* 0 *for $q_\lambda \to 1$ as $\lambda \to \infty$. As a depend of choosing q_λ this speed can be better (see* [3]*).*

3. Axillary results

We will give here some properties of the operators (1.1) and (1.2). In the classical case similar theorem for exponential weighted space given in [10].

Lemma 2. *(*[3]*) If $f \in L_{p,\ m}(\mathbb{R})$ for some $1 \le p < \infty$ and positive integer m, then*

$$\|P_\lambda(f,\ q;\ \cdot)\|_{p,\ m} \le A(\lambda,\ q,m)\,\|f\|_{p,\ m}$$

and

$$\|W_\lambda(f,\ q;\ \cdot)\|_{p,\ m} \le B(\lambda,\ q,m)\,\|f\|_{p,\ m}$$

for $0 < q < 1$ and $0 < \lambda < \infty$, where

$$A(\lambda,\ q,m) = 2^{2m-1}\left(1 + \frac{[2m]_q!}{[\lambda]_q^{2m}\, q^{m(2m+1)}}\right)$$

and

$$B(\lambda,\ q,m) = 2^{2m-1}\left(1 + \frac{q^{-\frac{m^2}{2}}\left(q^{1/2};q\right)_m}{[\lambda]_q^m}\right)$$

Following lemma follows immediately from the definitions (1.1)-(1.6).

Lemma 3. *For $x \in \mathbb{R}$ and $0 < q < 1$, the operators P_λ and W_λ satisfy the relations*

$$\begin{aligned}
&P_\lambda(1,\ x) = 1, && W_\lambda(1,\ x) = 1,\\
&P_\lambda(t,\ x) = x, && W_\lambda(t,\ x) = x,\\
&P_\lambda\left(t^2,\ x\right) = x^2 + \tfrac{[2]_q}{[\lambda]_q^2 q^3}, && W_\lambda\left(t^2,\ x\right) = x^2 + \frac{q^{-\frac{1}{2}}\left(q^{\frac{1}{2}};\, q\right)_1}{[\lambda]_q},\\
&P_\lambda\left(t^3,\ x\right) = x^3 + \tfrac{3[2]_q}{[\lambda]_q^2 q^3}x, && W_\lambda\left(t^3,\ x\right) = x^3 + \frac{3q^{-\frac{1}{2}}\left(q^{\frac{1}{2}};\, q\right)_1}{[\lambda]_q}x,\\
&P_\lambda\left(t^4,\ x\right) = x^4 + \tfrac{6[2]_q}{[\lambda]_q^2 q^3}x^2 + \tfrac{[4]_q!}{[\lambda]_q^4 q^{10}}, && W_\lambda\left(t^4,\ x\right) = x^4 + \frac{6q^{-\frac{1}{2}}\left(q^{\frac{1}{2}};\, q\right)_1}{[\lambda]_q}x^2\\
& && \qquad + \frac{q^{-2}\left(q^{\frac{1}{2}};\, q\right)_2}{[\lambda]_q^2}.
\end{aligned}$$

Lemma 4. *Let be $q_\lambda \in (0,\ 1)$ such that $q_\lambda \to 1$ as $\lambda \to \infty$. Suppose that x_0 is a fixed point in $\mathbb{R}$ and $h \in L_{\infty,\ m}(\mathbb{R})$. If*

$$\lim_{(t,\ q_\lambda)\to(x_0,\ 1)} h(t,\ x_0,\ q_\lambda) = 0 \tag{3.1}$$

then

$$\lim_{\lambda\to\infty} P_\lambda\left(h(t,\ x_0,\ q_\lambda),\ q_\lambda;\ x_0\right) = 0.$$

Proof. Let $\varepsilon > 0$ and the constant $A(\lambda,\ q_\lambda, m)$ be as in Lemma 2. Then from (3.1) and properties of $h(t,\ x_0,\ q_\lambda)$ there exist positive constants $\delta \equiv \delta(\varepsilon, A)$, $\widehat{q}_\lambda \in (0,\ 1)$ and M such that

$$\left(1 + t^{2m}\right)^{-1} |h(t,\ x_0,\ q_\lambda)| < \frac{\varepsilon}{2A(\lambda,\ q_\lambda, m)} \tag{3.2}$$

holds for $|t - x_0| < \delta$ and all $q_\lambda \in (\widehat{q}_\lambda,\ 1)$, and

$$\left(1 + t^{2m}\right)^{-1} |h(t,\ x_0,\ q_\lambda)| \le M \tag{3.3}$$

for $t \in \mathbb{R}$.

Also we can write

$$\begin{aligned}
&\left(1 + x_0^{2m}\right)^{-1} \left|P_\lambda\left(h(t,\ x_0,\ q_\lambda),\ q_\lambda;\ x_0\right)\right|\\
&\le \frac{\left(1 + x_0^{2m}\right)^{-1} [\lambda]_{q_\lambda}(1 - q_\lambda)}{2\ln q_\lambda^{-1}} \int_{-\infty}^{\infty} \frac{|h(t,\ x_0,\ q_\lambda)|}{E_{q_\lambda}\left([\lambda]_{q_\lambda}(1 - q_\lambda)\,|t - x_0|\right)} dt\\
&\le \frac{\left(1 + x_0^{2m}\right)^{-1} [\lambda]_{q_\lambda}(1 - q_\lambda)}{2\ln q_\lambda^{-1}} \left(\int_{|t-x_0|<\delta} \cdots + \int_{|t-x_0|\ge\delta} \cdots\right)\\
&= I_1 + I_2.
\end{aligned}$$

From (3.2), (1.4) and (1.5) we have

$$I_1 < 2^{2m-1}\left(1 + \frac{[2m]_q!}{[\lambda]_q^{2m}\, q^{m(2m+1)}}\right) \frac{\varepsilon}{2A(\lambda,\ q_\lambda, m)} = \frac{\varepsilon}{2}.$$

Moreover, by using (3.3) we get

$$\begin{aligned} I_2 &= \frac{\left(1+x_0^{2m}\right)^{-1}[\lambda]_{q_\lambda}(1-q_\lambda)}{2\ln q_\lambda^{-1}} \int\limits_{|t-x_0|\geq\delta} \frac{|h(t,\ x_0,\ q_\lambda)|}{E_{q_\lambda}\left([\lambda]_{q_\lambda}(1-q_\lambda)|t-x_0|\right)}dt \\ &\leq \frac{\left(1+x_0^{2m}\right)^{-1}[\lambda]_{q_\lambda}(1-q_\lambda)}{2\ln q_\lambda^{-1}} M \int\limits_{|t-x_0|\geq\delta} \frac{1+t^{2m}}{E_{q_\lambda}\left([\lambda]_{q_\lambda}(1-q_\lambda)|t-x_0|\right)}dt. \end{aligned}$$

Using the inequality

$$1+t^{2m} \leq 2^{2m-1}\left(1+(t-x_0)^{2m}\right)\left(1+x_0^{2m}\right),$$

(1.4) and (1.5), we get

$$\begin{aligned} I_2 &\leq \frac{2^{2m-1}[\lambda]_{q_\lambda}(1-q_\lambda)}{2\ln q_\lambda^{-1}} \frac{M}{\delta^2} \int\limits_{|t-x_0|\geq\delta} \frac{\left[1+(t-x_0)^{2m}\right](t-x_0)^2}{E_{q_\lambda}\left([\lambda]_{q_\lambda}(1-q_\lambda)|t-x_0|\right)}dt \\ &\leq 2^{2m-1}\frac{M}{\delta^2}\frac{1}{[\lambda]_{q_\lambda}^2}\left(\frac{[2]_{q_\lambda}}{q_\lambda^3}+\frac{1}{[\lambda]_{q_\lambda}^{2m}}\frac{[2m+2]_{q_\lambda}!}{q_\lambda^{(m+1)(2m+3)}}\right). \end{aligned}$$

Thus, for fixed positive numbers ε, δ, M and $x_0 \in \mathbb{R}$ there exists positive real number λ_0, depending on ε, δ and M such that

$$I_2 < \frac{\varepsilon}{2} \text{ for all } \lambda > \lambda_0,$$

i.e $\lim\limits_{\lambda\to\infty} P_\lambda(h(t,\ x_0,\ q_\lambda),\ q_\lambda;\ x_0) = 0$. This completes the proof of lemma 4. □

Lemma 5. *Let be $q_\lambda \in (0,\,1)$ such that $q_\lambda \to 1$ as $\lambda \to \infty$. Suppose that x_0 is a fixed point in $\mathbb{R}$ and $h \in L_{\infty,\ m}(\mathbb{R})$. If*

$$\lim_{(t,\ q_\lambda)\to(x_0,\ 1)} h(t,\ x_0,\ q_\lambda) = 0$$

then

$$\lim_{\lambda\to\infty} W_\lambda(h(t,\ x_0,\ q_\lambda),\ q_\lambda;\ x_0) = 0.$$

4. Voronovskaya type theorems

In this section we give an asymptotic formula of Voronovskaya type for the generalized operators defined by (1.1) and (1.2). We know that the Voronovskaya type asymptotic formula provides an exhaustive description of the speed of pointwise convergence of the considered operators. We will show that this asymptotic expression of the type $P_\lambda(f,\ q_\lambda;\ x) - f(x) = \mathcal{O}\left(\frac{1}{[\lambda]_{q_\lambda}^2}\right)$ for every $x \in \mathbb{R}$ assuming that the second $q-$derivative of f exists at such a point. From this result we say that the speed of pointwise convergence of $P_\lambda(f,\ q_\lambda;\ x)$ to $f(x)$ is $\frac{1}{[\lambda]_{q_\lambda}^2}$ thus, can be better depending of choosing of

q_λ and which results in higher speed rate than the classical case in which the rate is $\frac{1}{\lambda^2}$. Similar situation exists for the operator $W_\lambda(f;\cdot)$.

We recall that the $q-$derivative operator $\mathcal{D}_q$ is given by

$$(\mathcal{D}_q f)(x) := \frac{f(qx) - f(x)}{(q-1)x}, \ x \neq 0$$

and $(\mathcal{D}_q f)(0) := f'(0)$.

Higher $q-$derivatives are

$$\mathcal{D}_q^0 f := f, \ \mathcal{D}_q^n f := \mathcal{D}\left(\mathcal{D}_q^{n-1} f\right), \quad n = 1, \ 2, \ 3, \ldots.$$

Note that

$$\lim_{q\to 1} \left(\mathcal{D}_q^n f\right)(x)) = f^{(n)}(x), \ n = 1, \ 2, \ 3, \ldots. \tag{4.1}$$

if f is n times differentiable.

Theorem 3. *Let $q_\lambda \in (0,1)$ such that $q_\lambda \to 1$ as $\lambda \to \infty$. Suppose that f and its first and second q-derivatives belong to $L_{\infty,m}(\mathbb{R})$. Then for every fixed $x \in \mathbb{R}$ we have*

$$\lim_{\lambda\to\infty} [\lambda]_{q_\lambda}^2 \left(P_\lambda(f, \ q_\lambda; \ x) - f(x)\right) = \lim_{q_\lambda \to 1} \mathcal{D}_{q_\lambda}^2 f(x).$$

Proof. Let x_0 be a fixed point in $\mathbb{R}$. By the q-Taylor formula (see [11, thm 5.2]) we have

$$\begin{aligned} f(t) &= f(x_0) + \mathcal{D}_{q_\lambda} f(x_0)(t - x_0) + \frac{1}{[2]_{q_\lambda}} \mathcal{D}_{q_\lambda}^2 f(x_0)(t-x_0)(t - q_\lambda x_0) \\ &\quad + \varphi(t, \ x_0, q_\lambda)(t - x_0)(t - q_\lambda x_0), \end{aligned} \tag{4.2}$$

where $\varphi(t, \ x_0, q_\lambda) \equiv \varphi(t) \in L_{q_\lambda, \ m}(\mathbb{R})$ and

$$\lim_{(t, \ q_\lambda)\to(x_0, \ 1)} \varphi(t, \ x_0, q_\lambda) = 0.$$

From (4.2) and (1.1) we get

$$\begin{aligned} P_\lambda(f, q_\lambda; x_0) - f(x_0) &= \mathcal{D}_{q_\lambda} f(x_0) P_\lambda(t - x_0, q_\lambda; x_0) \\ &\quad + \frac{1}{[2]_{q_\lambda}} \mathcal{D}_{q_\lambda}^2 f(x_0) P_\lambda((t-x_0)(t - q_\lambda x_0), q_\lambda; x_0) \\ &\quad + P_\lambda(\varphi(t)(t-x_0)(t - q_\lambda x_0), q_\lambda; x_0). \end{aligned}$$

Using Lemma 3 we obtain

$$\begin{aligned} P_\lambda(f, q_\lambda; x_0) - f(x_0) &= \frac{1}{[\lambda]_{q_\lambda}^2 q_\lambda^3} \mathcal{D}_{q_\lambda}^2 f(x_0) + P_\lambda(\varphi(t)(t-x_0)(t - q_\lambda x_0), q_\lambda; x_0) \\ &= \frac{1}{[\lambda]_{q_\lambda}^2 q_\lambda^3} \mathcal{D}_{q_\lambda}^2 f(x_0) + P_\lambda\left(\varphi(t)(t-x_0)^2, q_\lambda; x_0\right) \\ &\quad + x_0(1 - q_\lambda) P_\lambda(\varphi(t)(t-x_0), q_\lambda; x_0). \end{aligned} \tag{4.3}$$

From the Hölder inequality, we can write

$$\left|[\lambda]_{q_\lambda}^2 P_\lambda\left(\varphi(t)(t-x_0)^2, q_\lambda; x_0\right)\right| \leq \left(P_\lambda\left(\varphi^2(t), q_\lambda x_0\right)\right)^{1/2} \times \left([\lambda]_{q_\lambda}^2 P_\lambda\left((t-x_0)^4, q_\lambda; x_0\right)\right)^{1/2} \quad (4.4)$$

and

$$\begin{aligned} &\left|x_0(1-q_\lambda)[\lambda]_{q_\lambda}^2 P_\lambda(\varphi(t)(t-x_0), q_\lambda; x_0)\right| \\ &\qquad \leq x_0\left((1-q_\lambda)^2[\lambda]_{q_\lambda}^4 P_\lambda\left((t-x_0)^2, q_\lambda; x_0\right)\right)^{1/2} \\ &\qquad \times \left(P_\lambda\left(\varphi^2(t), q_\lambda; x_0\right)\right)^{1/2} \end{aligned} \quad (4.5)$$

The function $h(t,\ x_0, q_\lambda) = \varphi^2(t,\ x_0, q_\lambda)$, $t \geq 0$, satisfies the conditions of Lemma 4; therefore

$$\lim_{\lambda\to\infty} P_\lambda\left(\varphi^2(t), q_\lambda; x_0\right) = 0. \quad (4.6)$$

Also using Lemma 3 we easily show that

$$P_\lambda\left((t-x_0)^4, q_\lambda; x_0\right) = \frac{[4]_{q_\lambda}!}{[\lambda]_{q_\lambda}^4 q_\lambda^{10}}$$

and

$$P_\lambda\left((t-x_0)^2, q_\lambda;\ x_0\right) = \frac{[2]_{q_\lambda}}{[\lambda]_{q_\lambda}^2 q_\lambda^3}.$$

Thus using these equalities we obtain

$$\lim_{\lambda\to\infty} [\lambda]_{q_\lambda}^4 P_\lambda\left((t-x_0)^4, q_\lambda; x_0\right) = 4! \quad (4.7)$$

and

$$\begin{aligned} \lim_{\lambda\to\infty}(1-q_\lambda)^2[\lambda]_{q_\lambda}^4 P_\lambda\left((t-x_0)^2, q_\lambda; x_0\right) &= \lim_{\lambda\to\infty}\frac{[2]_{q_\lambda}}{q_\lambda^3}(1-q_\lambda)^2[\lambda]_{q_\lambda}^2 \\ &= 2\lim_{\lambda\to\infty}\left(1-q_\lambda^\lambda\right)^2 \\ &= C, \end{aligned} \quad (4.8)$$

where $0 \leq C < 2$. Combining (4.8), (4.7), (4.6), (4.5) and (4.4) we have

$$\lim_{\lambda\to\infty} [\lambda]_{q_\lambda}^2 P_\lambda\left(\varphi(t)(t-x_0)^2, q_\lambda; x_0\right) = 0$$

and

$$\lim_{\lambda\to\infty} x_0(1-q_\lambda)[\lambda]_{q_\lambda}^2 P_\lambda(\varphi(t)(t-x_0), q_\lambda; x_0) = 0$$

From these equalities and (4.3) we have the desired result. □

In a similar fashion, using Theorem 3 and Lemma 5 we can prove the following Voronovskaya theorem for the operator $W_\lambda(f;\cdot)$.

Theorem 4. *Let $q_\lambda \in (0,\,1)$ such that $q_\lambda \to 1$ as $\lambda \to \infty$. Suppose that f and its first and second q-derivatives belong to $L_{\infty,\; m}(\mathbb{R})$. Then, for every fixed $x \in \mathbb{R}$, we have*

$$\lim_{\lambda\to\infty} [\lambda]_{q_\lambda} \left(W_\lambda\left(f,\; q_\lambda;\; x\right) - f\left(x\right)\right) = \lim_{q_\lambda \to 1} \mathcal{D}^2_{q_\lambda} f\left(x\right).$$

From (4.1) and in view of Theorem 3 and Theorem 4 we obtain following immediate results.

Corollary 2. *Let $q_\lambda \in (0,\,1)$ such that $q_\lambda \to 1$ as $\lambda \to \infty$. Suppose that f and its first and second q-derivatives belong to $L_{\infty,\; m}(\mathbb{R})$. Then, for every fixed $x \in \mathbb{R}$, we have*

$$\lim_{\lambda\to\infty} [\lambda]^2_{q_\lambda} \left(P_\lambda\left(f,\; q_\lambda;\; x\right) - f\left(x\right)\right) = f''\left(x\right).$$

Corollary 3. *Let $q_\lambda \in (0,\,1)$ such that $q_\lambda \to 1$ as $\lambda \to \infty$. Suppose that f and its first and second derivatives belong to $L_{\infty,\; m}(\mathbb{R})$. Then, for every fixed $x \in \mathbb{R}$, we have*

$$\lim_{\lambda\to\infty} [\lambda]_{q_\lambda} \left(W_\lambda\left(f,\; q_\lambda;\; x\right) - f\left(x\right)\right) = f''\left(x\right).$$

References

[1] R. Álvarez-Nodarse, M. K. Atakishiyeva and N. M. Atakishiyev, On $q-$extension of the Hermite Polynomials $H_n(x)$ with the continuous orthogonality property on R, *Bol. Soc. Mat. Mexicana* (3), 8, 127–139 (2002).

[2] G. A. Anastassiou and S. G. Gal, *Approximation Theory : Moduli of Continuity and Global Smoothness Preservation*, Birkhäuser, Boston, 2000.

[3] A. Aral, On the generalized Picard and Gauss Weierstrass Singular Integrals, *J. Comp. Anal. and Appl.* Vol 8, No:3, 249-261.

[4] A. Aral, On a generalized $\lambda-$Gauss-Weierstrass singular integrals, *Fasciculi Mathematici*, No:35, 23-33 (2005).

[5] A. Aral, On convergence of singular integrals with non-isotropic kernels, *Commun Fac. Sci. Univ. Ank. Series A1* 50 , 88–98 (2001).

[6] N. M. Atakishiyev and M. K. Atakishiyeva, A $q-$Analog of the Euler gamma integral, *Theo. and Math. Phys*, 129, No:1, 1325–1334 (2001).

[7] C. Berg, From discrete to absolutely continuous solution of indeterminate moment problems, *Arap. J. Math. Sci.*, 4, 2, 67–75 (1988).

[8] T. Ernst, A new notation for $q-$calculus and a new $q-$Taylor formula. ISSN 1101-3591 (1999).

[9] G. Gasper and M. Rahman, *Basic Hypergeometric series,* Cambridge: Cambiridge University Press,.1990.

[10] A. Lesniewicz, L. Rempulska, J. Wasiak, Approximation properties of the Picard singular integral in exponential weighted spaces, *Publicacions Matemàtiques*, 40, 233-242 (1996).

[11] P. M. Rajković, M. S. Stanković and S. D. Marinkovic, Mean value theorems in $q-$calculus, *Math. Vesnic.* 54 (2002), 171-178.

[12] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.

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REMARKS ON BROWDER AND PROLLA TYPE BEST APPROXIMATION PROBLEM

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Abstract. The aim of this paper is to present improved and extended version of the Browder and Prolla type best approximation and coincidence points problem by using relative almost p-quasiconvex map. Some important remarks are given as consequences.

1. INTRODUCTION

The well known best approximation theorem of Ky Fan [4] has been of great importance in Nonlinear analysis, approximation theory, mininmax theory, game theory, fixed point theory, and variational inequalities. Interesting extensions have been given by several researchers and variety of applications has also been given by a number of authors. For detail see [13].

One of the most interesting extensions were given by Prolla [11] for a pair of functions. Several extensions of Prolla's theorem have been given, and its application in fixed point theory, approximation and coincidence theory are also given by many authors. Most of the celebrated fixed point theorems in the field of nonlinear analysis were proved for the mappings whose domain and range were same(see [10]). The importance of Ky Fan's best approximation theorem [4] has attracted the analysts because many of these celebrated fixed point theorems were obtained under weaker assumption with the help of Ky Fan's theorem. Motivated by those new results, Park [9] generalize, improve and unify several existing Fan or Prolla type best approximation theorems for single- valued maps and applied them to get existence theorem of fixed or coincidence points.

On the other hand, Sehgal and Singh [12] applied KKM principle to obtain a theorem on the best approximation of a continuous function with respect to a p-affine type map. This result provides extensions of some well-known fixed point theorems of Browder [1] and many others.

Recently, Chen and Park [2] unify and extend the Browder type best approximation result of Sehgal and Singh [12, Theorem 5], and Prolla type best approximation and coincidence point problem of Park [9] and many others.

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The purpose of the this paper is to improve and extend the results of Chen and Park [2], Park [9], Sehgal and Singh [12] and many others by considering relative almost p-quasiconvex mappings with respect to multifunction. For this reason, we use Allen type variational inequality of Park [6, Theorem 0]. To proof the theorem, we follow the method of Chen and Park [2]. In the third section, we prove Browder type best approximation theorem for topological vector spaces and in the fourth section, we prove Prolla type best approximation theorem in the seminormed spaces for relative almost p-quasiconvex mappings with respect to multifunction in both. Some important results are obtained as consequences. In the fifth section, we present coincidence theorem in topological vector spaces having sufficient many linear functionals. Further, many important points on the recent work of Ding and Tarafdar [3] are obtained as remarks.

2. PRELIMINARIES

Let us recall the definitions:

For a subset $\mathcal{X}$ of a vector space $\mathcal{E}$ and $x \in \mathcal{E}$, the inward set of $\mathcal{X}$ at x is defined by

$$\mathcal{I}_{\mathcal{X}}(x) = \{x + r(u - x) \in \mathcal{E} : u \in \mathcal{X}, r > 0\}.$$

Let $\mathcal{X}$ be a convex space, $\mathcal{E}$ a topological vector space, and $p : \mathcal{X} \times \mathcal{E} \to \mathbb{R}$ a function which is convex in the second variable. A function $g : \mathcal{X} \to \mathcal{E}$ is said to be almost p-quasi convex [6] or p-affine [7] if for any $x, x_1, x_2 \in \mathcal{X}$, $z \in \mathcal{E}$, and $r \in [0, 1]$, we have

$$p(x, z - g(rx_1 + (1 - r)x_2)) \leq \max\{p(x, z - gx_i) : i = 1, 2\}.$$

Let $\mathcal{F}$ be a mapping from $\mathcal{X}$ to $2^{\mathcal{E}}$. A mapping g from $\mathcal{X}$ to $\mathcal{E}$ is said to be *almost p-quasiconvex with respect to* $\mathcal{F}$ [14], if

$$p(x, \mathcal{F}z - gy) \leq \max\{p(x, \mathcal{F}z - gx_i) : i = 1, 2\}.$$

where $y = rx_1 + (1 - r)x_2$, $x, x_1, x_2 \in \mathcal{X}, r \in [0, 1]$ and $z \in \mathcal{X}$.

A convex space $\mathcal{X}$ is a nonempty convex set with any topology that includes the Euclidean topology on the convex hulls of its subsets. For such spaces, we have the following version of the Allen type variational inequality which will be used to prove our main result in the sequel:

Theorem 2.1. [6, Theorem 0]. *Let $\mathcal{X}$ be a convex space, $\Psi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ a real function, and $\mathcal{K}$ a nonempty compact subset of $\mathcal{X}$. Suppose that*

(i) $\Psi(x, x) \leq 0$ for all $x \in \mathcal{X}$;

(ii) for each $y \in \mathcal{X}$, $\{x \in X : \Psi(x, y) > 0\}$ is compactly open;

(iii) for each $x \in \mathcal{X}$, $\{y \in \mathcal{X} : \Psi(x, y) > 0\}$ is convex or empty; and

(iv) for each $N \in < \mathcal{X} >$(the set of all nonempty finite subsets of $\mathcal{X}$), there exists a compact convex subset $\mathcal{L}_N$ of $\mathcal{X}$ containing N such that, for each

$x \in \mathcal{L}_N \backslash \mathcal{K}$, there exists $y \in \mathcal{L}_N$ satisfying $\Psi(x, y) > 0$.

Then there exists an $x_0 \in \mathcal{K}$ such that $\Psi(x_0, y) \leq 0$ for all $y \in \mathcal{X}$.

3. BROWDER TYPE BEST APPROXIMATION

We begin with the following `Browder type best approximation theorem` which improves and extends the main result of Chen and Park [2, Theorem 1] to a relative p-almost quasiconvex mapping with respect to multifunction:

Theorem 3.1. *Let $\mathcal{X}$ be a convex space, $\mathcal{K}$ a nonempty compact subset of $\mathcal{X}$, $\mathcal{E}$ a topological vector space, and $p : \mathcal{X} \times \mathcal{E} \to \mathbb{R}$ a continuous map, and $g : \mathcal{X} \to \mathcal{E}$ be a continuous map. Let $\mathcal{F} : \mathcal{X} \to \mathcal{E}$ be a continuous multifunction. Suppose that*

(i) for each $x \in \mathcal{X}$, $p(x, .)$ is a convex function on $\mathcal{E}$;

(ii) g is almost p-quasiconvex with respect to $\mathcal{F}$; and

(iii) for each $N \in < \mathcal{X} >$, there exists a compact convex subset $\mathcal{L}_N$ of $\mathcal{X}$ containing N such that, for each $x \in \mathcal{L}_N \backslash \mathcal{K}$, there exists a $y \in \mathcal{L}_N$ such that

$$p(x, \mathcal{F}x - gx) > p(x, \mathcal{F}x - gy).$$

Then there exists $u \in \mathcal{K}$ such that

$$p(u, \mathcal{F}u - gu) = \inf\{p(u, \mathcal{F}u - gy) : y \in \mathcal{X}\}.$$

Further, if $g(\mathcal{X})$ is convex, then

$$p(u, \mathcal{F}u - gu) = \inf\{p(u, \mathcal{F}u - z) : z \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}\}.$$

Proof. Define $\Psi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ by

$$\Psi(x, y) = p(x, \mathcal{F}x - gx) - p(x, \mathcal{F}x - gy)$$

for $(x, y) \in \mathcal{X} \times \mathcal{X}$. Then

By (i), $0 \in \Psi(x, x)$ for all $x \in \mathcal{X}$. For each fixed $y \in \mathcal{X}$, the set $\{x \in \mathcal{X} : \Psi(x, y) > 0\}$, i.e., $\{x \in \mathcal{X} : p(x, \mathcal{F}x - gx) > p(x, \mathcal{F}x - gy)\}$ is open since $x \to \Psi(x, y)$ is continuous. For each fixed $y \in \mathcal{X}$, the set $\{x \in \mathcal{X} : \Psi(x, y) > 0\}$, i.e., $\{x \in \mathcal{X} : p(x, \mathcal{F}x - gx) > p(x, \mathcal{F}x - gy)\}$ is convex. Indeed, let y_1 and y_2 are the elements in the set and $0 < \lambda < 1$, we have

$$p(x, \mathcal{F}x - gy_1) < p(x, \mathcal{F}x - gx)$$

and

$$p(x, \mathcal{F}x - gy_2) < p(x, \mathcal{F}x - gx).$$

Since g is almost p-quasiconvex with respect to $\mathcal{F}$, it follows that

$$p(x, \mathcal{F}x - g(\lambda y_1 + (1 - \lambda y_2) \leq \max\{p(x, \mathcal{F}x - gy_i) : i = 1, 2\}.$$

Hence

$$p(x, \mathcal{F}x - g(\lambda y_1 + (1 - \lambda y_2) < \{p(x, \mathcal{F}x - gy)\}$$

for each $x \in \mathcal{X}$. Thus the set $\{x \in \mathcal{X} : \Psi(x, y) > 0\}$ is convex.
Therefore, Theorem 2.1 guarantees that there exists $u \in \mathcal{K}$ such that

$$\Psi(u, y) = p(u, \mathcal{F}u - gu) - p(u, \mathcal{F}u - gy) \leq 0$$

holds for all $y \in \mathcal{X}$, i.e.,

$$p(u, \mathcal{F}u - gu) = \inf\{p(u, \mathcal{F}u - gy) : y \in \mathcal{X}\}.$$

Further, if $g(\mathcal{X})$ is convex, then for any $z \in \mathcal{I}_{g(\mathcal{X})}(gu)\backslash g(\mathcal{X})$, there exist $w \in g(\mathcal{X})$ and $\lambda > 1$ such that $z = gu + \lambda(w - gu)$. Suppose that $p(u, \mathcal{F}u - gu) > p(u, \mathcal{F}u - z)$. Since

$$w = \frac{1}{\lambda}z + (1 - \frac{1}{\lambda})gu \in g(\mathcal{X})$$

and p is convex(in the second variable) by (i), we have

$$p(u, \mathcal{F}u - w) \leq \tfrac{1}{\lambda}p(u, \mathcal{F}u - z) + (1 - \tfrac{1}{\lambda})p(u, \mathcal{F}u - gu)$$

$$< p(u, \mathcal{F}u - gu),$$

a contradiction. Therefore, $p(u, \mathcal{F}u - gu) \leq p(u, \mathcal{F}u - z)$ for all $z \in \mathcal{I}_{g(\mathcal{X})}(gu)$ and hence for all $z \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}$ by the continuity of p. Since $gu \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}$, we have

$$p(u, \mathcal{F}u - gu) = \inf\{p(u, \mathcal{F}u - z) : z \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}\}.$$

This completes the proof. □

Remark 3.2. In certain cases, the inward set in the conclusion can be replaced by the outward set. For detail see [1, 10].

Remark 3.3. The continuities of $\mathcal{F}$, g and p are used only to assure the compactly openness of $\{x \in \mathcal{X} : \Psi(x, y) > 0\}$ and that $p(u, \mathcal{F}u - gu) \leq p(u, \mathcal{F}u - z)$ holds for all $z \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}$.

A variant of Theorem 3.1 is as follows:

Theorem 3.4. *Under the hypothesis of Theorem 3.1, let $\mathcal{X}$ be a subset of $\mathcal{E}$. If $gu \in \mathcal{X}$, then we have*

$$p(u, \mathcal{F}u - gu) = \inf\{p(u, \mathcal{F}u - z) : z \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}\}.$$

Proof. If $gu \in \mathcal{X}$, then for any $z \in \mathcal{I}_{\mathcal{X}}(gu)\backslash\mathcal{X}$, there exits $w \in \mathcal{X}$ and $\lambda > 1$ such that $z = gu + \lambda(w - gu$. Therefore, as in the last part of the proof of Theorem 3.1, we have $p(u, \mathcal{F}u - gu) \leq p(u, \mathcal{F}u - z)$ for all $z \in \mathcal{I}_{g(\mathcal{X})}(gu)$ and hence for all $z \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}$. This completes the proof. □

Remark 3.5. In Theorem 3.4, $\mathcal{X}$ may not have the relative topology with respect to $\mathcal{E}$.

Remark 3.6. Theorem 3.4 generalizes the main result of Sehgal and Singh [12, Theorem 5], where $\mathcal{X}$ is a convex subset of a locally convex topological vector space $\mathcal{E}$, and $g : \mathcal{X} \to \mathcal{X}$. Moreover, they adopted a more strict coercivity condition than (iii).

Remark 3.7. As in [12], Theorem 3.4 can be applied to obtain some general forms of known results due to Fan, Browder, and Sehgal-Singh-Gastle.

4. PROLLA TYPE BEST APPROXIMATION

In the following, we prove `Prolla type best approximation theorem` in the setting of seminormed vector space which improves and extends the Theorem 3 of Chen and Park [2] to a relative p-almost quasiconvex mapping with respect to multifunction:

Theorem 4.1. *Let $\mathcal{X}$ be a convex space, $\mathcal{K}$ a nonempty compact subset of $\mathcal{X}$, $(\mathcal{E}, q)$ a seminormed vector space, where $q : \mathcal{E} \to [0, \infty)$ is a seminorm, and $g : \mathcal{X} \to (\mathcal{E}, q)$ be a continuous. Let $\mathcal{F} : \mathcal{X} \to \mathcal{E}$ be a continuous multifunction. Suppose that*

(i) g is almost q-quasiconvex with respect to $\mathcal{F}$; and

(ii) for each $N \in <\mathcal{X}>$, there exists a compact convex subset $\mathcal{L}_N$ of $\mathcal{X}$ containing N such that, for each $x \in \mathcal{L}_N \backslash \mathcal{K}$, there exists a $y \in \mathcal{L}_N$ satisfying

$$q(\mathcal{F}x - gx) > q(\mathcal{F}x - gy).$$

Then there exists $u \in \mathcal{K}$ such that

$$q(\mathcal{F}u - gu) = \inf\{q(\mathcal{F}u - gx) : x \in \mathcal{X}\}.$$

Further, if $g(\mathcal{X})$ is convex, then

$$q(\mathcal{F}u - gu) = \inf\{q(\mathcal{F}u - z) : z \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}\}.$$

In this case, $gu \in Bd\ \ g(\mathcal{X})$ and $\mathcal{F}u \notin \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}$ if $q(\mathcal{F}u - gu) > 0$.

Proof. Define $p : \mathcal{X} \times \mathcal{E} \to [0, \infty)$ by $p(x, z) = q(z)$ for $(x, z) \in \mathcal{X} \times \mathcal{E}$. Then the conclusion follows from Theorem 3.1. Note that if $gu \in Int\ \ g(\mathcal{X})$, then $\mathcal{I}_{g(\mathcal{X})}(gu) = \mathcal{E}$ and $\mathcal{F}u \in \mathcal{E}$, whence we have $q(\mathcal{F}u - gu) = 0$ This completes the proof. □

Remark 4.2. Theorem 4.1 includes earlier works of Fan, Prolla, Hadzic, Sehgal-Singh-Smithson, Sessa, Roux-Sing, Sehgal-Singh-Gastle, Carbone, Sessa-Singh, Park and Lin, even when $\mathcal{X}$ is a subset of $\mathcal{E}$(see [9]).

Remark 4.3. In Theorem 4.1, if $\mathcal{X} \subset \mathcal{E}$ and $g = \mathcal{I}_{\mathcal{X}}$, then we get Theorem 5 of Lin and Tian [5]. Also, other results in [5] are all consequences of corresponding ones of [6, 7].

A variant of Theorem 4.1 is as follows:

Theorem 4.4. *Under the hypothesis of Theorem 4.1, let $\mathcal{X}$ be a subset of $\mathcal{E}$. If $gu \in \mathcal{X}$, then we have*

$$p(u, \mathcal{F}u - gu) = \inf\{p(u, \mathcal{F}u - z) : z \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}\}.$$

In this case, $gu \in Bd\ \ g(\mathcal{X})$ and $\mathcal{F}u \notin \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}$ if $q(\mathcal{F}u - gu) > 0$.

Remark 4.5. In Theorem 4.1, $\mathcal{X}$ may not have the relative topology with respect to $\mathcal{E}$.

Remark 4.6. Same as theorem 4.1, Theorem 4.4 also has a lot of known particular form in the literature.

5. COINCIDENCE THEOREMS

In the following, we prove `Coincidence theorem` in the setting of topological vector space having sufficient many linear functionals which improves and extends the Theorem 5 of Chen and Park [2] to a relative p-almost quasiconvex mapping with respect to multifunction, using Theorem 4.1:

Let $(\mathcal{E}, \tau)$ be a topological vector space, $\mathcal{E}^*$ its topological dual, w the weak topology of $\mathcal{E}$, and $\mathcal{S}(\mathcal{E}, w)$ the family of all continuous seminorms on $(\mathcal{E}, w)$.

Theorem 5.1. *Let $\mathcal{X}$ be a convex space, $\mathcal{K}$ a nonempty compact subset of $\mathcal{X}$, $(\mathcal{E}, \tau)$ a topological vector space on which $\mathcal{E}^*$ separates points and $g : \mathcal{X} \to (\mathcal{E}, w)$ be a continuous. Let $\mathcal{F} : \mathcal{X} \to (\mathcal{E}, w)$ be a continuous multifunction. Suppose that*

(i) g is almost q-quasiconvex with respect to $\mathcal{F}$ for each $q \in \mathcal{S}(\mathcal{E}, w)$; and

(ii) for each $q \in \mathcal{S}(\mathcal{E}, w)$ and $N \in< \mathcal{X} >$, there exists a compact convex subset $\mathcal{L}_N$ of $\mathcal{X}$ containing N such that, for each $x \in \mathcal{L}_N \backslash \mathcal{K}$, there exists a $y \in \mathcal{L}_N$ satisfying

$$q(\mathcal{F}x - gx) > q(\mathcal{F}x - gy).$$

Then either

(a) $\mathcal{F}$ and g have a coincidence point $u \in \mathcal{K}$, or

(b) there exist a $q \in \mathcal{S}(\mathcal{E}, w)$ and a $u \in \mathcal{K}$ such that $gu \in Bd\ \ g(\mathcal{X})$(Bd denotes the boundary) and

$$0 < q(\mathcal{F}u - gu) = \inf\{q(\mathcal{F}u - z) : gu \in Bd\ \ g(\mathcal{X})\}.$$

Further, if $g(\mathcal{X})$ is convex, the above inequality in (b) can be replaced by

$$0 < q(\mathcal{F}u - gu) = \inf\{q(\mathcal{F}u - z) : z \in \overline{\mathcal{I}_{g(\mathcal{X})}(gu)}\}.$$

Proof. It follows by simply modifying the proof of [9, Theorem 7]. □

Remark 5.2. Theorem 5.1 is a slightly different version of [9, Theorem 7] and a sharpened form of Ding and Tarafdar [3, Theorem 3.1].

Remark 5.3. We can deduce coincidence theorems like [9, Theorem 8], from Theorem 5.1, which generalizes [3, Theorem 3.2].

Remark 5.4. Similarly, other results in [3] can be improved with much simpler proofs. For example, [3, Lemma 2.1] is a simple consequence of the well-known result of Berge.

References

[1] F. E. Browder, *On a sharpened form of Schauder fixed point theorem,* Proc. Nat. Acad. Sci. U. S. A. 76(1977), 4749 - 4751.

[2] M. P. Chen and S. Park, *On the Browder type best approximation theorem,* Indian J. Math. 39(1)(1997), 91 - 97.

[3] X. P. Ding and E. Tarafdar, *Some further generalizations of Ky Fan's best approximation theorem,* J. Approx. Theory 81(1995), 406 - 420.

[4] Ky Fan, *Existence of two fixed point theorems of F. E. Browder,* Math. Z. 112(1969), 234 - 240.

[5] Y. J. Lin and G. Tian, *Minimax inequalities equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz theorem,* Appl. Math. Optim. 28(1993), 173 - 179.

[6] Sehie Park, *Remarks on some variational inequalities,* Bull. Korean Math. Soc. 28(1991), 163 - 174.

[7] Sehie Park, *On minimax inequalities on spaces having certain contractible subsets,* Bull. Austral. Math. Soc. 47(1993), 25 - 40.

[8] Sehie Park, *Foundation of the KKM theory via coincidence of composites of upper semicontinuous maps,* J. Korean Math. Soc. 31(1994), 493 - 519.

[9] Sehie Park, *Remarks on generalizations of best approximation theorems,* Honam Math. J. 16(1994), 27 - 39.

[10] S. Park and M. P. Chen, *Best approximations and fixed points of multifunctions whose domains and ranges have different topologies,* Comm. Appl. Nonlinear Anal. 6(1999), No. 3, 77 - 101.

[11] J. B. Prolla, *Fixed point theorem for set-valued mappings and existence of best approximations,* Numer. Funct. Anal. Optimiz. 5 (1982- 83), No.4, 449 - 455 .

[12] V. M. Sehgal and S. P. Singh, *The best approximation and an extension of a fixed point theorem of F. E. Browder,* Int. J. Math. Math. Sci. 18(1995), 745 - 748.

[13] S. P. Singh, B. Watson and P. Srivastava, *Fixed point theory and best approximation: The KKM-Map Principle,* Vol.424, Kluwer Academic Publishers, 1997.

[14] P. Vijayaraju and M. Marudai, *Best appriximations and fixed points for multivalued mappings,* Indian J. Math. 44(2002) No.2, 233-241.

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p-ADIC q-INTEGRALS AND BASIC q-ZETA FUNCTION

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Abstract. The purpose of this paper is to introduce the some interesting properties of q-Euler numbers and polynomials, cf. [1, 2, 5]. Finally, we will consider the " sums of products of q-Euler polynomials".

1. Introduction

Let p be a fixed odd prime, and let $\mathbb{C}_p$ denote the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. For d a fixed positive integer with $(p,d)=1$, let

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$, (cf. [1], [2], [15]).

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The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. Let q be variously considered as an indeterminate a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we always assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper, we use the following notation :

$$[x]_q = [x : q] = \frac{1-q^x}{1-q}.$$

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$– and denote this property by $f \in UD(\mathbb{Z}_p)$– if the difference quotients

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y},$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$, cf. [1, 11, 12]. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^i f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p), \text{ cf. } [2, 4],$$

representing q-analogue of Riemann sums for f.

The integral of f on $\mathbb{Z}_p$ will be defined as limit $(n \to \infty)$ of these sums, when it exists. An invariant p-adic q-integral of a function $f \in \mathrm{UD}(\mathbb{Z}_p)$ on $\mathbb{Z}_p$ is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} f(j) q^j.$$

Note that if $f_n \to f$ in $UD(Z_p)$; then

$$\int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) \to \int_{\mathbb{Z}_p} f(x) d\mu_q(x).$$

It was well known that the ordinary Euler numbers are defined by

$$F(t) = \frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

where we use the technique method notation by replacing E^m by E_m $(m \geq 0)$, symbolically, cf.[2, 6]. In this paper, we introduce the definitions and properties of q-Euler numbers and polynomials. Finally we introduce formulae for the “ sums of products of q-Euler polynomials” .

§2. q-Euler and Genocchi numbers associated with p-adic q-integrals

The Euler polynomials are defined by means of the following generating function: $\frac{2}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$. Note that $E_n(0) = E_n$. From these Euler polynomials, we can evaluate the value of the below alternating sums of powers of consecutive integers:
(1)
$$-1^m + 2^m - 3^m + \cdots + (-1)^{m-1}(n-1)^m = \frac{1}{2}\left((-1)^{n+1}E_m(n) - E_m\right), \text{ cf. [11, 2, 3] .}$$

In the meaning of fermionic, we now consider the below p-adic q-integrals:

$$\int_{X_f} [x]_q^k d\mu_{-q}(x) = \int_{\mathbb{Z}_p} [x]_q^k d\mu_{-q}(x) = E_{k,q} \text{ for } k, f \in \mathbb{N}, \text{ cf. [7, 8, 11, 13, 14] .} \tag{2}$$

From the computation of this p-adic q-integral, we derive the below Eq.(3):

$$E_{k,q} = [2]_q \left(\frac{1}{1-q}\right)^k \sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{1}{1+q^{l+1}}, \text{ cf. [8, 11],} \tag{3}$$

where $\binom{k}{i}$ is the binomial coefficient. Note that $\lim_{q\to 1} E_{k,q} = E_k$. Hence, $E_{k,q}$ is q-extension of Euler numbers which are called q-Euler numbers. Let $F_q(t) = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}$ be the generating function of q-Euler numbers. Then we easily see that

$$F_q(t) = e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{[2]_q}{[2]_{q^{j+1}}} \left(\frac{1}{q-1}\right)^j \frac{t^j}{j!} = [2]_q \sum_{l=0}^{\infty} (-q)^l e^{[l]_q t}, \text{ cf. [6, 8, 9, 10.]} \tag{4}$$

By using an invariant p-adic q-integral on $\mathbb{Z}_p$, we can also consider the q-extension of ordinary Euler polynomials which are called q-Euler polynomials. For $x \in \mathbb{Z}_p$, we define q-Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} [x+y]_q^k d\mu_{-q}(y) = E_{k,q}(x), \text{ cf. [7, 8].} \tag{5}$$

By (5), we easily see that

$$E_{k,q}(x) = \sum_{n=0}^{k} \binom{k}{n} [x]_q^{k-n} q^{nx} E_{n,q}.$$

In Eq.(5), it is easy to see that

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{xk} \frac{1}{1+q^{k+1}}.$$

By using the definition of Eq.(5), we will give the distribution of q-Euler polynomials. From the definition of p-adic q-integral , , we derive the below formula:

$$\int_{X_m} [x+y]_q^n d\mu_{-q}(y) = \frac{[m]_q^n}{[m]_{-q}} \sum_{a=0}^{m-1} (-1)^a q^a \int_{\mathbb{Z}_p} [\frac{a+x}{m} + y]_{q^m}^n d\mu_{-q^m}(y), \text{ if } m \text{ is odd.}$$

Thus, if m is the odd integer, then we have

$$E_{n,q}(x) = \frac{[m]_q^n}{[m]_{-q}} \sum_{a=0}^{m-1} (-1)^a q^a E_{n,q^m}(\frac{a+x}{m}).$$

From the definition of the q-Euler polynomials, we note that

$$F_q(x,t) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t}, \text{ cf. [8].}$$

It is well known that Genocchi numbers are defined by

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}.$$

Thus, we easily see that $G_n = \sum_{l=0}^{n-1} \binom{n}{l} 2^l B_l$, where B_l are ordinary Bernoulli numbers. We now define the q-extension of Genocchi number which are called q-Genocchi numbers as follows:

$$F_q^*(t) = [2]_q t \sum_{l=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}. \tag{6}$$

By Eq.(6), we easily see that
(7)

$$G_{n,q} = n[2]_q \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{[2]_{q^{l+1}}} = [2]_q n \int_{\mathbb{Z}_p} [x]_q^{n-1} d\mu_q(x), \text{ when } n \text{ is odd .}$$

From Eq.(6), we can also derive the definition of q-Genocchi polynomials as follows:

$$F_q^*(x,t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}, \text{ when } n \text{ is odd.} \tag{8}$$

Let $a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k$ be positive integers . For $w \in \mathbb{Z}_p$, we define multiple Daehee q-Euler polynomials by using the invariant p-adic q-integrals as follows, cf. [7]:
(9)

$$E_n^{(k)}(w,q|a_1,a_2,\cdots,a_k : b_1,b_2,\cdots,b_k) = \int_{\mathbb{Z}_p^k} q^{\sum_{j=1}^k (b_j-1)x_j} [w + \sum_{j=1}^k a_j x_j]_q^n d\mu_{-q}(x),$$

and

$$E_n^{(k)}(q|a_1,\cdots,a_k : b_1,\cdots,b_k) = \int_{\mathbb{Z}_p^k} q^{\sum_{j=1}^k (b_j-1)x_j} [\sum_{j=1}^k a_j x_j]_q^n d\mu_{-q}(x),$$

where

$$\int_{\mathbb{Z}_p^k} f(x) d\mu_{-q}(x) = \underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} f(x) d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r).$$

From the Eq.(9), we can derive the below theorem:

Proposition. *Let $a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k$ be positive integers. Then we have*

$$E_n^{(k)}(w,q|a_1,\cdots,a_k : b_1,\cdots,b_k) = \frac{[2]_q^k}{(1-q)^n} \sum_{r=0}^{n} \binom{n}{r} (-q^w)^r \prod_{j=1}^{k} \left(\frac{1}{[2]_{q^{b_j + r a_j}}} \right). \tag{10}$$

Given elements $\alpha_1, \cdots, \alpha_m \in \mathbb{C}_p$ and positive integers $N_1, \cdots, N_m, n$, it is easy to see that

$$\begin{aligned}
&[N_1(x_1+\alpha_1) + \cdots + N_m(x_m+\alpha_m)]_q^n \\
&= \sum_{\substack{i_1,\cdots,i_m \geq 0 \\ i_1+\cdots+i_m = n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \cdots \sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}} \\
&\quad \times \binom{n}{i_1,\cdots,i_m} \binom{n-i_1}{k_1} \binom{n-i_1-i_2}{k_2} \cdots \binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}} \\
&\quad \times (q-1)^{k_1+\cdots+k_{m-1}} [N_1]_q^{i_1+k_1} \cdots [N_{m-1}]_q^{i_{m-1}+k_{m-1}} [N_m]_q^{i_m} \\
&\quad \times [x_1+\alpha_1 : q^{N_1}]^{k_1+i_1} \cdots [x_{m-1}+\alpha_{m-1} : q^{N_{m-1}}]^{k_{m-1}+i_{m-1}} [x_m+\alpha_m : q^{N_m}]^{i_m}.
\end{aligned} \tag{11}$$

Hence, we have

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{m \text{ times}}[N_1(x_1+\alpha_1)+\cdots+N_m(x_m+\alpha_m)]_q^n d\mu_{-q^{N_1}}(x_1)\cdots d\mu_{-q^{N_m}}(x_m)$$

$$\begin{aligned}
&= \sum_{\substack{i_1,\cdots,i_m\geq 0\\ i_1+\cdots+i_m=n}} \sum_{k_1=0}^{n-i_1}\sum_{k_2=0}^{n-i_1-i_2}\cdots\sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}}\\
&\times \binom{n}{i_1,\cdots,i_m}\binom{n-i_1}{k_1}\binom{n-i_1-i_2}{k_2}\cdots\binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}}\\
&\times (q-1)^{k_1+\cdots+k_{m-1}}[N_1]_q^{i_1+k_1}\cdots[N_{m-1}]_q^{i_{m-1}+k_{m-1}}[N_m]_q^{i_m}\\
&\times E_{k_1+i_1,q^{N_1}}(\alpha_1)\cdots E_{k_{m-1}+i_{m-1},q^{N_{m-1}}}(\alpha_{m-1})E_{i_m,q^{N_m}}(\alpha_m).
\end{aligned} \tag{12}$$

From (9), (10), (11) and (12), we can derive the below theorem:

Theorem. *(Sums of products of q-Euler polynomials)*
Given elements $\alpha_1,\cdots,\alpha_m\in\mathbb{C}_p$ *and positive integers* $N_1,\cdots,N_m,n$,

$$\begin{aligned}
&\sum_{\substack{i_1,\cdots,i_m\geq 0\\ i_1+\cdots+i_m=n}} \sum_{k_1=0}^{n-i_1}\sum_{k_2=0}^{n-i_1-i_2}\cdots\sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}}\\
&\times \binom{n}{i_1,\cdots,i_m}\binom{n-i_1}{k_1}\binom{n-i_1-i_2}{k_2}\cdots\binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}}\\
&\qquad\times (q-1)^{k_1+\cdots+k_{m-1}}[N_1]_q^{i_1+k_1}\cdots[N_{m-1}]_q^{i_{m-1}+k_{m-1}}[N_m]_q^{i_m}\\
&\qquad\quad\times E_{k_1+i_1,q^{N_1}}(\alpha_1)\cdots E_{k_{m-1}+i_{m-1},q^{N_{m-1}}}(\alpha_{m-1})E_{i_m,q^{N_m}}(\alpha_m)\\
&= E_n^{(m)}(N_1\alpha_1+\cdots+N_m\alpha_m,q|N_1,\cdots,N_m:1,1,\cdots,1),
\end{aligned}$$

where $\binom{n}{i_1,\cdots,i_m}$ *is multinomial coefficient.*

§3. Further Remarks and Observations

In this section, we assume that $q\in\mathbb{C}$ with $|q|<1$. Let $\Gamma(s)$ be the ordinary gamma function given by $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt$, $s\in\mathbb{C}$. From (8) and complex integral, we can derive the below formula:

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}F_q^*(x,-t)dt = [2]_q\sum_{n=0}^\infty \frac{(-1)^{n+1}q^{n+x}}{[n+x]_q},\ \text{for } s\in\mathbb{C}. \tag{13}$$

For $s \in \mathbb{C}$, we define the(Hurwitz's type) q-Genocchi zeta function as follows:

$$\zeta_{q,G}(s,x) = [2]_q\sum_{n=0}^\infty \frac{(-1)^{n+1}q^{x+n}}{[n+x]_q^s},\ \text{where } x\in\mathbb{R} \text{ with } 0<x<1. \tag{14}$$

By (8), (13) and (14), we easily see that

$$\zeta_{q,G}(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}F_q^*(x,-t)dt = \sum_{n=0}^\infty \frac{G_{n,q}(x)}{n!}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{n+s-2}dt\right). \tag{15}$$

By using the residue theorem in Eq.(15), we easily see that

$$\zeta_{q,G}(1-n,x) = \frac{(-1)^{n-1}}{n}G_{n,q}(x),\ n\in\mathbb{N}\ .$$

References

[1] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), 288-299.
[2] T. Kim, *On a q-analogue of the p-adic* log *gamma functions and related integrals*, J. Number Theory **76** (1999), 320-329.
[3] T.Kim, *Sums powers of consecutive q-integers*, Advan. Stud. Contemp. Math. **9** (2004), 15-18.
[4] T.Kim, *Power series and asymptotic series associated with the q-analog of the two variable p-adic L-function*, Russian J. Math. Phys. **12 no. 2** (2005), 186-196.
[5] T.Kim, *Non-archimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials*, Russian J. Math. Phys. **10 no. 1** (2003), 91-98.
[6] T.Kim, *p-adic q-integral associated with the Changhee-Barnes' q-Bernoulli polynomials*, Integral Transforms Spec. Funct. **15** (2004), 415-420.
[7] T.Kim, *An invariant p-adic integral associated with Daehee numbers*, Integral Transforms Spec. Funct. **13** (2002), 65-69.
[8] T.Kim, *A note on q-Volkenborn integration*, Proc. Jangjeon Math. Soc. **8** (2005), 13-17.
[9] T.Kim, S. H. Rim, *On Changhee-Barnes' q-Euler numbers and polynomials*, Advan. Stud. Contemp. Math. **9** (2004), 81-86.
[10] T.Kim, *A note on q-zeta functions*, Proceedings of the 15th international conference of the Jangjeon Mathematical Society(Hapcheon, South Korea, August 5-7, 2004). **1** (2004), 110-114.
[11] T. Kim, *On the q-extension of Euler and Genocchi numbers*, J. Math. Anal. Appl. **326** (2007), 1458-1465.
[12] T. Kim, S.-H. Rim, Y. Simsek, *A note on the alternating sums of powers of consecutive q-integers*, Advan. Stud. Contemp. Math. **13** (2006), 159-164.

[13] M. Cenkci, M. Can, *Some results on q-analogue of the Lerch zeta function,*, Advan. Stud. Contemp. Math. **12** (2006), 213-223.

[14] Y. Simsek, *Transformation of four Titchmarsh-type infinite integrals and generalized Dedekind sums associated with Lambert series*, Advan. Stud. Contemp. Math. **9** (2004), 195-202.

[15] H. Ozden, Y. Simsek, S.H. Rim, I. Cangul, *A note on p-adic q-Euler measure*, Advan. Stud. Contemp. Math. **14** (2007), 233-239.

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Calculating zeros of (h, q)-extension of the Euler numbers and polynomials

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Abstract In this paper, our goal is to investigate the roots of the $E_{n,q}^{(h)}(z)$ for values of the index n by using computer. By numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the $E_{n,q}^{(h)}(z)$. Finally, we give a table for the solutions of the (h, q)-extension of the q-Euler polynomials $E_{n,q}^{(h)}(z)$.

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Key words- Euler numbers and polynomials, q-Euler numbers and polynomials, (h, q)-extension of the q-Euler numbers and polynomials

1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

$$[x]_q = [x : q] = \frac{1-q^x}{1-q}, \text{ cf. } [1,2,3,4,5,9]\ .$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. The Euler numbers E_n are usually defined by means of the following generating function:

$$F(t) = e^{Et} = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \text{ cf. } [1,3,5,6,7]$$

where the symbol E_n is interpreted to mean that E^n must be replaced by E_n when we expand the one on the left. These numbers are classical and important in mathematics and in various places like analysis, number theory. The Euler polynomials $E_n(z)$ are usually defined by means of the following generating function:

$$F(t, z) = \frac{2}{e^t+1} e^{zt} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}.$$

For $g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiaable function $\}$, the p-adic q-integral was defined by [6]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{0 \leq x < p^N} g(x) q^x, \text{ cf. } [1,2,3,4,5,8,9].$$

Now, we consider the case $q \in (-1, 0)$ corresponding to q-deformed fermionic certain and annihilation operators and the literature given therein [2,3,4,6,7]. The expression for the $I_q(g)$ remains same, so it is tempting to consider the limit $q \to -1$. That is,

$$I_{-1}(g) = \lim_{q \to -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \le x < p^N} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

Ryoo, Kim and Agarwal [6] treated analogue of Euler numbers, which is called q-Euler numbers in this paper. By using p-adic q-integral, we defined the q-Euler numbers $E_{n,q}$ as follows:

$$E_{n,q} = \int_{\mathbb{Z}_p} [t]_q^n d\mu_{-q}(t), \text{ for } n \in \mathbb{N}. \tag{1.3}$$

The q-Euler numbers $E_{n,q}$ are defined by means of the generating function

$$F_q(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t}. \tag{1.4}$$

Similarly, the generating function $F_q(t, z)$ of the q-Euler polynomials $E_{n,q}(z)$ is defined analogously as follows:

$$F_q(t, z) = \sum_{n=0}^{\infty} E_{n,q}(z) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+z]_q t}, \tag{1.5}$$

see [6], for details. In this paper, we introduce the (h, q)-extension of the q-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(z)$. We study some properties which are related to the (h, q)-extension of the q-Euler numbers and polynomials. Our aim in this paper is to we investigate the roots of the $E_{n,q}^{(h)}(z)$ for values of the index n by using computer.

2. (h, q)-extension of the q-Euler numbers and polynomials

Our primary goal of this section is to construct the (h, q)-extension of the q-Euler numbers and polynomials. We also find generating functions of the (h, q)-extension of the q-Euler numbers and polynomials. For $h \in \mathbb{Z}, q \in \mathbb{C}_p$ with $|1 - p|_p \le p^{-\frac{1}{p-1}}$, the (h, q)-extension of the q-Euler numbers $E_{n,q}^{(h)}$ are defined by

$$E_{n,q}^{(h)} = \int_{\mathbb{Z}_p} q^{x(h-1)} [x]_q^n d\mu_{-q}(x). \tag{2.1}$$

By using p-adic q-integral, we have

$$E_{n,q}^{(h)} = \frac{[f]_q^n}{[f]_{-q}} \sum_{a=0}^{f-1} (-1)^a q^{ha} E_{n,q^f}^{(h)} \quad \frac{a}{f} \quad .$$

Note that

$$\int_{\mathbb{Z}_p} q^{x(h-1)} [x]_q^n d\mu_{-q}(x) = [2]_q \quad \frac{1}{1-q} \quad {}^{n} \sum_{l=0}^{n} \genfrac{}{}{0pt}{}{n}{l} \; (-1)^l \frac{1}{1 + q^{(h+l)}}.$$

Hence we have

$$E_{n,q}^{(h)} = [2]_q \quad \frac{1}{1-q} \quad {}^{n} \sum_{l=0}^{n} \genfrac{}{}{0pt}{}{n}{l} \; (-1)^l \frac{1}{1 + q^{(h+l)}}.$$

By using above equation, we have

$$
\begin{aligned}
F_q^{(h)}(t) &= \sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} \left(\frac{1}{1-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{h+1}} \Bigg) \frac{t^n}{n!} \\
&= [2]_q \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m q^{hm} [m]_q^n \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^{hm} e^{[m]_q t}
\end{aligned}
\tag{2.2}
$$

Here is the list of the first the (h, q)-extension of the q-Euler numbers $E_{n,q}^{(h)}$.

$$
\begin{aligned}
E_{1,q}^{(h)} &= -\frac{q^h(1+q)}{(1+q^h)(1+q^{1+h})}, \quad E_{2,q}^{(h)} = \frac{q^h(1+q)(-1+q^{1+h})}{(1+q^h)(1+q^{1+h})(1+q^{2+h})}, \\
E_{3,q}^{(h)} &= -\frac{q^h(1+q)(1-2q^{1+h}-2q^{2+h}+q^{3+2h})}{(1+q^h)(1+q^{1+h})(1+q^{2+h})(1+q^{3+h})}, \\
E_{4,q}^{(h)} &= \frac{q^h(1+q)(-1+q^{2+h})(1-3q^{1+h}-4q^{2+h}-3q^{3+h}+q^{4+2h})}{(1+q^h)(1+q^{1+h})(1+q^{2+h})(1+q^{3+h})(1+q^{4+h})}, \cdots
\end{aligned}
$$

We display the shapes of the (h, q)-extension of the q-Euler numbers $E_{n,q}^{(h)}$. For $n = 1, \cdots, 10$, we can draw a curve of $E_{n,q}^{(h)}, 1/10 \le q \le 9/10$, respectively. This shows the ten curves combined into one. We display the shape of $E_{n,q}^{(h)}$. (Figures 1, 2). Thus (h, q)-extension of the

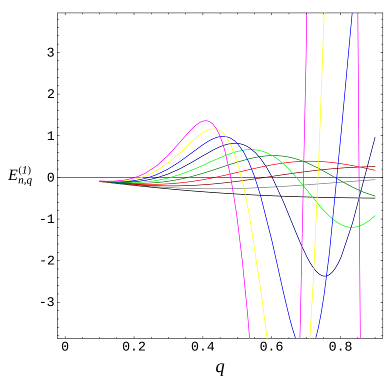

Figure 1: Curvers of $E_{n,q}^{(1)}$

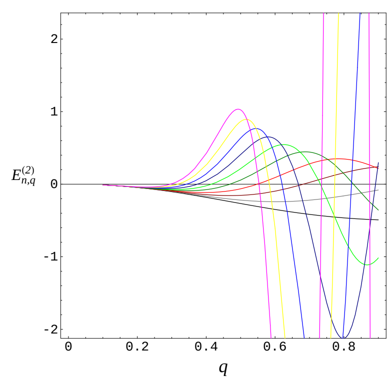

Figure 2: Curvers of $E_{n,q}^{(2)}$

q-Euler numbers $E_{n,q}^{(h)}$ are defined by means of the generating function

$$
F_q^{(h)}(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[n]_q t}. \tag{2.3}
$$

Note that, if $q \to 1$, then $F_q^{(h)}(t) \to F(t)$. By using (2.1), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{x(h-1)} [x]_q^n d\mu_{-q}(x) \frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} q^{x(h-1)} [x]_q^n \frac{t^n}{n!} d\mu_{-q}(x) = \int_{\mathbb{Z}_p} q^{x(h-1)} e^{[x]_q t} d\mu_{-q}(x).
\end{aligned}
\tag{2.4}
$$

By (2.2), (2.4), we have

$$
\int_{\mathbb{Z}_p} q^{x(h-1)} e^{[x]_q t} d\mu_{-q}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^{hm} e^{[m]_q t}.
$$

Observe that if $h = 1$, then (2.3) reduces (1.4). Next, we introduce the (h, q)-extension of the q-Euler polynomials $E_{n,q}^{(h)}(z)$. The (h, q)-extension of the q-Euler polynomials $E_{n,q}^{(h)}(z)$ are defined by

$$E_{n,q}^{(h)}(z) = \int_{\mathbb{Z}_p} q^{x(h-1)}[z+x]_q^n d\mu_{-q}(x). \tag{2.5}$$

By using p-adic q-integral, we obtain

$$E_{n,q}^{(h)}(z) = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{zl} \frac{1}{1+q^{(h+l)}}. \tag{2.6}$$

Here is the list of the first the (h, q)-extension of the q-Euler polynomials $E_{n,q}^{(h)}(z)$.

$$\begin{aligned}
E_{1,q}^{(h)}(z) &= -\frac{(1+q)(1+q^{1+h}-q^z-q^{h+z})}{(-1+q)(1+q^h)(1+q^{1+h})},\\
E_{2,q}^{(h)}(z) &= \frac{(1+q)(1+q^{1+h}+q^{2+h}+q^{3+2h}}{(-1+q)^2(1+q^h)(1+q^{1+h})(1+q^{2+h}))}\\
&\quad+\frac{(1+q)(-2q^z+q^{2z}-2q^{h+z}-2q^{2+h+z}-2q^{2+2h+x}+q^{h+2z}+q^{1+h+2z}+q^{1+2h+2z})}{(-1+q)^2(1+q^h)(1+q^{1+h})(1+q^{2+h})},\\
&\quad\cdots.
\end{aligned}$$

We set

$$F_q^{(h)}(t,z) = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(z)\frac{t^n}{n!}. \tag{2.7}$$

By using (2.6) and (2.7), we obtain

$$F_q^{(h)}(t,z) = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(z)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty}(-1)^m q^{hm} e^{[m+z]_q t}.$$

Observe that, if $q \to 1$, then $F_q^{(h)}(t,z) \to F(t,z)$. Similarly, the generating function $F_q^{(h)}(t,z)$ of the (h, q)-extension of the q-Euler polynomials $E_{n,q}^{(h)}(z)$ is defined analogously as follows:

$$F_q^{(h)}(t,z) = \sum_{n=0}^{\infty} E_{n,q}(z)\frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty}(-1)^n q^{hn} e^{[n+z]_q t}. \tag{2.8}$$

By (2.5), we obtain

$$E_{n,q}^{(h)}(z) = \frac{[f]_q^n}{[f]_{-q}} \sum_{a=0}^{f-1}(-1)^a q^{ha} E_{n,q^f}^{(h)}\left(\frac{a+z}{f}\right).$$

Note that, if $h = 1$, then (2.8) reduces (1.5). By using (2.8), we easily see that

$$\int_{\mathbb{Z}_p} q^{x(h-1)} e^{[z+x]_q t} d\mu_{-q}(x) = [2]_q \sum_{n=0}^{\infty}(-1)^n q^{hn} e^{[n+z]_q t}.$$

Since $[x+z]_q = [z]_q + q^z[x]_q$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}^{(h)}(z)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{x(h-1)} e^{[x+z]_q t} d\mu_{-q}(x)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} q^{lz}[z]_q^{n-l}\int_{\mathbb{Z}_p} q^{x(h-1)}[x]_q^l d\mu_{-q}(x)\right)\frac{t^n}{n!}.
\end{aligned}$$

By using comparing coefficients $\frac{t^n}{n!}$, we easily see that

$$E_{n,q}^{(h)}(z) = \sum_{l=0}^{n}\binom{n}{l} q^{lz}[z]_q^{n-l} E_{n,q}^{(h)}.$$

Observe that, if $q \to 1$, then $E_{n,q}^{(h)} \to E_n$ and $E_{n,q}^{(h)}(z) \to E_n(z)$, where $E_n(z)$ are the Euler polynomials.

3. Distribution and Structure of the Zeros

In this section, we investigate the zeros of the (h, q)-extension of the q-Euler polynomials $E^{(h)}_{n,q}(z)$ by using computer. For $n = 1, \cdots, 10$, we can draw a curve of $E^{(h)}_{n,q}(z), q = 1/2, -1 \leq z \leq 1$, respectively. This shows the ten curves combined into one. We display the shape of $E^{(h)}_{n,q}(z)$. (Figures 3, 4). We plot the zeros of $E^{(h)}_{n,q}(z), z \in \mathbb{C}$ for $n = 10, 20, 30, 40, q = 1/2$,

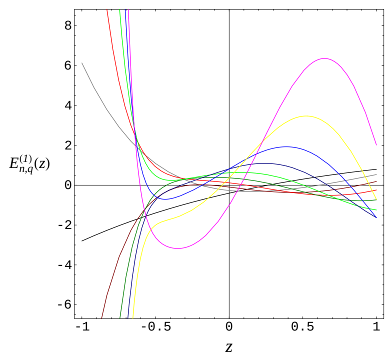

Figure 3: Curvers of $E^{(1)}_{n,q}(z)$

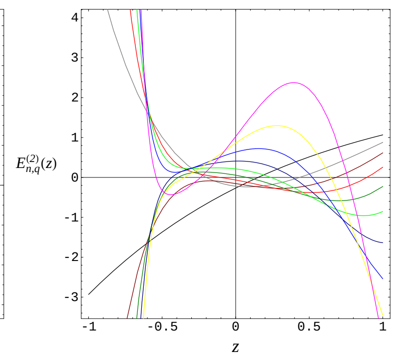

Figure 4: Curvers of $E^{(2)}_{n,q}(z)$

and $h = 4$. (Figures 5, 6, 7, and 8). Next, we plot the zeros of $E^{(h)}_{n,q}(z), z \in \mathbb{C}$ for

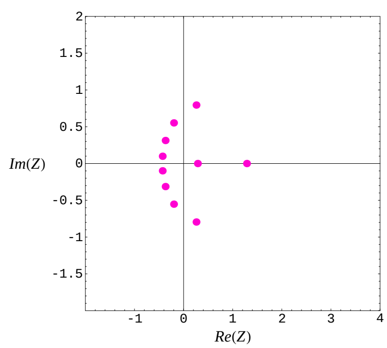

Figure 5: Zeros of $E^{(4)}_{10,q}$

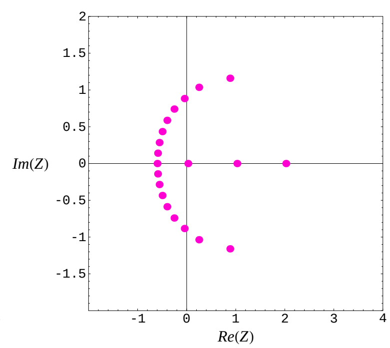

Figure 6: Zeros of $E^{(4)}_{20,q}$

$n = 40, h = 5, 7, 9, 11, q = 1/2$. (Figures 9, 10, 11, and 12). In Figures 5, 6, 7, 8, 9, 10, 11, and 12, $E^{(h)}_{n,q}(z), z \in \mathbb{C}$, has $Im(z) = 0$ reflection symmetry. This translates to the following open problem: Prove or disprove that $E^{(h)}_{n,q}(z), z \in \mathbb{C}$, has $Im(z) = 0$ reflection symmetry. Our numerical results for numbers of real and complex zeros of $E^{(h)}_{n,q}(z)$ are displayed in Table 1. In general, how many roots does $E^{(h)}_{n,q}(z)$ have ? Prove or disprove: $E^{(h)}_{n,q}(z)$ has n distinct solutions. Find the numbers of complex zeros $C_{E^{(h)}_{n,q}(z)}$ of $E^{(h)}_{n,q}(z), Im(z) \neq 0$. Prove or give a

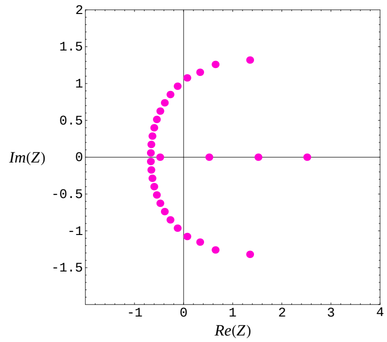

Figure 7: Zeros of $E_{30,q}^{(4)}$

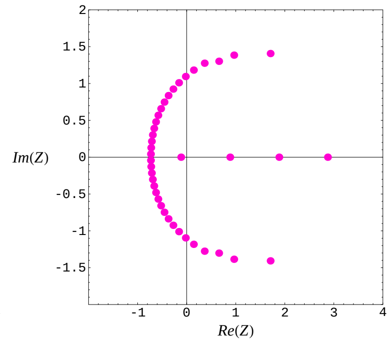

Figure 8: Zeros of $E_{40,q}^{(4)}$

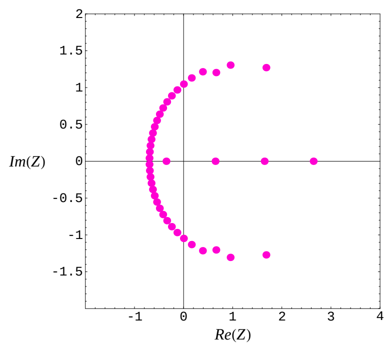

Figure 9: Zeros of $E_{40,q}^{(5)}$

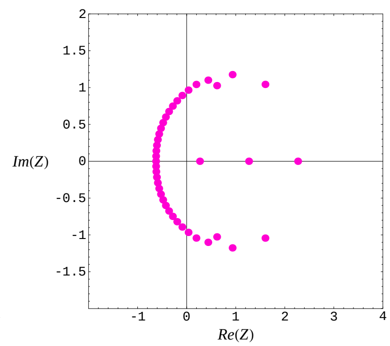

Figure 10: Zeros of $E_{40,q}^{(7)}$

counterexample: *Conjecture*: Since n is the degree of the polynomial $E_{n,q}^{(h)}(z)$, the number of real zeros $R_{E_{n,q}^{(h)}(z)}$ lying on the real plane $Im(z) = 0$ is then $R_{E_{n,q}^{(h)}(z)} = n - C_{E_{n,q}^{(h)}(z)}$, where $C_{E_{n,q}^{(h)}(z)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q}^{(h)}(z)}$ and $C_{E_{n,q}^{(h)}(z)}$. We calculated an approximate solution satisfying $E_{n,q}^{(h)}(z), q = 1/2, z \in \mathbb{R}$. The results are given in Table 2 and Table 3.

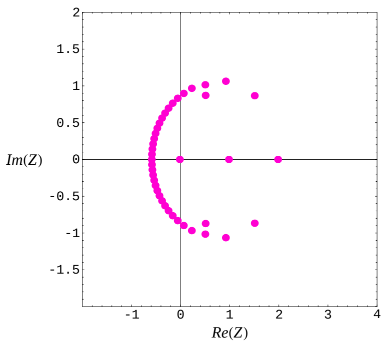

Figure 11: Zeros of $E_{40,q}^{(9)}$

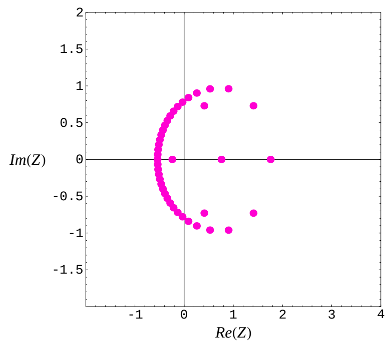

Figure 12: Zeros of $E_{40,q}^{(11)}$

Table 1. Numbers of real and complex zeros of $E_{n,q}^{(h)}(z)$

degree n	$h=5$		$h=7$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	1	2	1	2
4	2	2	2	2
5	1	4	1	4
6	2	4	2	4
7	3	4	1	6
8	2	6	2	6
9	3	6	1	8
10	2	8	2	8
11	3	8	3	8
12	2	10	2	10
13	3	10	3	10

Table 2. Approximate solutions of $E_{n,q}^{(h)}(z)=0, h=5, z\in\mathbb{R}$

degree n	z
1	0.0220263
2	-0.108814, 0.141981
3	-0.108814
4	-0.21906, 0.432345
5	0.566711
6	-0.242223, 0.691324
7	-0.286074, -0.19714, 0.806877
8	-0.0856786, 0.914321
9	-0.379364, 0.0145838, 1.01458
10	0.108493, 1.10849

Table 3. Approximate solutions of $E_{n,q}^{(h)}(z) = 0, h = 7, z \in \mathbb{R}$

degree n	z
1	0.00560271
2	−0.059284, 0.0676963
3	0.166937
4	−0.173491, 0.274515
5	0.380837
6	−0.234947, 0.482881
7	0.579819
8	−0.255598, 0.671612
9	0.75851
10	−0.159105, 0.84086

Finally, we shall consider the more general problems. Find the equation of envelope curves bounding the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs. We can draw a plot of zeros of the $E_{n,q}^{(h)}(z)$, respectively (Figures 5, 6, 7, 8, 9, 10, 11, and 12) . These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of roots of the $E_{n,q}^{(h)}(z)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the $E_{n,q}^{(h)}(z)$ to appear in mathematics and physics. For related topics the interested reader is referred to [6], [7], [8].

References

[1] M. Cenkci, M. Can, V. Kurt, *q-adic interpolation functions and kummer type congruence for q-twisted and q-generalized twisted Euler numbers*, Advan. Stud. Contemp. Math., 2004(9), 203-216.

[2] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., 2002(9), 288-299.

[3] T. Kim, *On the analogue of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$* , Journal of Mathematical Analysis and Applications, in Press.

[4] T. Kim, *A note on p-adic invariant integral in the rings of p-adic integers*, Adv. Stud. Contemp. Math., 2006(13), 95-99.

[5] C. S. Ryoo, T. Kim, *An analogue of the zeta function and its applications*, Applied Mathematics Letters, 2006(19), 1068-1072.

[6] C. S. Ryoo, T. Kim, R.P. Agarwal, *A numerical investigation of the roots of q-polynomials*, Inter. J. Comput. Math., 2006(83), 223-234.

[7] C. S. Ryoo, T. Kim, L. C. Jang, *A note on generalized Euler numbers and polynomials* , Inter. J. Comput. Math., in Press.

[8] C. S. Ryoo, H. Song, R.P. Agarwal, *On the roots of the q-analogue of Euler-Barnes' polynomials*, Adv. Stud. Contemp. Math., 2004(9), 153-163.

[9] Y. Simsek, *Theorem on twisted L-function and twisted Bernoulli numbers*, Advan. Stud. Contemp. Math., 2006(12), 237-246.

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FIXED POINT THEOREM IN $\mathcal{M}$-FUZZY METRIC SPACES FOR A CLASS MAPS

SHABAN SEDGHI AND NABI SHOBE

ABSTRACT. In this paper, we give some new definitions of $\mathcal{M}$-fuzzy metric spaces and we prove a common fixed point theorem for six mappings under the condition of weak compatible mappings of type first or second in complete $\mathcal{M}$-fuzzy metric spaces. We get some improved versions of several fixed point theorems in complete $\mathcal{M}$-fuzzy metric spaces.

1. Introduction and Preliminaries

Dhage [1] introduced the notion of generalized metric or D-metric as follows:
Let X be a nonempty set. A generalized metric (or D-metric) on X is a function $D : X^3 \longrightarrow [0, \infty)$ that satisfies the following conditions for each $x, y, z, a \in X$,

(1) $D(x, y, z) = 0$ if and only if $x = y = z$,

(2) $D(x, y, z) = D(p\{x, y, z\})$, where p is a permutation function,(symmetry)

(3) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ (Tetrahedral inequality).

The pair (X, D) is called a generalized metric space or a D-metric space .
A sequence $\{x_n\}$ in (X, D) is said to be convergent to x in X if and only if $D(x_n, x_m, x) \to 0$ as $n, m \to \infty$. $\{x_n\}$ in X is called D- Cauchy sequence if $D(x_n, x_m, x_p) \to 0$ as $n, m, p \to \infty$. The D-metric space (X, D) is said to be complete if every D-Cauchy sequence in X is convergent in X.
Let (X, D) be a D-metric space. For x_o in X and $r > 0$,the set

$$B(x_o, r) = \{y \in X : D(x_o, y, y) < r\}$$

is called an open ball with center x_o and radius r.
Dhage [2] claimed that D-metric convergence defines a Hausdorff topology and D-metric is sequentially continuous in all the three variables. Several authors have taken these claims for granted and used them in proving some fixed point theorems in D-metric spaces. Unfortunately, almost all theorems in D-metric spaces are either false or of doubtful validity (see [7, 8, 9]). Recently Sedghi,Rao and Shobe [6] introduced D^*-metric space by modifying the tetrahedral inequality in D-metric space and proved some basic results in it, which are not true in D-metric space.In this paper with using D^*-metric concepts, we define $\mathcal{M}$-fuzzy metric space and prove some results in it. We need some definitions and lemmas given by Sedghi,Rao and Shobe [6].Let $\mathbb{R}^+$ be the set of all positive real numbers and $\mathbb{N}$ be the set of all natural numbers.

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Definition 1.1. Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function: $D^* : X^3 \longrightarrow [0, \infty)$ that satisfies the following conditions for each $x, y, z, a \in X$,

(1) $D^*(x, y, z) = 0$ if and only if $x = y = z$,
(2) $D^*(x, y, z) = D^*(p\{x, y, z\})$,(symmetry) where p is a permutation function,
(3) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a generalized metric (or D^*-metric) .

Immediate examples of such a function are

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = \mathbb{R}^n$ then we define

$$D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for every $p \in [1, \infty)$.

(d) If $X = \mathbb{R}^+$ then we define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise }, \end{cases}$$

Remark 1.2. In a D^*-metric space, we have $D^*(x, x, y) = D^*(x, y, y)$.

Remark 1.3. Let (X, D^*) be a D^*-metric space. If we define $f : X^2 \longrightarrow [0, \infty)$ as $f(x, y) = D^*(x, x, y)$ for all $x, y \in X$ then f is an ordinary metric on X.

Let (X, D^*) be a D^*-metric space. For $r > 0$ define

$$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}.$$

Example 1.4. Let $X = \mathbb{R}$. Denote $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_{D^*}(1, 2) &= \{y \in \mathbb{R} : D^*(1, y, y) < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| < 1\} = (0, 2). \end{aligned}$$

Definition 1.5. Let (X, D^*) be a D^*-metric space and $A \subset X$.

(1) If for every $x \in A$ there exists $r > 0$ such that $B_{D^*}(x, r) \subset A$, then subset A is called open subset of X.

(2) Subset A of X is said to be D^*-bounded if there exists $r > 0$ such that $D^*(x, y, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $D^*(x_n, x_n, x) \to 0$ as $n \to \infty$.

(4) A sequence $\{x_n\}$ in X is called a Cauchy sequence $D^*(x_n, x_m, x_p) \to 0$ as $n, m, p \to \infty$. The D^*-metric space (X, D^*) is said to be complete if every Cauchy sequence in X is convergent in X .

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_{D^*}(x, r) \subset A$. Then τ is a topology on X (induced by the D^*-metric D^*).

Lemma 1.6. *Let (X, D^*) be a D^*-metric space. If $r > 0$, then the ball $B_{D^*}(x, r)$ with center $x \in X$ and radius r is an open subset of X.*

Lemma 1.7. *Let (X, D^*) be a D^*- metric space. Then D^* is continuous function on X^3.*

Lemma 1.8. *Let (X, D^*) be a D^*-metric space. If sequence $\{x_n\}$ in X converges to x,then x is unique.*

Lemma 1.9. *Let (X, D^*) be a D^*-metric space. If sequence $\{x_n\}$ in X is converges to x ,then sequence $\{x_n\}$ is a Cauchy sequence.*

Definition 1.10. [4] A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0, 1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.
Also the following definitions and lemmas given by Sedghi,Rao and Shobe [6].

Definition 1.11. A 3-tuple $(X, \mathcal{M}, *)$ is called a $\mathcal{M}$-fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,
(1) $\mathcal{M}(x, y, z, t) > 0$,
(2) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,
(3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$,(symmetry) where p is a permutation function,
(4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,
(5) $\mathcal{M}(x, y, z, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Remark 1.12. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. We prove that for every $t > 0$, $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$. Because for each $\epsilon > 0$ by triangular inequality we have
(i) $\mathcal{M}(x, x, y, \epsilon + t) \geq \mathcal{M}(x, x, x, \epsilon) * \mathcal{M}(x, y, y, t) = \mathcal{M}(x, y, y, t)$
(ii) $\mathcal{M}(y, y, x, \epsilon + t) \geq \mathcal{M}(y, y, y, \epsilon) * \mathcal{M}(y, x, x, t) = \mathcal{M}(y, x, x, t)$.
By taking limits of (i) and (ii) when $\epsilon \longrightarrow 0$, we obtain $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space . For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_{\mathcal{M}}(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.
A sequence $\{x_n\}$ in X converges to x if and only if $\mathcal{M}(x, x, x_n, t) \longrightarrow 1$ as $n \longrightarrow \infty$,for each $t > 0$. It is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$. The $\mathcal{M}$-fuzzy metric $(X, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence is convergent.

Example 1.13. Let X is a nonempty set and D^* is the D^*-metric on X. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in]0, \infty[$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$$

for all $x, y, z \in X$. It is easy to see that $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space.

Lemma 1.14. *Let $(X, M, *)$ is a fuzzy metric space. If we define $\mathcal{M} : X^3 \times (0, \infty) \longrightarrow [0, 1]$ by*

$$\mathcal{M}(x, y, z, t) = M(x, y, t) * M(y, z, t) * M(z, x, t)$$

for every x, y, z in X, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space.

Definition 1.15. Let $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space, then $\mathcal{M}$ is called of $first\ type$ if for every $x, y \in X$ we have

$$\mathcal{M}(x, x, y, t) \geq \mathcal{M}(x, y, z, t)$$

for every $z \in X$.
Also it is called of *second type* if for every $x, y, z \in X$ we have

$$\mathcal{M}(x, y, z, t) = M(x, y, t) * M(y, z, t) * M(z, x, t).$$

Let $a * b = \min(a, b)$ for every $a, b \in [0, 1]$ in this case it is easy to see that, if $\mathcal{M}$ is second type then $\mathcal{M}$ is first type.

Example 1.16. If we define $\mathcal{M}(x, y, z, t) = \frac{t}{t+D^*(x,y,z)}$ where $D^*(x, y, z) = d(x, y) + d(y, z) + d(x, z)$, or define $\mathcal{M}(x, y, z, t) = \frac{t}{t+\max\{x,y,z\}}$ then $\mathcal{M}$ is first type .
If $(X, M, *)$ is a fuzzy metric and $M(x, y, t) = \frac{t}{t+d(x,y)}$, then

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + d(x, y)} * \frac{t}{t + d(y, z)} * \frac{t}{t + d(x, z)}$$

is second type.

Remark 1.17. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. If $\mathcal{M}$ is *second* type, sequence $\{x_n\}$ in X converges to x if and only if $\mathcal{M}(x, x, x_n, t) \longrightarrow 1$ or if and only if $M(x, x_n, t) \longrightarrow 1$.
For

$$\mathcal{M}(x, x, x_n, t) = M(x, x, t) * M(x, x_n, t) * M(x, x_n, t) = M(x, x_n, t) * M(x, x_n, t)$$

Lemma 1.18. *Let $(X, M, *)$ be a fuzzy metric space. If we define $E_{\lambda,M} : X^2 \rightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,M}(x, y) = \inf\{t > 0 \ : \ M(x, y, t) > 1 - \lambda\}$$

for each $\lambda \in (0, 1)$ and $x, y \in X$. Then we have

(i) *For any $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that*

$$E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)$$

for any $x_1, x_2, ..., x_n \in X$.

(ii) *The sequence $\{x_n\}_{n\in N}$ is convergent in fuzzy metric space $(X, M, *)$ if and only if $E_{\lambda,M}(x_n, x) \rightarrow 0$. Also the sequence $\{x_n\}_{n\in N}$ is Cauchy sequence if and only if it is Cauchy with $E_{\lambda,M}$.*

Proof. See [5] □

Lemma 1.19. *Let (X,M,*) be a fuzzy metric space. If*

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)$$

for some $k > 1$ and for every $n \in \mathbb{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

2. The Main Results

Lemma 2.1. [6] *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is nondecreasing with respect to t, for all x, y, z in X.*

Definition 2.2. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. M is said to be continuous function on $X^3 \times (0, \infty)$ if

$$\lim_{n\to\infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t)$$

Whenever a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z, t) \in X^3 \times (0, \infty)$ i.e.

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z \text{ and } \lim_{n\to\infty} \mathcal{M}(x, y, z, t_n) = \mathcal{M}(x, y, z, t)$$

Lemma 2.3. [6]*Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X^3 \times (0, \infty)$.*

In 1998, Jungck and Rhoades [3] introduced the following concept of weak compatibility.

Definition 2.4. Let A and S be mappings from a $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 2.5. Let A and S be mappings from a $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} \mathcal{M}(ASx_n, SAx_n, SAx_n, t) = 1, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

Lemma 2.6. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. If we define $E_{\lambda,\mathcal{M}} : X^3 \to \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,\mathcal{M}}(x, y, z) = \inf\{t > 0 \ : \ \mathcal{M}(x, y, z, t) > 1 - \lambda\}$$

for every $\lambda \in (0, 1)$, then

(i) *for each $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that*

$$E_{\mu,\mathcal{M}}(x_1, x_1, x_n) \le E_{\lambda,\mathcal{M}}(x_1, x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_2, x_3) + \cdots + E_{\lambda,\mathcal{M}}(x_{n-1}, x_{n-1}, x_n)$$

for any $x_1, x_2, ..., x_n \in X$

(ii) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ if and only if $E_{\lambda,\mathcal{M}}(x_n, x_n, x) \to 0$. Also the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy sequence if and only if it is Cauchy with $E_{\lambda,\mathcal{M}}$.*

Lemma 2.7. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. If $\mathcal{M}$ is first type and*

$$\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t) \ge \mathcal{M}(x_0, x_1, x_2, k^n t)$$

for some $k > 1$ and for every $n \in \mathbb{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. Since $\mathcal{M}$ is first type hence for every $\lambda \in (0,1)$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned} E_{\lambda,\mathcal{M}}(x_n, x_n, x_{n+1}) &= \inf\{t>0 \ : \ \mathcal{M}(x_n, x_n, x_{n+1}, t) > 1-\lambda\} \\ &\leq \inf\{t>0 \ : \ \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t) > 1-\lambda\} \\ &\leq \inf\{t>0 \ : \ \mathcal{M}(x_0, x_1, x_2, k^n t) > 1-\lambda\} \\ &= \inf\{\frac{t}{k^n}>0 \ : \ \mathcal{M}(x_0, x_1, x_2, t) > 1-\lambda\} \\ &= \frac{1}{k^n}\inf\{t>0 \ : \ \mathcal{M}(x_0, x_1, x_2, t) > 1-\lambda\} \\ &= \frac{1}{k^n}E_{\lambda,\mathcal{M}}(x_0, x_1, x_2). \end{aligned}$$

By Lemma 2.6, for every $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} E_{\mu,\mathcal{M}}(x_n, x_n, x_m) &\leq E_{\lambda,\mathcal{M}}(x_n, x_n, x_{n+1}) + E_{\lambda,\mathcal{M}}(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,\mathcal{M}}(x_{m-1}, x_{m-1}, x_m) \\ &\leq \frac{1}{k^n}E_{\lambda,\mathcal{M}}(x_0, x_1, x_2) + \frac{1}{k^{n+1}}E_{\lambda,\mathcal{M}}(x_0, x_1, x_2) + \cdots + \frac{1}{k^{m-1}}E_{\lambda,\mathcal{M}}(x_0, x_1, x_2) \\ &= E_{\lambda,\mathcal{M}}(x_0, x_1, x_2)\sum_{j=n}^{m-1}\frac{1}{k^j} \longrightarrow 0. \end{aligned}$$

Hence sequence $\{x_n\}$ is Cauchy sequence. □

Lemma 2.8. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. If $\mathcal{M}$ is second type and*

$$\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t) \geq \mathcal{M}(x_0, x_1, x_2, k^n t)$$

for some $k > 1$ and for every $n \in \mathbb{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. Since $\mathcal{M}$ is second type hence for every $\lambda \in (0,1)$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned} E_{\lambda,M}(x_n, x_{n+1}) &= \inf\{t>0 \ : \ M(x_n, x_{n+1}, t) > 1-\lambda\} \\ &\leq \inf\{t>0 \ : \ \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t) > 1-\lambda\} \\ &\leq \inf\{t>0 \ : \ \mathcal{M}(x_0, x_1, x_2, k^n t) > 1-\lambda\} \\ &= \inf\{\frac{t}{k^n}>0 \ : \ \mathcal{M}(x_0, x_1, x_2, t) > 1-\lambda\} \\ &= \frac{1}{k^n}\inf\{t>0 \ : \ \mathcal{M}(x_0, x_1, x_2, t) > 1-\lambda\} \\ &= \frac{1}{k^n}E_{\lambda,\mathcal{M}}(x_0, x_1, x_2). \end{aligned}$$

By Lemma 1.18, for every $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} E_{\mu,M}(x_n, x_m) &\leq E_{\lambda,M}(x_n, x_{n+1}) + E_{\lambda,M}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,M}(x_{m-1}, x_m) \\ &\leq \frac{1}{k^n}E_{\lambda,\mathcal{M}}(x_0, x_1, x_2) + \frac{1}{k^{n+1}}E_{\lambda,\mathcal{M}}(x_0, x_1, x_2) + \cdots + \frac{1}{k^{m-1}}E_{\lambda,\mathcal{M}}(x_0, x_1, x_2) \\ &= E_{\lambda,\mathcal{M}}(x_0, x_1, x_2)\sum_{j=n}^{m-1}\frac{1}{k^j} \longrightarrow 0. \end{aligned}$$

Hence sequence $\{x_n\}$ is Cauchy sequence w.r.t M .Since

$$\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t) = M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * M(x_n, x_{n+2}, t)$$

Thus $\{x_n\}$ is Cauchy sequence w.r.t $\mathcal{M}$. □

Remark 2.9. Let $(X,\mathcal{M},*)$ be a $\mathcal{M}$-fuzzy metric space. If $\mathcal{M}$ is first or second type and

$$\mathcal{M}(x_n,x_{n+1},x_{n+2},t)\geq\mathcal{M}(x_0,x_1,x_2,k^nt)$$

for some $k>1$ and for every $n\in\mathbb{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

A class of implicit relation. Let Φ denotes a family of mappings such that each $\phi\in\Phi$, $\phi:[0,1]\longrightarrow[0,1]$, such that ϕ is continuous and $\phi(s)>s$ for every $s\in[0,1)$.

Theorem 2.10. *Let A,B,C,S,T and R be self-mappings of a complete $\mathcal{M}$-fuzzy metric space $(X,\mathcal{M},*)$ where $\mathcal{M}$ is first or second type with :*

(i)*$A(X)\subseteq R(X)$, $B(X)\subseteq S(X)$, $C(X)\subseteq T(X)$ and $A(X)$ or $B(X)$ or $C(X)$ is a closed subset of X,*

(ii) *$\mathcal{M}(Ax,By,Cz,t)\geq\phi(\mathcal{M}(Tx,Ry,Sz,kt))$, for every $x,y,z\in X$,$k>1$ and $\phi\in\Phi$,*

(iii) *the pair (A,T) , (B,R) and (S,C) are weak compatible.*

Then A,B,C,S,T and R have a unique common fixed point in X.

Proof. Let $x_0\in X$ be an arbitrary point . By (i), there exists $x_1,x_2,x_3\in X$ such that

$$Ax_0=Rx_1=y_0,\ Bx_1=Sx_2=y_1 \textit{ and } Cx_2=Tx_3=y_2.$$

Inductively, construct sequence $\{y_n\}$ in X such that

$$y_{3n}=Ax_{3n}=Rx_{3n+1},\ y_{3n+1}=Bx_{3n+1}=Sx_{3n+2} \textit{ and } y_{3n+2}=Cx_{3n+2}=Tx_{3n+3},$$

for $n=0,1,2,\cdots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. Let $d_m(t)=\mathcal{M}(y_m,y_{m+1},y_{m+2},t)$. Then, we have

$$\begin{aligned}
d_{3n}(t) &= \mathcal{M}(y_{3n},y_{3n+1},y_{3n+2},t)=\mathcal{M}(Ax_{3n},Bx_{3n+1},Cx_{3n+2},t)\\
&\geq \phi(\mathcal{M}(Tx_{3n},Rx_{3n+1},Sx_{3n+2},kt))\\
&= \phi(\mathcal{M}(y_{3n-1},y_{3n},y_{3n+1},kt))\\
&\geq \mathcal{M}(y_{3n-1},y_{3n},y_{3n+1},kt)=d_{3n-1}(kt)
\end{aligned}$$

$$\begin{aligned}
d_{3n+1}(t) &= \mathcal{M}(y_{3n+1},y_{3n+2},y_{3n+3},t)=\mathcal{M}(y_{3n+3},y_{3n+1},y_{3n+2},t)=\mathcal{M}(Ax_{3n+3},Bx_{3n+1},Cx_{3n+2},t)\\
&\geq \phi(\mathcal{M}(Tx_{3n+3},Rx_{3n+1},Sx_{3n+2},kt))\\
&= \phi(\mathcal{M}(y_{3n+2},y_{3n},y_{3n+1},kt))\\
&\geq \mathcal{M}(y_{3n+2},y_{3n},y_{3n+1},kt)=d_{3n}(kt)
\end{aligned}$$

$$\begin{aligned}
d_{3n+2}(t) &= \mathcal{M}(y_{3n+2},y_{3n+3},y_{3n+4},t)=\mathcal{M}(y_{3n+3},y_{3n+4},y_{3n+2},t)=\mathcal{M}(Ax_{3n+3},Bx_{3n+4},Cx_{3n+2},t)\\
&\geq \phi(\mathcal{M}(Tx_{3n+3},Rx_{3n+4},Sx_{3n+2},kt))\\
&= \phi(\mathcal{M}(y_{3n+2},y_{3n+3},y_{3n+1},kt))\\
&\geq \mathcal{M}(y_{3n+2},y_{3n+3},y_{3n+1},kt)=d_{3n+1}(kt)
\end{aligned}$$

Hence for every $n\in\mathbb{N}$ we have $d_n(t)\geq d_{n-1}(kt)$. That is

$$d_n(t)=\mathcal{M}(y_n,y_{n+1},y_{n+1},t)\geq\mathcal{M}(y_{n-1},y_n,y_{n+1},kt)\geq\cdots\geq\mathcal{M}(y_0,y_1,y_2,k^nt)$$

Since $\mathcal{M}$ is a first or second type , hence by Remark 2.9 $\{y_n\}$ is Cauchy and the completeness of X, $\{y_n\}$ converges to y in X. That is,$\lim_{n\to\infty} y_n = y$

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty} Ax_{3n} = \lim_{n\to\infty} Bx_{3n+1} = \lim_{n\to\infty} Cx_{3n+2}\\ &= \lim_{n\to\infty} Tx_{3n+3} = \lim_{n\to\infty} Rx_{3n+1} = \lim_{n\to\infty} Sx_{3n+2} = y\end{aligned}$$

Let $C(X)$ be a closed subset of X , hence there exist $u \in X$ such that $Tu = y$. We prove that $Au = y$. For

$$\mathcal{M}(Au, Bx_{3n+1}, Cx_{3n+2}, t) \geq \phi(\mathcal{M}(Tu, Rx_{3n+1}, Sx_{3n+2}, kt))$$

On making $n \longrightarrow \infty$ we get

$$\mathcal{M}(Au, y, y, t) \geq \phi(\mathcal{M}(y, y, y, kt)) = \phi(1) = 1.$$

Thus $Au = y$. By weak compatible the pair (T, A) we have $ATu = TAu$, hence $Ay = Ty$. We prove that $Ay = y$, if $Ay \neq y$, then

$$\mathcal{M}(Ay, Bx_{3n+1}, Cx_{3n+2}, t) \geq \phi(\mathcal{M}(Ty, Rx_{3n+1}, Sx_{3n+2}, kt))$$

As $n \longrightarrow \infty$ we have

$$\mathcal{M}(Ay, y, y, t) \geq \phi(\mathcal{M}(Ay, y, y, kt)) > \mathcal{M}(Ay, y, y, kt)$$

is a contradiction. Therefore,$Ty = Ay = y$, that is, y is a common fixed of T, A . Since $y = Ay \in A(X) \subseteq R(X)$, hence there exist $v \in X$ such that $Rv = y$. We prove that $Bv = y$. For

$$\mathcal{M}(y, Bv, Cx_{3n+2}, t) = \mathcal{M}(Ay, Bv, Cx_{3n+2}, t) \geq \phi(\mathcal{M}(Ty, Rv, Sx_{3n+2}, kt))$$

On making $n \longrightarrow \infty$ we get

$$\mathcal{M}(y, Bv, y, t) \geq \phi(\mathcal{M}(y, y, y, kt)) = \phi(1) = 1.$$

Thus $Bv = y$. By weak compatible the pair (B, R) we have $RBv = BRv$, hence $By = Ry$. We prove that $By = y$, if $By \neq y$, then

$$\mathcal{M}(Ay, By, Cx_{3n+2}, t) \geq \phi(\mathcal{M}(Ty, Ry, Sx_{3n+2}, kt))$$

As $n \longrightarrow \infty$ we have

$$\mathcal{M}(y, By, y, t) \geq \phi(\mathcal{M}(y, By, y, kt)) > \mathcal{M}(y, By, y, kt)$$

is a contradiction. Therefore,$By = Ry = y$, that is, y is a common fixed of B, R . Similarly, since $y = By \in B(X) \subseteq S(X)$,hence there exist $w \in X$ such that $Sw = y$. We prove that $Cw = y$. For

$$\mathcal{M}(y, y, Cw, t) = \mathcal{M}(Ay, By, Cw, t) \geq \phi(\mathcal{M}(Ty, Ry, Sw, kt)) = \phi(1) = 1$$

Thus $Cw = y$. By weak compatible the pair (C, S) we have $CSw = SCw$, hence $Cy = Sy$. We prove that $Cy = y$, if $Cy \neq y$, then

$$\mathcal{M}(y, y, Cy, kt) = \mathcal{M}(Ay, By, Cy, t) \geq \phi(\mathcal{M}(Ty, Ry, Sy, kt)) > \mathcal{M}(y, y, Cy, kt)$$

is a contradiction. Therefore,$Cy = Sy = y$, that is, y is a common fixed of C, S.Thus

$$Ay = Sy = Ty = By = Cy = Ry = y$$

Uniqueness, let v be another common fixed point of T, A, B, C, R, S.
If $\mathcal{M}(y, v, v, t) < 1$, hence

$$\begin{aligned}\mathcal{M}(y, v, v, t) &= \mathcal{M}(Ay, Bv, Cv, t)) \geq \phi(\mathcal{M}(Ty, Rv, Sv, kt)) \\ &> \mathcal{M}(y, v, v, kt)\end{aligned}$$

a contradiction. Therefore, $y = v$ is the unique common fixed point of self-maps T, A, B, C, R, S. □

Corollary 2.11. *Let S, T, R and $\{A_\alpha\}_{\alpha \in I}$, $\{B_\beta\}_{\beta \in J}$ and $\{C_\gamma\}_{\gamma \in K}$ be the set of all self-mappings of a complete $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$, where $\mathcal{M}$ is first or second type satisfying :*

(i)*there exists $\alpha_0 \in I$, $\beta_0 \in J$ and $\gamma_0 \in K$ such that $A_{\alpha_0}(X) \subseteq R(X)$, $B_{\beta_0}(X) \subseteq S(X)$ and $C_{\gamma_0}(X) \subseteq T(X)$,*

(ii)*$A_{\alpha_0}(X)$ or $B_{\beta_0}(X)$ or $C_{\gamma_0}(X)$ is a closed subset of X,*

(iii) *$\mathcal{M}(A_\alpha x, B_\beta y, C_\gamma z, t) \geq \phi(\mathcal{M}(Tx, Ry, Sz, kt))$ for every $x, y, z \in X$, some $k > 1$ and every $\alpha \in I, \beta \in J$,$\gamma \in K$, $\phi \in \Phi$,*

(iv) *the pair (A_{α_0}, T) or (B_{β_0}, R) or (C_{γ_0}, S) are weak compatible.*

Then A, B, C, S, T and R have a unique common fixed point in X.

Proof. By Theorem 2.10 R, S, T and A_{α_0} , B_{β_0} and C_{γ_0} for some $\alpha_0 \in I, \beta_0 \in J, \gamma_0 \in K$ have a unique common fixed point in X. That is there exist a unique $a \in X$ such that $R(a) = S(a) = T(a) = A_{\alpha_0}(a) = B_{\beta_0}(a) = C_{\gamma_0}(a) = a$. Let there exist $\lambda \in J$ such that $\lambda \neq \beta_0$ and $\mathcal{M}(a, B_\lambda a, a, t) < 1$ then we have

$$\mathcal{M}(a, B_\lambda a, a, t) = \mathcal{M}(A_{\alpha_0} a, B_\lambda a, C_{\gamma_0} a, t) \geq \phi(\mathcal{M}(Ta, Ra, Sa, kt)) = \phi(1) = 1$$

is a contradiction. Hence for every $\lambda \in J$ we have $B_\lambda(a) = a$. Similarly for every $\delta \in I$ and $\eta \in K$ we get $A_\delta(a) = C_\eta(a) = a$. Therefore for every $\delta \in I, \lambda \in J$ and $\eta \in K$ we have

$$A_\delta(a) = B_\lambda(a) = C_\eta(a) = R(a) = S(a) = T(a) = a.$$

□

References

[1] B.C.Dhage, Generalised metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc.84(1992),no.4,329-336.

[2] B.C.Dhag,some results on common fixed points-I,Indian.J.Pure.Appl.Math. 30(8),(1999),827-837.

[3] Jungck G and Rhoades B. E, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. **29**(1998), no. 3,227-238.

[4] Kramosil I, Michalek J. Fuzzy metric and statistical metric spaces. Kybernetica 1975; 11:326–34.

[5] Saadati R, Razani A, and Adibi H. A common fixed point theorem in $\mathcal{L}$-fuzzy metric spaces. Chaos, Solitons and Fractals, doi:10.1016/j.chaos.2006.01.023.

[6] S.Sedghi,K.P.R.Rao and N.Shobe, A related fixed point theorem in three M-fuzzy metric spaces,communicated.

[7] S.V.R.Naidu,K.P.R.Rao and N.Srinivasa Rao,On the topology of D-metric spaces and the generation of D-metric spaces from metric spaces, Internat.J.Math. Math.Sci. 2004(2004),No.51,2719-2740.

[8] S.V.R.Naidu,K.P.R.Rao and N.Srinivasa Rao:-On the concepts of balls in a D-metric space, Internat.J.Math.Math.Sci.,2005,No.1 (2005)133-141.

[9] S.V.R.Naidu,K.P.R.Rao and N.Srinivasa Rao:-On convergent sequences and fixed point theorems in D-Metric spaces, Internat.J.Math.Math.Sci., 2005:12(2005),1969-1988.

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Statistical Limit Points of Sequences on Intuitionistic Fuzzy Normed Spaces

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Abstract

Karakuş, Demirci, Duman [Chaos, Solitons and Fractals (2006), doi: 10.1016/j.chaos.2006.05.046.] has recently introduced the notion of statistical convergence on intuitionistic fuzzy normed spaces. Using this notion, we study the concept of statistical limit points, statistical cluster points on intuitionistic fuzzy normed spaces and then we give the relations between these and limit points of sequence on intuitionistic fuzzy normed spaces and also we give some topological properties.

1 Introduction[1]

The fuzzy sets theory first introduced by Zadeh [28] in 1965. Then many authors have developed the theory of fuzzy set and applications. The fuzzy logic has been used in metric and topological spaces [6] , [7] , [13] , [16], in the theory of functions [4] , [15] , [27] and also in computer programming [14], in the quantum physics [19] and preference ordering have been the object of extensive study in econometrics and other fields [1] , [2] , [3] , [5] , [21]. Furthermore, Park [22] introduced the notion of intuitionistic fuzzy metric space and also recently the concept of intuitionistic fuzzy normed space have been given by Saadati and Park [23].

In this paper, we introduced the statistical limit points, statistical cluster points and limit points on intuitionistic fuzzy normed spaces. Then we give the relations between these.

Now we give some definitions which we use through the paper:

[1] *Key words and phrases.* Natural density, statistical convergence, statistical limit point,statistical cluster point, limit point, continuous $t-$norm, continuous $t-$conorm, intuitionistic fuzzy normed space.

Definition 1 [25]*A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous* $t-$*norm if it satisfies the following conditions:*

(a) $*$ *is associative and commutative,*

(b) $*$ *is continuous,*

(c) $a * 1 = a$ *for all* $a \in [0,1]$,

(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

We can give two examples for continuous $t-$norm which are $a * b = ab$, $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$.

Definition 2 [24]*A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous* $t-$*conorm if it satisfies the following conditions:*

(a) $\diamond$ *is associative and commutative,*

(b) $\diamond$ *is continuous,*

(c) $a \diamond 0 = a$ *for all* $a \in [0,1]$,

(d) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

We can give two examples for continuous $t-$conorm which are $a \diamond b = \min\{a + b, 1\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0,1]$.

Now we give the definition of intuitionistic fuzzy normed space which have been introduced by Saadati and Park [23]

Definition 3 [23]*The* $5-$*tuple* $(V, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy normed space* $(IFNS)$ *if* V *is a vector space,* $*$ *is a continuous* $t-$*norm,* $\diamond$ *is a continuous* $t-$*conorm, and* μ, ν *fuzzy sets on* $V \times (0, \infty)$ *satisfying the following conditions for every* $x, y \in V$ *and* $s, t > 0$:

(a) $\mu(x,t) + \nu(x,t) \leq 1$,

(b) $\mu(x,t) > 0$,

(c) $\mu(x,t) = 1$ *if and only if* $x = 0$,

(d) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x + y, t + s)$,

(f) $\mu(x, \cdot) : (0, \infty) \to [0,1]$ *is continuous,*

(g) $\lim\limits_{t \to \infty} \mu(x,t) = 1$ *and* $\lim\limits_{t \to 0} \mu(x,t) = 0$,

(h) $\nu(x,t) < 1$,

(*i*) $\nu(x,t) = 0$ *if and only if* $x = 0$,

(*j*) $\nu(\alpha x,t) = \nu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(*k*) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$,

(*l*) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ *is continuous*,

(*m*) $\lim\limits_{t\to\infty} \nu(x,t) = 0$ *and* $\lim\limits_{t\to 0} \nu(x,t) = 0$.

In this case (μ,ν) is called an *intuitionistic fuzzy norm.* We can give an example as follows:

Let $(V, \|\cdot\|)$ be a normed space, and let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x, y \in V$ and every $t > 0$, consider

$$\mu_0(x,t) := \frac{t}{t + \|x\|} \text{ and } \nu_0(x,t) := \frac{\|x\|}{t + \|x\|}.$$

Then $(V, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy normed space.

Definition 4 [23]*Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. A sequence $x = \{x_k\}$ is said to be convergent to $L \in V$ with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L) > 1 - \varepsilon$ and $\nu(x_k - L) < \varepsilon$ for all $k \geq k_0$. In this case we write $(\mu, v) - \lim x = L$ or $x_k \overset{(\mu,\nu)}{\to} L$ as $k \to \infty$.*

2 Statistical Convergence on IFNS

Fast introduced an extension of the usual concept of sequential limits which is called statistical convergent [9]. Schoenberg gave some basic properties of statistical convergence [24]. In this section we give some fundemantel properties of statistical convergence on IFNS.

Natural density δ of a set K of positive integers:

$$\delta(K) : \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in K\}\right|$$

(see [20]) . The natural density may not exist for each set K. But the upper density $\overline{\delta}$ always exists for each set K identified as follows

$$\overline{\delta}(\{K\}) := \limsup_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in K\}\right|.$$

Moreover, the natural density of K is different from zero which means $\overline{\delta}(\{K\}) > 0$. It is clear that finite sets have zero density and $\delta(K^c) = 1 - \delta(K)$ wherever either side exista and $K^c = \mathbb{N} \diagdown K$. The sequence x is statistical convergent to L, denoted $st - \lim x = L$, if for every $\varepsilon > 0$, $\delta\{k : |x_k - L| \geq \varepsilon\} = 0$ [10], [26].

Statistical convergence has been examined in number theory [8], trigonometric series [29] and summability theory [12]. It has also been considered in local convex spaces [18].

Definition 5 [17]*Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. A sequence $x = \{x_k\}$ is statistically convergent to $L \in V$ with respect to the intuitionistic fuzzy norm (μ, ν) provided that, for every $\varepsilon > 0$ and $t > 0$,*

$$\delta\{k \in \mathbb{N} : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - L, t) \geq \varepsilon\} = 0, \tag{1}$$

i.e.;

$$\lim_n \frac{1}{n} |\{k \leq n : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - L, t) \geq \varepsilon\}| = 0.$$

In this case we write $st_{(\mu,\nu)} - \lim x = L$, where L is said to be $st_{(\mu,\nu)}-$limit.

By using (1) and the well-known properties of the density, we easily get the following lemma.

Lemma 6 [17]*Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. Then, for every $\varepsilon > 0$ and $t > 0$, the following statements are equivalent:*

(i) $st_{\mu,\nu} - \lim x = L$

(ii) $\delta\{k \in \mathbb{N} : \mu(x_k - L, t) \leq 1 - \varepsilon\} = \delta\{k \in \mathbb{N} : \nu(x_k - L, t) \geq \varepsilon\} = 0.$

(iii) $\delta\{k \in \mathbb{N} : \mu(x_k - L, t) > 1 - \varepsilon \text{ and } \nu(x_k - L, t) < \varepsilon\} = 1.$

(iv) $\delta\{k \in \mathbb{N} : \mu(x_k - L, t) > 1 - \varepsilon\} = \delta\{k \in \mathbb{N} : \nu(x_k - L, t) < \varepsilon\} = 1.$

(v) $st - \lim \mu(x_k - L, t) = 1$ and $st - \lim \nu(x_k - L, t) = 0.\ddot{u}$

We show that statistically convergence on IFNS has some arithmetical properties similar to properties of the usual convergence on $\mathbb{R}$.

Lemma 7 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. If $st_{(\mu,\nu)} - \lim x_k = \xi$ and $st_{(\mu,\nu)} - \lim y_k = \eta$ then $st_{(\mu,\nu)} - \lim (x_k + y_k) = \xi + \eta$.*

Proof. Let $st_{(\mu,\nu)} - \lim x_k = \xi$, $st_{(\mu,\nu)} - \lim y_k = \eta$, $t > 0$ and $\varepsilon \in (0, 1)$. Choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) \geq 1 - \varepsilon$ and $r \diamond r \leq \varepsilon$. Then we define the following sets:

$$\begin{aligned}
K_{\mu,1}(r, t) &: = \{k \in \mathbb{N} : \mu(x_k - \xi, t) \leq 1 - r\}, \\
K_{\mu,2}(r, t) &: = \{k \in \mathbb{N} : \mu(y_k - \eta, t) \leq 1 - r\}, \\
K_{\nu,1}(r, t) &: = \{k \in \mathbb{N} : \nu(x_k - \xi, t) \geq r\}, \\
K_{\nu,2}(r, t) &: = \{k \in \mathbb{N} : \nu(y_k - \eta, t) \geq r\}.
\end{aligned}$$

Since $st_{\mu,\nu} - \lim x = \xi$, we have

$$\delta\{K_{\mu,1}(\varepsilon, t)\} = \delta\{K_{\nu,1}(\varepsilon, t)\} = 0 \quad \text{for all } t > 0.$$

Similarly, since $st_{\mu,\nu} - \lim y_k = \eta$, we get

$$\delta\{K_{\mu,2}(\varepsilon,t)\} = \delta\{K_{\nu,2}(\varepsilon,t)\} = 0 \quad \text{for all } t > 0.$$

Now let $K_{\mu,\nu}(\varepsilon,t) := \{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\} \cap \{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$. Then observe that $\delta\{K_{\mu,\nu}(\varepsilon,t)\} = 0$ which implies $\delta\{\mathbb{N}/K_{\mu,\nu}(\varepsilon,t)\} = 1$. If $k \in \mathbb{N}/K_{\mu,\nu}(\varepsilon,t)$, then we have two possible cases. The former is the case of $k \in \mathbb{N}/\{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\}$; and the letter is $k \in \mathbb{N}/\{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$. We first consider that $k \in \mathbb{N}/\{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\}$. Then we have

$$\begin{aligned} \mu((x_k - \xi) + (y_k - \eta), t) &\geq \mu(x_k - \xi, \frac{t}{2}) * \mu(y_k - \eta, \frac{t}{2}) \\ &> (1-r) * (1-r) \geq 1 - \varepsilon. \end{aligned}$$

On the other hand, if $k \in \mathbb{N}/\{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$, then we can write that

$$\begin{aligned} \nu((x_k - \xi) + (y_k - \eta), t) &\leq \nu(x_k - \xi, \frac{t}{2}) \diamond \nu(x_k - \eta, \frac{t}{2}) \\ &< r \diamond r < \varepsilon. \end{aligned}$$

This show that

$$\delta(\{k \in \mathbb{N} : \mu((x_k - \xi) + (y_k - \eta), t) \leq 1 - \varepsilon \quad or \quad \nu((x_k - \xi) + (y_k - \eta), t) \geq \varepsilon\}) = 0$$

so $st_{(\mu,\nu)} - \lim (x_k + y_k) = \xi + \eta$. ■

Lemma 8 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. If $st_{(\mu,\nu)} - \lim x_k = \xi$ and $\alpha \in \mathbb{R}$, $(\alpha \neq 0)$ then $st_{(\mu,\nu)} - \lim \alpha x_k = \alpha\xi$.*

Proof. Let $st_{(\mu,\nu)} - \lim x_k = \xi$, $\varepsilon \in (0,1)$ and $t > 0$. From definition we can write

$$\delta(\{k \in \mathbb{N} : \mu(x_k - \xi, t) \leq 1 - \varepsilon \quad or \quad \nu(x_k - \xi, t) \geq \varepsilon\}) = 0.$$

So, if we define the sets:

$$\begin{aligned} K_{\mu,1}(\varepsilon,t) &: = \{k \in \mathbb{N} : \mu(x_k - \xi, t) \leq 1 - \varepsilon\} \\ K_{\nu,1}(\varepsilon,t) &: = \{k \in \mathbb{N} : \nu(x_k - \xi, t) \geq \varepsilon\} \end{aligned}$$

then we can say $\delta\{K_{\mu,1}(\varepsilon,t)\} = \delta\{K_{\nu,1}(\varepsilon,t)\} = 0$ for all $t > 0$. Now let $K_{\mu,\nu}(\varepsilon,t) = K_{\mu,1}(\varepsilon,t) \cup K_{\nu,1}(\varepsilon,t)$ then $\delta\{K_{\mu,\nu}(\varepsilon,t)\} = 0$ which implies $\delta\{\mathbb{N} \diagdown K_{\mu,\nu}(\varepsilon,t)\} = 1$. If $k \in \mathbb{N} \diagdown K_{\mu,\nu}(\varepsilon,t)$ then we have

$$\begin{aligned} \mu(\alpha x_k - \alpha\xi, t) &= \mu(x_k - \xi, \frac{t}{|\alpha|}) \\ &\geq \mu(x_k - \xi, t) * \mu(0, \frac{t}{|\alpha|} - t) \\ &= \mu(x_k - \xi, t) * 1 \\ &= \mu(x_k - \xi, t) > 1 - \varepsilon. \end{aligned}$$

for $\alpha \in \mathbb{R}$ $(\alpha \neq 0)$.

Similarly, we observe that

$$\begin{aligned} \nu(\alpha x_k - \alpha\xi, t) &= \nu(x_k - \xi, \frac{t}{|\alpha|}) \\ &\leq \nu(x_k - \xi, t) \diamond \nu(0, \frac{t}{|\alpha|} - t) \\ &= \nu(x_k - \xi, t) \diamond 0 \\ &= \nu(x_k - \xi, t) < \varepsilon. \end{aligned}$$

for $\alpha \in \mathbb{R}$ $(\alpha \neq 0)$.This show that

$$\delta\left(\{k \in \mathbb{N} : \mu(\alpha x_k - \alpha\xi, t) \leq 1 - \varepsilon \quad or \quad \nu(\alpha x_k - \alpha\xi, t) \geq \varepsilon\}\right) = 0$$

so $st_{(\mu,\nu)} - \lim \alpha x_k = \alpha\xi$. ■

Lemma 9 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. If $st_{(\mu,\nu)} - \lim x_k = \xi$ and $st_{(\mu,\nu)} - \lim y_k = \eta$ then $st_{(\mu,\nu)} - \lim (x_k - y_k) = \xi - \eta$.*

Proof. The proof is clear from Lemma 8 and Lemma 9. ■

Definition 10 [23]*Let $(V, \mu, \nu, *, \diamond)$ be an IFNS . We say that a sequence $x = (x_k)$ is IF-bounded if there exist $t > 0$ and $0 < r < 1$ such that $\mu(x_k, t) > 1 - r$ and $\nu(x_k, t) < r$ for every $k \in \mathbb{N}$.*

Definition 11 [23]*Let $(V, \mu, \nu, *, \diamond)$ be an IFNS . For $t > 0$, we define open ball $B(x, r, t)$ with center $x \in V$ and radious $0 < r < 1$, as*

$$B(x, r, t) = \{y \in V : \mu(x - y, t) > 1 - r \ , \ \nu(x - y, t) < r\}.$$

It follows from Lemma 8, Lemma 9 and Lemma 10, that the set of all IF-bounded statistically convergent sequences on IFNS is a linear subspace of the space $\ell_\infty^{(\mu,\nu)}(V)$ of all IF-bounded sequences on IFNS.

Theorem 12 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS and the space of statistical convergence IF-bounded sequences show with $S_b^{(\mu,\nu)}(V)$. Then the set $S_b^{(\mu,\nu)}(V)$ is a closed linear subspace of the set $\ell_\infty^{(\mu,\nu)}(V)$.*

Proof. It is clear that $S_b^{(\mu,\nu)}(V) \subset \overline{S_b^{(\mu,\nu)}(V)}$. Now we show that $\overline{S_b^{(\mu,\nu)}(V)} \subset S_b^{(\mu,\nu)}(V)$. Let $y \in \overline{S_b^{(\mu,\nu)}(V)}$. Since $B(y, r, t) \cap S_b^{(\mu,\nu)}(V) \neq \varnothing$, there is a $x \in B(y, r, t) \cap S_b^{(\mu,\nu)}(V)$.

Let $t > 0$ and $\varepsilon \in (0, 1)$. Choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) \geq 1 - \varepsilon$ and $r \diamond r \leq \varepsilon$. Since $x \in B(y, r, t) \cap S_b^{(\mu,\nu)}(V)$, there is a set $K \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that

$$\mu\left(y_k - x_k, \frac{t}{2}\right) > 1 - r \quad and \quad \nu\left(y_k - x_k, \frac{t}{2}\right) < r$$

and

$$\mu\left(x_k, \frac{t}{2}\right) > 1 - r \quad and \quad \nu\left(x_k, \frac{t}{2}\right) < r$$

for all $k \in K$. Then we have

$$\begin{aligned}
\mu\left(y_k, t\right) &= \mu\left(y_k - x_k + x_k, t\right) \\
&\geq \mu\left(y_k - x_k, \frac{t}{2}\right) * \mu\left(x_k, \frac{t}{2}\right) \\
&> (1 - r) * (1 - r) \geq 1 - \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\nu\left(y_k, t\right) &= \nu\left(y_k - x_k + x_k, t\right) \\
&\leq \nu\left(y_k - x_k, \frac{t}{2}\right) \diamond \nu\left(x_k, \frac{t}{2}\right) \\
&< r \diamond r \leq \varepsilon
\end{aligned}$$

for all $k \in K$. Hence

$$\delta\left(\{k \in \mathbb{N} : \mu\left(y_k, t\right) > 1 - \varepsilon \quad and \quad \nu\left(y_k, t\right) < \varepsilon\}\right) = 1$$

and thus $y \in S_b^{(\mu,\nu)}(V)$. ■

3 Statistical Limit Points and Statistical Cluster Points on IFNS.

Fridy [11] introduced the concepts of statistical limit points and statistical cluster points of real number sequences and gave some properties of the sets of statistical limit and cluster points. Now we study analogues of these on intuitionistic fuzzy normed spaces and then we give the relations between these and limit points of sequence on intuitionistic fuzzy normed spaces.

Definition 13 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. $\ell \in V$ is called a limit point of the sequence $x = \{x_k\}$ with respect to the intuitionistic fuzzy norms (μ, ν) provided that there is a subsequence of x that converges to ℓ with respect to the intuitionistic fuzzy norms (μ, ν). Let $L_{(\mu,\nu)}(x)$ denote the set of all limit points of the sequence x.*

Definition 14 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. If $\{x_{k(j)}\}$ is a subsequence of $x = (x_k)$ and $K := \{k(j) \in \mathbb{N} : j \in \mathbb{N}\}$ then we abbreviate $\{x_{k(j)}\}$ by $\{x\}_K$ which in case $\delta(\{K\}) = 0$. $\{x\}_K$ is called a subsequence of density zero or thin subsequence. On the other hand, $\{x\}_K$ is a nonthin subsequence of x if K does not have density zero.*

Definition 15 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. Then $\lambda \in V$ is called a statistical limit point of sequence $x=(x_k)$ with respect to the intuitionistic fuzzy norms (μ,ν) provided that there is a nonthin subsequence of x that converges to λ with respect to the intuitionistic fuzzy norms (μ,ν). In this case we say λ is a $st_{(\mu,\nu)}$-limit point of seqence $x=(x_k)$. Let $\Lambda_{(\mu,\nu)}(x)$ denote the set of statistical limit points of the sequence x.*

Definition 16 *Let $(V,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy normed space.Then $\gamma \in V$ is called a statistical cluster point of sequence $x=(x_k)$ with respect to the intuitionistic fuzzy norms (μ,ν) provided that for every $\varepsilon>0$ and $t>0$,*

$$\bar{\delta}\left(\{k \in \mathbb{N} : \mu(x_k-\gamma,t) > 1-\varepsilon \quad and \quad \nu(x_k-\gamma,t)<\varepsilon\}\right) > 0.$$

In this case we say γ is a $st_{(\mu,\nu)}$-cluster point of seqence $x=(x_k)$. Let $\Gamma_{(\mu,\nu)}(x)$ denote the set of all statistical cluster points of the sequence x.

Theorem 17 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. For any sequence $x \in V$, $\Lambda_{(\mu,\nu)}(x) \subset \Gamma_{(\mu,\nu)}(x)$.*

Proof. Suppose $\lambda \in \Lambda_{(\mu,\nu)}(x)$, then there is a nonthin subsequence $\left(x_{k(j)}\right)$ of (x_k) that converges to λ with respect to the intuitionistic fuzzy norms (μ,ν), i.e.

$$\delta\left(\left\{k(j) \in \mathbb{N} : \mu\left(x_{k(j)}-\lambda,t\right) > 1-\varepsilon \quad and \quad \nu\left(x_{k(j)}-\lambda,t\right)<\varepsilon\right\}\right) = d > 0.$$

Since

$$\begin{gathered}\{k \in \mathbb{N} : \mu(x_k-\lambda,t) > 1-\varepsilon \quad and \quad \nu(x_k-\lambda,t)<\varepsilon\} \supset \\ \left\{k(j) \in \mathbb{N} : \mu\left(x_{k(j)}-\lambda,t\right) > 1-\varepsilon \quad and \quad \nu\left(x_{k(j)}-\lambda,t\right)<\varepsilon\right\}\end{gathered}$$

for every $\varepsilon>0$, we have

$$\{k \in \mathbb{N} : \mu(x_k-\lambda,t) > 1-\varepsilon \quad and \quad \nu(x_k-\lambda,t)<\varepsilon\} \supseteq$$

$$\{k(j) \in \mathbb{N} : j \in \mathbb{N}\} \setminus \left\{k(j) \in \mathbb{N} : \mu\left(x_{k(j)}-\lambda,t\right) \le 1-\varepsilon \quad or \quad \nu\left(x_{k(j)}-\lambda,t\right) \ge \varepsilon\right\}.$$

Since $\left(x_{k(j)}\right)$ converges to λ with respect to the intuitionistic fuzzy norms (μ,ν), the set

$$\left\{k(j) \in \mathbb{N} : \mu\left(x_{k(j)}-\lambda,t\right) \le 1-\varepsilon \quad or \quad \nu\left(x_{k(j)}-\lambda,t\right) \ge \varepsilon\right\}$$

is finite for any $\varepsilon>0$. Therefore,

$$\bar{\delta}\left(\{k \in \mathbb{N} : \mu(x_k-\lambda,t) > 1-\varepsilon \quad and \quad \nu(x_k-\lambda,t)<\varepsilon\}\right) \ge$$

$$\bar{\delta}\{k(j) \in \mathbb{N} : j \in \mathbb{N}\} \setminus - \bar{\delta}\left\{k(j) \in \mathbb{N} : \mu\left(x_{k(j)}-\lambda,t\right) \le 1-\varepsilon \quad or \quad \nu\left(x_{k(j)}-\lambda,t\right) \ge \varepsilon\right\}.$$

Hence

$$\bar{\delta}\left(\{k \in \mathbb{N} : \mu(x_k-\lambda,t) > 1-\varepsilon \quad and \quad \nu(x_k-\lambda,t)<\varepsilon\}\right) > 0$$

which means that $\lambda \in \Gamma_{(\mu,\nu)}(x)$. ■

Theorem 18 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. For any sequence $x\in V$, $\Gamma_{(\mu,\nu)}(x)\subseteq L_{(\mu,\nu)}(x)$.*

Proof. Let $\gamma\in\Gamma_{(\mu,\nu)}(x)$, then

$$\delta\left(\{k\in\mathbb{N}:\mu(x_k-\gamma,t)>1-\varepsilon\quad and\quad \nu(x_k-\gamma,t)<\varepsilon\}\right)\neq 0$$

for every $\varepsilon>0$ and $t>0$. We set $\{x\}_K$ a nonthin subsequence of x such that

$$K:=\left\{k(j)\in\mathbb{N}:\mu\left(x_{k(j)}-\gamma,t\right)>1-\varepsilon\quad and\quad \nu\left(x_{k(j)}-\gamma,t\right)<\varepsilon\right\}$$

for every $\varepsilon>0$ and $\delta(K)\neq 0$. Since there are infinitely many element in K, $\gamma\in L_{(\mu,\nu)}(x)$. ∎

Theorem 19 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. For a sequence $x=(x_k)$, $st_{(\mu,\nu)}-\lim x=x_0$ then $\Lambda_{(\mu,\nu)}(x)=\Gamma_{(\mu,\nu)}(x)=\{x_0\}$.*

Proof. First we show that $\Lambda_{(\mu,\nu)}(x)=\{x_0\}$. We suppose that $\Lambda_{(\mu,\nu)}(x)=\{x_0,y_0\}$ such that $\mu(x_0-y_0,t)<1-2\varepsilon$ and $\nu(x_0-y_0,t)>2\varepsilon$ for every $\varepsilon>0$ and $t>0$. In this case, there exist $\left\{x_{k(j)}\right\}$ and $\left\{x_{l(i)}\right\}$ nonthin subsequences of $x=(x_k)$ that $(\mu,\nu)-$convergence to x_0,y_0 respectively. Since $\left\{x_{l(i)}\right\}$ $(\mu,\nu)-$convergence to y_0 for every $\varepsilon>0$

$$K:=\left\{l(i)\in\mathbb{N}:\mu\left(x_{l(i)}-y_0,t\right)\leq 1-\varepsilon\quad or\quad \nu\left(x_{l(i)}-y_0,t\right)\geq\varepsilon\right\}$$

is a finite set so $\delta(K)=0$.Then we observe that

$$\{l(i)\in\mathbb{N}:i\in\mathbb{N}\}=\left\{l(i)\in\mathbb{N}:\mu\left(x_{l(i)}-y_0,t\right)>1-\varepsilon\quad and\quad \nu\left(x_{l(i)}-y_0,t\right)<\varepsilon\right\}$$
$$\cup\left\{l(i)\in\mathbb{N}:\mu\left(x_{l(i)}-y_0,t\right)\leq 1-\varepsilon\quad or\quad \nu\left(x_{l(i)}-y_0,t\right)\geq\varepsilon\right\}$$

which implies that

$$\delta\left(\left\{l(i)\in\mathbb{N}:\mu\left(x_{l(i)}-y_0,t\right)>1-\varepsilon\quad or\quad \nu\left(x_{l(i)}-y_0,t\right)<\varepsilon\right\}\right)\neq 0.\tag{2}$$

Since $st_{(\mu,\nu)}-\lim x=x_0$

$$\delta\left(\{k\in\mathbb{N}:\mu(x_k-x_0,t)\leq 1-\varepsilon\quad or\quad \nu(x_k-x_0,t)\geq\varepsilon\}\right)=0\tag{3}$$

for every $\varepsilon>0$. Therfore, we can write

$$\delta\left(\{k\in\mathbb{N}:\mu(x_k-x_0,t)>1-\varepsilon\quad and\quad \nu(x_k-x_0,t)<\varepsilon\}\right)\neq 0.$$

For every $\mu(x_0-y_0,t)<1-2\varepsilon$ and $\nu(x_0-y_0,t)>2\varepsilon$

$$\left\{l(i)\in\mathbb{N}:\mu\left(x_{l(i)}-y_0,t\right)>1-\varepsilon\quad and\quad \nu\left(x_{l(i)}-y_0,t\right)<\varepsilon\right\}$$
$$\cap\{k\in\mathbb{N}:\mu(x_k-x_0,t)>1-\varepsilon\quad and\quad \nu(x_k-x_0,t)<\varepsilon\}=\varnothing.$$

Hence

$$\left\{l(i)\in\mathbb{N}:\mu\left(x_{l(i)}-y_0,t\right)>1-\varepsilon\quad and\quad \nu\left(x_{l(i)}-y_0,t\right)<\varepsilon\right\}$$

$$\subseteq \{k \in \mathbb{N} : \mu(x_k - x_0, t) \leq 1 - \varepsilon \quad or \quad \nu(x_k - x_0, t) \geq \varepsilon\}.$$

Therefore

$$\bar{\delta}\left(\left\{l(i) \in \mathbb{N} : \mu\left(x_{l(i)} - y_0, t\right) > 1 - \varepsilon \quad and \quad \nu\left(x_{l(i)} - y_0, t\right) < \varepsilon\right\}\right)$$

$$\leq \bar{\delta}\left(\{k \in \mathbb{N} : \mu(x_k - x_0, t) \leq 1 - \varepsilon \quad or \quad \nu(x_k - x_0, t) \geq \varepsilon\}\right) = 0.$$

This contradicts to (2). Hence $\Lambda_{(\mu,\nu)}(x) = \{x_0\}$.

Now we assume that $\Gamma_{(\mu,\nu)}(x) = \{x_0, z_0\}$ such that $\mu(x_0 - z_0, t) < 1 - 2\varepsilon$ and $\nu(x_0 - z_0, t) > 2\varepsilon$ for some $\varepsilon > 0$ and all $t > 0$. Then

$$\bar{\delta}\left(\{k \in \mathbb{N} : \mu(x_k - z_0, t) > 1 - \varepsilon \quad and \quad \nu(x_k - x_0, t) < \varepsilon\}\right) \neq 0. \tag{4}$$

Since

$$\{k \in \mathbb{N} : \mu(x_k - x_0, t) > 1 - \varepsilon \quad and \quad \nu(x_k - x_0, t) < \varepsilon\}$$

$$\cap \{k \in \mathbb{N} : \mu(x_k - z_0, t) > 1 - \varepsilon \quad and \quad \nu(x_k - z_0, t) < \varepsilon\} = \varnothing$$

for every $\mu(x_k - z_0, t) < 1 - 2\varepsilon$ and $\nu(x_k - z_0, t) > 2\varepsilon$ so

$$\{k \in \mathbb{N} : \mu(x_k - x_0, t) \leq 1 - \varepsilon \quad or \quad \nu(x_k - x_0, t) \geq \varepsilon\}$$

$$\supseteq \{k \in \mathbb{N} : \mu(x_k - z_0, t) > 1 - \varepsilon \quad and \quad \nu(x_k - z_0, t) < \varepsilon\}.$$

Therefore

$$\bar{\delta}\left(\{k \in \mathbb{N} : \mu(x_k - x_0, t) \leq 1 - \varepsilon \quad or \quad \nu(x_k - x_0, t) \geq \varepsilon\}\right)$$

$$\geq \bar{\delta}\left(\{k \in \mathbb{N} : \mu(x_k - z_0, t) > 1 - \varepsilon \quad and \quad \nu(x_k - z_0, t) < \varepsilon\}\right). \tag{5}$$

From (4), the right side of (5) is greater than zero and from (3), the left side of (5) equals to zero. This is a contradiction. Hence $\Gamma_{(\mu,\nu)}(x) = \{x_0\}$. ∎

Theorem 20 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. Then the set $\Gamma_{(\mu,\nu)}$ is closed in V for each $x = (x_k)$ of elements of V.*

Proof. Let $y \in \overline{\Gamma_{(\mu,\nu)}(x)}$. Take $0 < r < 1$ and $t > 0$. There exists $\gamma \in \Gamma_{(\mu,\nu)}(x) \cap B(y, r, t)$ such that

$$B(y, r, t) = \{z \in V : \mu(y - z, t) > 1 - r \quad and \quad \nu(y - z, t) < r\}.$$

Choose $\eta > 0$ such that $B(\gamma, \eta, t) \subset B(y, r, t)$. We have

$$\{k \in \mathbb{N} : \mu(y - x_k, t) > 1 - r \quad and \quad \nu(y - x_k, t) < r\} \supset$$

$$\{k \in \mathbb{N} : \mu(\gamma - x_k, t) > 1 - \eta \quad and \quad \nu(\gamma - x_k, t) < \eta\}$$

hence

$$\delta\left(\{k \in \mathbb{N} : \mu(y - x_k, t) > 1 - r \quad and \quad \nu(y - x_k, t) < r\}\right) \neq 0$$

and $y \in \Gamma_{(\mu,\nu)}$. ∎

REFERENCES

[1] Aliprantis CD., Border KC. Infinite dimensional analysis. Springer Verlag, Heidelberg 1994.

[2] Aliprantis CD. Brown DJ: and Burkinshaw O. Edgeworth equilibria. Econometrica 1997;5:1109–37.

[3] Billot A. Economic theory of fuzzy equilibria. Springer - Verlag, Berlin - 1992.

[4] Burgin M. Theory of fuzzy limits. Fuzzy Sets Syst. 2000;115:433-43.

[5] Debreu G. Mathematical Economics: Twenty papers of Gerard Debreu. Cambridge University Press, Cambridge-1983.

[6] Deng Z. Fuzzy pseudo-metric space, J. Math. Anal. Appl. 1982;86: 74–95.

[7] Erceg MA. Metric spaces in fuzzy set theory. J. Math. Anal. Appl. 1979;69:205–30.

[8] Erdös P., Tenenbaum G. Sur les densities de certaines suites d'entries. Proc. London Math. Soc. 1989;59:417-38.

[9] Fast H. Sur la convergence statistique. Colloq. Math. 1951;2:241-44.

[10] Fridy JA. On statistical convergence. Analysis 1985;5:301–13.

[11] Fridy JA. Statistical limit points. Proc. Amer. Math. Soc. 1993;118:1187-92.

[12] Fridy J., Orhan C. Lacunary statistical summability. J. Math.Anal.Appl. 1993:173:497-504.

[13] George A, Veeramani P. On some result in fuzzy metric space. Fuzzy Sets Syst. 1994;64:395-9.

[14] Giles R. A computer program for fuzzy reasoning. Fuzzy Sets Syst. 1980;4:221-34.

[15] Jäger G. Fuzzy uniform convergence and equicontinuity. Fuzzy Sets Syst. 2000;109:187-98.

[16] Kaleva O., Seikkala S. On fuzzy metric spaces. Fuzzy Sets and Systems1984; 12:215–29.

[17] Karakuş S., Demirci K., Duman O. Statistical convergence on intuitionistic fuzzy normed spaces. Chaos, Solitons and Fractals (2006), doi: 10.1016/j.chaos. 2006.05.046.

[18] Maddox IJ. Statistical convergence in a locally convex space. Math. Proc. Cambridge Phil. Soc. 1988;104:141-5.

[19] Madore J. Fuzzy physics. Annals of Physics 1992;219:187-98.

[20] Niven I., Zuckerman HS., Montgomery H. An introduction to statistical convergence. Trans.Amer.Math.Soc. 1995;347;1811-19.

[21] Orlovsky SA. Decision making with a fuzzy preference relation. Fuzzy Sets and Systems 1978;1:155–67.

[22] Park JH. Intuitionistic fuzzy metric spaces. Chaos, Solutions & Fractals 2004;22:1039–46.

[23] Saadati R, Park JH. On the intuitionistic fuzzy topological spaces. Chaos, Solitons & Fractals 2006;27:331–44.

[24] Schoenberg J. The integrability of certain functions and related summability. Amer.Math.Monthly. 1959;66:361-375.

[25] Schweizer B, Sklar A. Statistical metric spaces. Pacific J. Math. 1960;10:314–44.

[26] Steinhaus H. Sur la convergence ordinaire et la convergence asymptotique. Colloq. Math. 1951;2:73–74.

[27] Wu K. Convergences of fuzzy sets based on decomposition theory and fuzzy polynomial function. Fuzzy Sets Syst. 2000;109:173-85.

[28] Zadeh LA. Fuzzy sets. Inform. Control 1965;8:338-53.

[29] Zygmund A. Trigonometric Series. Cambridge Univ. Press; 1979.

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ON THE MECHANICAL QUADRATURE METHOD FOR SOLVING SINGULAR INTEGRAL EQUATIONS WITH HILBERT KERNEL

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Abstract

In this paper, a mechanical quadrature method has been used for solving a class of nonlinear singular integral equations with Hilbert kernel in generalized Hölder spaces. The rate of convergence of approximate solution has been determined. The method has been applied to a nonlinear and a linear singular integral equation with known exact solution. The error has been calculated.

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Introduction

The theory of singular integral equations has been developed significant importance during the last years, it is arise in many problems of mathematical physics, such as the theory of elasticity, hydrodynamics, biological problems, population genetics and others. Also, nonlinear singular integral equations with Hilbert and Cauchy kernel and its related Rimann-Hilbert problems have been studied in works of Pogorzelski W. [16], Guseinov A. I. and Mukhtarov Kh. Sh. [8], Wolfersdorf L. V. [21] and Wegert E. [20] and others.

Existence results and approximate solutions for certain classes of nonlinear singular integral equations are studied in ([2,3,5-7,9]) and others. The theory of approximation methods and its application for the solution of linear and nonlinear singular integral equations has been developed by many authors, ([4,8,10,11,13]).

It is well known that the nonlinear singular integral equations are the much-complicated forms of the nonlinear integral equations. The mechanical quadrature method is one of the basic tools to investigate the approximate solutions of many classes of nonlinear and linear equations involving integral operator. In this paper we applied the mechanical quadrature method to a certain class of nonlinear singular integral equation (NSIE) with Hilbert kernel in generalized Hölder spaces. The method has been applied to a nonlinear and a linear singular integral equation (SIE) with known exact solution and the error has been calculated. The obtained results of the mechanical quadrature method of SIE are compared with the obtained results of the Toeplitz matrix method and the product Nystrom method that have been applied in [1] to obtain the approximate solution of the same problem.

1- Formulation of the problem

This paper is devoted to investigate the approximate solution of the following nonlinear singular integral equation:

$$u(t)=F(t,u(t),v(t)), \tag{1.1}$$

where

$$v(t)=\frac{\lambda}{2\pi}\int_0^{2\pi} g(t,\tau,u(\tau))\cot\frac{\tau-t}{2}\,d\tau, \tag{1.2}$$

in generalized Hölder spaces $H_{\varphi,m}$ and $H_{\varphi,m}^{(N)}$, λ is a numerical parameter, under the following assumptions:

Assumption I:

Suppose that the function $g(t,\tau,u(\tau))$ is defined on the domain

$$D=\{(t,\tau,u);\ 0\le t,\tau\le 2\pi,\ \ |u|\le M, M>0\},$$

that has partial derivatives up to $(m-1)-$order and satisfy the following Hölder-Lipschitz condition for arbitrary $t_n,\tau_n\in[0,2\pi],u_n\in[-M,M]$, $(n=1,2)$

$$\left|\frac{\partial^\beta g(t_1,\tau_1,u_1)}{\partial t^i\partial\tau^j\partial u^k}-\frac{\partial^\beta g(t_2,\tau_2,u_2)}{\partial t^i\partial\tau^j\partial u^k}\right|\le\eta(\beta)\left[\varphi(|t_1-t_2|)+\varphi^*(|\tau_1-\tau_2|)+|u_1-u_2|\right] \tag{1.3}$$

where φ,φ^* are non-decreasing functions belong to the class Φ, $i+j+k=\beta$, $\beta=0,1,2,...,m-1$ and $\eta(\beta)$ is a constant depends on β.

Assumption II:

Suppose that the function $F(t,u(t),v(t))$ is defined on the domain

$$D^*=\{(t,u,v);\ 0\le t\le 2\pi,\ |u|\le M, |v|\le M;\ M>0\},$$

that has partial derivatives up to $(m-1)-$ order and satisfy the following condition for arbitrary $t_n\in[0,2\pi]$, $u_n,v_n\in[-M,M]$, $M>0$, $(n=1,2)$

$$\left|\frac{\partial^\upsilon F(t_1,u_1,v_1)}{\partial t^p\partial u^q\partial v^r}-\frac{\partial^\upsilon F(t_2,u_2,v_2)}{\partial t^p\partial u^q\partial v^r}\right|\le\xi(\upsilon)\left[\varphi_1(|t_1-t_2|)+|u_1-u_2|+|v_1-v_2|\right] \tag{1.4}$$

for $p+q+r=\upsilon$,$\upsilon=0,1,2,...,m-1$, where $\varphi_1\in\Phi$ and $\xi(\upsilon)$ is a constant depends on υ. Equation (1.1) with Cauchy kernel has been studied by the collocation method in [4], the special cases of equation (1.1) have been found in [17,18].

2- Some basic definitions and auxiliary results:

In this section we introduce some definitions and results which will be used in the sequel

Definition 2.1 [8,14].

a- We denote by Φ to the class of all continuous almost increasing functions φ defined on $(0,\pi]$ such that $\varphi(\delta) > 0,\ \lim\limits_{\delta\to 0^+} \varphi(\delta) = 0$.

b- We denote by Φ^m to the class of all functions $\varphi \in \Phi$ such that $0 < \delta_1 < \delta_2 < \pi$

implies $\delta_1^m \varphi(\delta_2) \le c(m)\delta_2^m \varphi(\delta_1)$, where $c(m)$ is a constant depends on m.

c- We denote by $c_{2\pi}$ to the space of $2\pi-$periodic continuous functions with the norm

$$\| u \|_c = \max_{t\in[-\pi,\pi]} |u(t)|.$$

d- The generalized Hölder space $H_{\varphi,m}$ is defined as

$$H_{\varphi,m} = \left\{ u \in c_{2\pi} : \omega_u^m(\delta) = O(\varphi(\delta)),\ \varphi \in H\Phi^m \right\},$$

where $\omega_u^m(\delta)$ is the modulus of continuity of order m of the function u and

$$H\Phi^m = \left\{ \varphi \in \Phi^m : \int_0^\delta \frac{\varphi(\xi)}{\xi} d\xi + \delta^m \int_\delta^\pi \frac{\varphi(\xi)}{\xi^{m+1}} d\xi \le \tilde{c}(m)\varphi(\delta) \right\},$$

where $\tilde{c}(m)$ is a constant depends on m.

e- For $u \in H_{\varphi,m}$ we define

$$\| u \|_{\varphi,m} = \| u \|_c + \sup_{0<\delta\le\pi} \frac{\omega_u^m(\delta)}{\varphi(\delta)}$$

and

$$H_{\varphi,m}(M) = \left\{ u \in H_{\varphi,m} : \| u \|_{\varphi,m} \le M,\ M > 0 \right\}$$

as a subspace of $H_{\varphi,m}$

Definition 2.2 [17,19]

a- Let the generalized Hölder space $H_{\varphi,m}^{(N)}$, $\varphi \in \Phi^m$, be the space of 2N-dimensional vectors $z = (z_0, z_1, \dots, z_{2N-1})$ with the norm

$$\| z \|_{\varphi,m}^{(N)} = \max\left\{ \max_{i=o,\dots,2N-1} |z_i| ,\ \max_{\substack{p\in X\\ p\neq 0}} \frac{\omega_X^m(z,p)}{\varphi(\pi\, p/N)} ,\ \max_{\substack{p\in X\\ p\neq 0}} \frac{\omega_Y^m(z,p)}{\varphi(\pi\, p/N)} \right\},$$

where

$$\omega_X^m(z,p) = \max_{\substack{i\in[0,2N-1-mh]\cap X\\ h\in[0,p]\cap X}} \left| \Delta_h^m z_i \right|$$

and

$$\omega_Y^m(z,p) = \max_{\substack{i\in[0,2N-1-mh]\cap Y\\ h\in[0,p]\cap X}} \left| \Delta_h^m z_i \right|$$

are the modulus of continuity of order m of the vector z with respect to the two sets $X=\{0,2,...,2N-2\}$, $Y=\{1,3,...,2N-1\}$ and $p\in X$,

$$\Delta_h^m z_i = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\ z_{i+kh}\ .$$

For $z\in H_{\varphi,m}^{(N)}$ we define

$$H_{\varphi,m}^{(N)}(M)=\left\{z\in H_{\varphi,m}^{(N)}:\ \|z\|_{\varphi,m}^{(N)}\le M,\ \ M>0\right\}$$

as a subspace of $H_{\varphi,m}^{(N)}$.

b- We denote the norm in the space $L_p^{(N)}$ by

$$\|z\|_{L_p^{(N)}}=\left(\frac{\pi}{N}\sum_{k=0}^{2N-1}|z_k|^p\right)^{1/p},\quad p>1.$$

Theorem 2.1 [15,16]

Let $\varphi\in H\Phi^m$, then the operator

$$(Au)(t)=\frac{1}{2\pi}\int_0^{2\pi}u(\tau)\cot\frac{\tau-t}{2}d\tau$$

Transforms $H_{\varphi,m}(M)$ into $H_{\varphi,m}(\widetilde{M})$ where

$$\widetilde{M}=M\left\{e_1(m)\int_0^{\pi}\frac{\varphi(\delta)}{\delta}d\delta+e_1(m)+e_2(m)\widetilde{c}(m)\right\}$$

where $e_1(m),\ e_2(m)\ and\ \widetilde{c}(m)$ are constants depend on m

Lemma 2.1 [17]

Let the condition (1.3) is satisfied and $u(t)\in H_{\varphi,m}$ then $g(t,\tau,u(\tau))\in H_{\varphi,m}$.

Lemma 2.2 [18]

Let the condition (1.4) is satisfied and $u(t),\ v(t)\in H_{\varphi,m}$ then $F(t,u,v)\in H_{\varphi,m}$

3- The approximate solution in the space $H_{\varphi,m}$

Theorem 3.1

Let the function $F(t,u(t),v(t))$ satisfy the condition (1.4) and the function $g(t,\tau,u(\tau))$ satisfy the condition (1.3), then for $|\lambda|<\lambda_0$, (λ_0 sufficiently small), the equation (1.1) has a unique solution in $H_{\varphi,m}(M)$. The solution is uniformly convergent and can be obtained by the method of successive approximations.

Proof.

Let $u,v \in H_{\varphi,m}(M)$. Then by Lemmas 2.1, 2.2 and Theorem 2.1, the operator

$$(Pu)(t) = F(t,u(t),v(t)) \tag{3.1}$$

transforms the space $H_{\varphi,m}(M)$ into the space $H_{\varphi,m}(|\lambda|\, R)$. Therefore if $|\lambda| R \le M$, the operator P transforms $H_{\varphi,m}(M)$ into itself. Using M. Riesz's Theeorem [12,17],

$$\|\tilde{u}\|_{L_p} \le \gamma(p)\|u\|_{L_p}\ , \ p > 1 \tag{3.2}$$

where

$$\tilde{u}(t) = \frac{1}{2\pi}\int_0^{2\pi} u(\tau)\cot\frac{\tau - t}{2}d\tau\ ,$$

and from the conditions (1.3) , (1.4), we obtain

$$\| Pu_1 - Pu_2\|_{L_p} \le \xi(0)\big(1 + |\lambda|\gamma(P)\eta(0)\big)\| u_1 - u_2\|_{L_p}\ . \tag{3.3}$$

Choosing

$$|\lambda| < \min\left\{\frac{M}{R}, \frac{1-\xi(0)}{\gamma(P)\eta(0)\xi(0)}\right\} = \lambda_0$$

and

$$\xi(0)\big(1 + |\lambda|\gamma(P)\eta(0)\big) < 1,$$

then the operator P is a contraction mapping. From the completeness of $H_{\varphi,m}(M)$ in L_p, $P > 1$, the equation (1.1) has a unique solution in the subspace $H_{\varphi,m}(M)$ and this solution can be found by the method of successive approximations.

4- The approximate solution in the space $H^{(N)}_{\varphi,m}$

By the mechanical quadrature formula, [19], the integral

$$(I\,u)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\tau)\cot\frac{\tau - t}{2}d\tau \tag{4.1}$$

takes the following form

$$(I\,u)(t) = \frac{1}{N}\sum_{i=0}^{2N-1} u_i \sin^2\frac{t - t_i}{2}\cot\frac{t_i - t}{2} \tag{4.2}$$

where $u_i = u(t_i)$, $\quad t_i = \dfrac{i\pi}{N}$,

the formula (4.2) at node points t_i takes the form:

$$(I\,u)(t_j) = \frac{1}{2N}\sum_{\substack{i=0\\ i\ne j}}^{2N-1} u_i\,\alpha_{i-j}\cot\frac{t_i - t_j}{2} \tag{4.3}$$

where $\quad \alpha_{i-j} = \begin{cases} 0 & ,\ i-j\ even, \\ 2 & ,\ i-j\ odd. \end{cases}$

Applying the quadrature formula (4.3) to the equation (1.1) at the node points, we obtain

$$u(t_j)=F(t_j,u(t_j),\frac{\lambda}{2N}\sum_{\substack{i=0\\ i\neq j}}^{2N-1}g(t_j,t_i,u(t_i))\alpha_{i-j}\cot\frac{t_i-t_j}{2}+R_N(g,t_j)),$$

where $R_N(g,t_j)$ is the remainder term, $j=\overline{0,2N-1}$. If we put $u(t_j)=z_j$, and the $R_N(g,t_j)$ is negligible, we obtain the following system of nonlinear algebraic equations

$$z_j=F\left(t_j,z_j,\frac{\lambda}{2N}\sum_{\substack{i=0\\ i\neq j}}^{2N-1}g(t_j,t_i,z_i)\alpha_{i-j}\cot\frac{t_i-t_j}{2}\right). \tag{4.4}$$

Lemma 4.1 [6,15]

If the function $g(t,\tau,u)$ and its derivative $g_t(t,\tau,u)$ satisfy the condition (1.3), then the function

$$\chi(t,\tau,u)=g(t,\tau,u)-g\tau,\tau,u)$$

satisfies the following condition

$$\left|\chi(t,\tau,u_1)-\chi(t,\tau,u_2)\right|\le\eta(1)\,|t-\tau|\,|u_1-u_2|, \tag{4.5}$$

where $u_1,u_2\in[-M,M]$.

Theorem 4.1

Let the function $F(t,u,v)$ satisfy the condition (1.4) and the function $g(t,\tau,u)$ satisfy the condition (1.3), then the system (4.4) has a unique solution in the space $H_{\varphi,m}^{(N)}(M)$ for arbitrary $N\ge 3$, and this solution can be found by the method of successive approximations.

Proof.

From Definition 2.2, we have

$$H_{\varphi,m}^{(N)}(M)=\left\{z\in H_{\varphi,m}^{(N)}:\|z\|_{\varphi,m}^{(N)}\le M,\ \ M>0\right\},$$

where

$$z=(z_0,z_1,...,z_{2N-1}).$$

Putting

$$Jz=\left(g(t_0,\tau_0,z_0),...,g(t_{2N-1},\tau_{2N-1},z_{2N-1})\right),$$

since the space $H_{\varphi,m}^{(N)}(M)$ of vectors of bounded norms is a closed subspace of $L_p^{(N)}$ and the function $g(t,\tau,z)$ satisfies the condition of Lemmas 2.1, 2.2 and Theorem 3.1 hence $Jz\in H_{\varphi,m}^{(N)}(R')$.

Taking

$$P^{(N)}(z)=\left(F\left(t_0,z_0,K_0^{(N)}z\right),...,\ F\left(t_{2N-1},z_{2N-1},K_{2N-1}^{(N)}z\right)\right),$$

$$K^{(N)}z=\left(K_o^{(N)}z,...,K_{2N-1}^{(N)}z\right),$$

where

$$K_j^{(N)} z = \frac{\lambda}{2N} \sum_{\substack{i=0 \\ i \neq j}}^{2N-1} g(t_j, t_i, z_i) \alpha_{i-j} \cot \frac{t_i - t_j}{2},$$

let

$$K^{(N)} z = \lambda E^{(N)} J z,$$

where

$$E^{(N)} z = \left(E_0^{(N)} z, \ldots, E_{2N-1}^{(N)} z \right),$$

$$E_j^{(N)} z = \frac{1}{2N} \sum_{\substack{i=0 \\ i \neq j}}^{2N-1} z_{j,i} \, \alpha_{i-j} \cot\left(\frac{i-j}{2N} \right) \pi,$$

and

$$\left\| E^{(N)} \right\|_{\varphi,m}^{(N)} \leq \theta(m), \text{ [17,18]},$$

where $\theta(m)$ is a constant depends on m. Thus we have

$$\left\| K^{(N)} z \right\|_{\varphi,m}^{(N)} \leq |\lambda| R' \theta(m),$$

Now, let

$$z^{(k)} = \left(z_0^{(k)}, z_1^{(k)}, \ldots, z_{2N-1}^{(k)} \right) \in H_{\varphi,m}^{(N)}(M), \; k = 1,2$$

Hence

$$\left\| P^{(N)} z^{(1)} - P^{(N)} z^{(2)} \right\|_{L_p^{(N)}} = \left\| F\left(t, z^{(1)}, K^{(N)} z^{(1)} \right) - F\left(t, z^{(2)}, K^{(N)} z^{(2)} \right) \right\|_{L_p^{(N)}}$$

$$\leq \xi(0) \left(\left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}} + \left\| K^{(N)} z^{(1)} - K^{(N)} z^{(2)} \right\|_{L_p^{(N)}} \right) \quad (4.6)$$

since

$$\left\| K^{(N)} \right\|_{L_p^{(N)}} \leq q(p), \; p > 1, \text{ [19]}, \quad (4.7)$$

where $q(p)$ is a constant depends on p.

Hence from condition (1.3), Lemma 4.1 and from [17], we obtain

$$\left\| K^{(N)} z^{(1)} - K^{(N)} z^{(2)} \right\|_{L_p^{(N)}} \leq |\lambda| q(p) \eta(0) \left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}} + \\ + \left\{ \frac{|\lambda| \eta(1)}{\pi^{1/q}} \left[\int_0^{11\pi/6} \left(\frac{x/2}{\sin x/2} \right)^q dx \right]^{1/q} \right\} \left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}}, \quad (4.8)$$

substituting from inequality (4.8) into (4.6), we get

$$\left\| P^{(N)} z^{(1)} - P^{(N)} z^{(2)} \right\|_{L_p^{(N)}} \leq \xi(0) \left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}} + \\ + \xi(0) |\lambda| \left\{ q(p) \eta(0) + \frac{\eta(1)}{\pi^{1/q}} \left[\int_0^{11\pi/6} \left(\frac{x/2}{\sin x/2} \right)^q dx \right]^{1/q} \right\} \left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}}. \quad (4.9)$$

From boundedness of the operator $K^{(N)}$ in $L_p^{(N)}$ and by using the principle of contraction mapping at

$$|\lambda| < \min\left\{ \frac{M}{R'\theta(m)}, \frac{1-\xi(0)}{\xi(0)} \left(q(p)\eta(0) + \frac{\eta(1)}{\pi^{1/q}} \left[\int_0^{11\pi/6} \left(\frac{x/2}{\sin x/2} \right)^q dx \right]^{1/q} \right)^{-1} \right\}, \quad (4.10)$$

the system (4.4) has a unique solution in $H_{\varphi,m}^{(N)}(M)$ for arbitrary $N \geq 3$, hence the theorem is proved.

5- The rate of convergence of the approximate solution

From inequality (4.10), the equation (1.1) has a unique solution $u^*(t) \in H_{\varphi,m}(M)$ and the system (4.4) has a unique solution $z^* = \left(z_0^*, z_1^*, ..., z_{2N-1}^*\right) \in H_{\varphi,m}^{(N)}(M)$.
The relation

$$u_N^*(t) = F\left(t, u_N^*(t), \frac{\lambda}{N} \sum_{i=0}^{2N-1} g\left(t, t_i, z_i^*\right) \sin^2 \frac{t-t_i}{2} \cot \frac{t_i - t}{2} \right), \quad (5.1)$$

at $t = t_j$ is called the approximate solution of the equation (1.1), $u_N^*(t) = z_j^*$, $\left(j = \overline{0, 2N-1}\right)$. The norm of the difference of the vectors z^* and u^* where $u^* = \left(u(t_0), u(t_1), ..., u(t_{2N-1})\right)$ in $L_p^{(N)}$ can be determined as follows:
Applying the quadrature formula (4.3) to equation (1.1) at node points t_j, we obtain

$$u^*(t_j) = F\left(t_j, u^*(t_j), \frac{\lambda}{2N} \sum_{i=0}^{2N-1} g\left(t_j, t_i, u^*(t_i)\right) \alpha_{i-j} \cot \frac{t_i - t_j}{2} + R_N\left(g, t_j\right) \right), \quad (5.2)$$

putting $z^{(1)} = u^*, z^{(2)} = z^*$ in (4.9) and using the inequality (4.10), we get

$$\left\| u^* - z^* \right\|_{L_p^{(N)}} \leq \xi(0) \left\| u^* - z^* \right\|_{L_p^{(N)}} + |\lambda| \xi(0) \left\| R_N(g,t) \right\|_c$$
$$+ \xi(0) |\lambda| \left\{ q(p)\eta(0) + \frac{\eta(1)}{\pi^{1/q}} \left[\int_0^{11\pi/6} \left(\frac{x/2}{\sin x/2} \right)^q dx \right]^{1/q} \right\} \left\| u^* - z^* \right\|_{L_p^{(N)}},$$

consequently, we have

$$\left\| u^* - z^* \right\|_{L_p^{(N)}} \leq \frac{|\lambda| \xi(0) \left\| R_N(g,t) \right\|_c}{1 - \xi(0) - \xi(0) |\lambda| \left\{ q(p)\eta(0) + \frac{\eta(1)}{\pi^{1/q}} \left[\int_0^{11\pi/6} \left(\frac{x/2}{\sin x/2} \right)^q dx \right]^{1/q} \right\}}. \quad (5.3)$$

To evaluate $\left\| u^*(t) - u_N^*(t) \right\|_c$, we state the following two lemmas:

Lemma 5.1 [19]

Let $z=(z_0,z_1,...,z_{2N-1})\in H_{\varphi,m}^{(N)}(M)$, $\varphi\in\Phi^m$. Then for arbitrary natural number h, $0<h<\dfrac{N}{2(m+1)}$, we get

$$\max_i |z_i|\le l(m,M)\left[\left(\frac{N}{h}\right)^{1/p}\| z\|_{L_p^{(N)}}+\varphi\left(\frac{\pi\ h}{N}\right)\right].$$

Lemma 5.2 [17]

$$\frac{1}{N}\sum_{i=0}^{2N-1}\left|\sin^2\frac{t-t_i}{2}\cot\frac{t_i-t}{2}\right|\le 2(1+\pi)(1+\ln(2N)).$$

Applying formula (4.2) on the equation (1.1) and from equation (5.1), we obtain

$$\left\|u^*(t)-u_N^*(t)\right\|_c\le\xi(0)\left\|u^*(t)-u_N^*(t)\right\|_c+\xi(0)\eta(0)|\lambda|\max_i\left|u^*(t_i)-z_i^*\right|$$
$$\left(\frac{1}{N}\sum_{i=0}^{2N-1}\left|\sin^2\frac{t-t_i}{2}\cot\frac{t_i-t}{2}\right|\right)+\xi(0)|\lambda|\|R_N\|_c$$

Using Lemma 5.2

$$\begin{aligned}\left\|u^*(t)-u_N^*(t)\right\|_c &\le\frac{2\xi(0)\eta(0)}{1-\xi(0)}|\lambda|(1+\pi)(1+\ln 2N)\max_i\left|u^*(t_i)-z_i^*\right|\\ &+\frac{\xi(0)}{1-\xi(0)}|\lambda|\|R_N\|_c,\end{aligned}\tag{5.4}$$

from Lemma 5.1 and inequality (5.3), we get

$$\max_i\left|u^*(t_i)-z_i^*\right|\le l(m,M)\min_{2\le h\le N/2(m+1)}\left[\xi(0)|\lambda|\left(\frac{N}{h}\right)^{1/p}\|R_N(g,t)z\|_c+\varphi\left(\frac{\pi\ h}{N}\right)\right],$$

since

$$\|R_N(g,t)\|_c\le l(m)\varphi(\pi/N)\ln N,\ [15,19],\tag{5.5}$$

therefore,

$$\max_i\left|u^*(t_i)-z_i^*\right|\le l(m,M)|\lambda|\min_{2\le h\le N/2(m+1)}\left[\xi(0)\left(\frac{N}{h}\right)^{1/p}\varphi\left(\frac{\pi}{N}\right)\ln N+\varphi\left(\frac{\pi\ h}{N}\right)\right],$$

taking $h=N^{\alpha}$, $p^{-1}<\alpha<1$, then

$$\max_i\left|u^*(t_i)-z_i^*\right|\le const\left[\xi(0)(\varphi(\pi/N)\ln N)\,N^{(p^{-1}-\alpha)}+\varphi\left(\frac{\pi}{N^{1-\alpha}}\right)\right],\tag{5.6}$$

consequently, from (5.4), (5.5) and (5.6), we obtain

$$\left\| u^*(t) - u_N^*(t) \right\|_c \le \frac{2\xi(0)\eta(0)}{1-\xi(0)} |\lambda| (1+\pi)(1+\ln 2N)\left[\xi(0)(\ln N / N^{\alpha-1/p})\varphi(\pi/N) + \varphi(\pi/N^{1-\alpha})\right]$$
$$+ \frac{\xi(0)}{1-\xi(0)} |\lambda| \, l(m)\varphi(\pi/N)\ln N.$$

Hence,

$$\left\| u^*(t) - u_N^*(t) \right\|_c \le const \left(\ln^2(N) / N^{\alpha - p^{-1}} \right),$$

Now we present the following examples.

Example 1. Consider the integral equation

$$u(t) = \frac{\lambda}{2\pi} \int_0^{2\pi} g(t,\tau,u(\tau)) \cot\frac{\tau - t}{2} d\tau + f(t), \tag{5.7}$$

where

$$g(t,\tau,u(\tau)) = \sin t \, \sin u(\tau), \qquad f(t) = t - \sin t \cos t.$$

It is easy to check that $u^*(t) = t$ is the exact solution of equation (5.7) at $\lambda = 1$. Applying the quadrature formula (4.3) to equation (5.7) at node points, we obtain the following system of nonlinear algebraic equations:

$$z_j^* = \frac{\lambda}{2N} \sum_{\substack{i=0 \\ i \ne j}}^{2N-1} \sin t_j \, \sin z_i^* \, \alpha_{i-j} \cot\frac{t_i - t_j}{2} + t_j - \sin t_j \cos t_j,$$

where

$$t_j = j\pi / N, \; j = 0,1,\ldots,2N-1.$$

Table 1 displays the exact solution, the approximate solution and error between them for the equation (5.7) by using the mechanical quadrature method with $N = 20, \lambda = 1$ and at initial values $z_j^* = 0$, $j = 0,1,\ldots,2N-1$.

Now, we apply the mechanical quadrature method to a class of LSIE.

Example 2. Consider the integral equation

$$u(t) - \lambda \int_{-\pi}^{\pi} u(\tau) \cot\frac{\tau - t}{2} d\tau = f(t), \tag{5.8}$$

under the condition

$$u(\pm\pi) = 0, \tag{5.9}$$

where

$$f(t) = \sin t - 2\pi \, \cos t.$$

It is clear that $u^*(t) = \sin t$ is the exact solution of equation (5.8) at $\lambda = 1$. Applying the quadrature formula (4.3) to equation (5.8) under the condition (5.9) at the node points t_j, we obtain the following system of linear algebraic equations:

$$z_j^* = \frac{\lambda}{2N} \sum_{\substack{i=-N+1 \\ i \neq j}}^{N-1} z_i^* \alpha_{i-j} \cot \frac{t_i - t_j}{2} + \sin t_j - 2\pi \, \cos t_j ,$$

where

$$t_j = j\pi / N, \; j = -N+1, ..., N-1.$$

The condition $u(\pm\pi) = 0$ reduces the node points t_j to $2N-1$ points.

Table 2 displays the exact solution, the approximate solution and the error between them for the equation (5.8) under the condition (5.9) by using the mechanical quadrature method with $N = 20$, $\lambda = 1$.

Conclusions

1. Tables 1, 2 display that the mechanical quadrature method gives accurate results with respect to NSIE and LSIE, these results are very acceptable compared to the exact solution.

2. The Toeplitz matrix method and the product Nystrom method have been applied to the same equation (5.8) under the condition (5.9) in [1], table 3 displays the values of exact solution $u(t) = \sin t$, approximate solution $u_n^{(T)}$ and the error $R^{(T)}$ at the interior points by using the Toeplitz matrix method with $N = 20, \lambda = 1$. Also it shows the approximate solution $u_n^{(N)}$ and the error $R^{(N)}$ at the interior points by using the product Nystrom method with $N = 40, \lambda = 1$.

3. Its found that, the obtained results of the mechanical quadrature method are better than the obtained results of the Toeplitz matrix method and the product Nystrom method that have been applied in [1] to obtain the approximate solution of the same problem (5.8) under the same condition (5.9).

t	u^*	z^*	E
.0000000	.0000000	.0000000	0 .000000000
.1570796	.1570796	.1570795	1.490116E-07
.3141593	.3141593	.3141592	2.980232E-08
.4712389	.4712389	.4712395	5.960464E-07
.6283185	.6283185	.6283190	4.172325E-07
.7853982	.7853982	.7853984	2.384186E-07
.9424778	.9424778	.9424775	2.980232E-07
1.099557	1.099557	1.099557	4.768372E-07
1.256637	1.256637	1.256638	4.768372E-07
1.413717	1.413717	1.413717	4.768372E-07
1.570796	1.570796	1.570797	4.768372E-07
1.727876	1.727876	1.727875	1.072884E-06
1.884956	1.884956	1.884955	5.960464E-07
2.042035	2.042035	2.042036	4.768372E-07
2.199115	2.199115	2.199115	2.384186E-07
2.356194	2.356194	2.356194	4.768372E-07
2.513274	2.513274	2.513274	2.384186E-07
2.670354	2.670354	2.670354	0.000000E+00
2.827434	2.827434	2.827434	4.768372E-07
2.984513	2.984513	2.984513	0.000000E+00
3.141593	3.141593	3.141593	2.384186E-07
3.298672	3.298672	3.298672	0.000000E+00
3.455752	3.455752	3.455752	0.000000E+00
3.612832	3.612832	3.612832	2.384186E-07
3.769911	3.769911	3.769912	2.384186E-07
3.926991	3.926991	3.926991	4.768372E-07
4.084071	4.084071	4.084071	4.768372E-07
4.241150	4.241150	4.241152	1.907349E-06
4.398230	4.398230	4.398227	2.384186E-06
4.555309	4.555309	4.555309	0.000000E+00
4.712389	4.712389	4.712389	0.000000E+00
4.869469	4.869469	4.869469	0.000000E+00
5.026548	5.026548	5.026549	4.768372E-07
5.183628	5.183628	5.183627	9.536743E-07
5.340708	5.340708	5.340708	4.768372E-07
5.497787	5.497787	5.497787	0.000000E+00
5.654867	5.654867	5.654868	9.536743E-07
5.811946	5.811946	5.811946	0.000000E+00
5.969026	5.969026	5.969025	9.536743E-07
6.126106	6.126106	6.126106	0.000000E+00

Table 1: The results for the Eq. (5.7).
by using the mechanical quadrature method

t	u^*	z^*	E
-.298451E+01	-.1564343E+00	-.1564343E+00	1.192093E-07
-.282743E+01	-.3090165E+00	-.3090165E+00	2.682209E-07
-.267035E+01	-.4539907E+00	-.4539907E+00	2.980232E-07
-.251327E+01	-.5877851E+00	-.5877851E+00	5.960464E-08
-.235619E+01	-.7071072E+00	-.7071072E+00	4.768372E-07
-.219911E+01	-.8090172E+00	-.8090172E+00	2.384186E-07
-.204204E+01	-.8910056E+00	-.8910056E+00	8.940697E-07
-.188496E+01	-.9510571E+00	-.9510571E+00	6.556511E-07
-.172788E+01	-.9876884E+00	-.9876884E+00	0.000000E+00
-.157080E+01	-.1000000E+01	-.9999997E+00	2.980232E-07
-.141372E+01	-.9876881E+00	-.9876881E+00	2.384186E-07
-.125664E+01	-.9510569E+00	-.9510569E\+00	3.576279E-07
-.109956E+01	-.8910068E+00	-.8910068E+00	2.384186E-07
-.942478E+00	-.8090168E+00	-.8090168E+00	1.788139E-07
-.785398E+00	-.7071068E+00	-.7071068E+00	0.000000E+00
-.628319E+00	-.5877856E+00	-.5877856E+00	3.576279E-07
-.471239E+00	-.4539903E+00	-.4539903E+00	1.788139E-07
-.314159E+00	-.3090171E+00	-.3090171E+00	1.192093E-07
-.157080E+00	-.1564343E+00	-.1564343E+00	1.788139E-07
.000000E+00	.0000000E+00	-.9783179E-07	9.783179E-08
.157080E+00	.1564345E+00	.1564345E+00	5.960464E-08
.314159E+00	.3090171E+00	.3090171E+00	1.192093E-07
.471239E+00	.4539906E+00	.4539906E+00	5.960464E-08
.628319E+00	.5877854E+00	.5877854E+00	1.192093E-07
.785399E+00	.7071070E+00	.7071070E+00	2.384186E-07
.942478E+00	.8090170E+00	.8090170E+00	0.000000E+00
.109956E+01	.8910061E+00	.8910061E+00	4.768372E-07
.125664E+01	.9510570E+00	.9510570E+00	4.172325E-07
.141372E+01	.9876889E+00	.9876889E+00	5.364418E-07
.157080E+01	.1000000E+01	.9999995E+00	5.364418E-07
.172788E+01	.9876882E+00	.9876882E+00	1.192093E-07
.188496E+01	.9510567E+00	.9510567E+00	2.384186E-07
.204204E+01	.8910065E+00	.8910065E+00	5.960464E-08
.219912E+01	.8090163E+00	.8090163E+00	7.152557E-07
.235619E+01	.7071071E+00	.7071071E+00	3.576279E-07
.251327E+01	.5877852E+00	.5877852E+00	0.000000E+00
.267035E+01	.4539907E+00	.4539907E+00	2.980232E-07
.282743E+01	.3090197E+00	.3090169E+00	1.192093E-07
.298451E+01	.1564343E+00	.1564343E+00	1.341105E-07

Table 2: The results of Eq. (5.8)
by using the mechanical quadrature method.

t	u	$u_n^{(T)}$	$R^{(T)}$	$u_n^{(N)}$	$R^{(N)}$
-.298451E+01	-.156434E+00	-.157566E+00	.113183E-02	-.157904E+00	.146956E-02
-.282743E+01	-.309017E+00	-.306774E+00	.224258E-02	-.307179E+00	.183761E-02
-.267035E+01	-.453990E+00	-.456815E+00	.282412E-02	-.454282E+00	.291620E-03
-.251327E+01	-.587785E+00	-.587984E+00	.198949E-03	-.587120E+00	.665654E-03
-.235619E+01	-.707107E+00	-.710086E+00	.297899E-02	-.706959E+00	.147790E-03
-.219911E+01	-.809017E+00	-.810573E+00	.155627E-02	-.808990E+00	.265871E-04
-.204204E+01	-.891007E+00	-.894161E+00	.315436E-02	-.890639E+00	.367293E-03
-.188496E+01	-.951057E+00	-.953450E+00	.239302E-02	-.951288E+00	.231250E-03
-.172788E+01	-.987688E+00	-.990983E+00	.329504E-02	-.987275E+00	.413104E-03
-.157080E+01	-.100000E+01	-.100286E+00	.285655E-02	-.100020E+01	.197142E-03
-.141372E+01	-.987688E+00	-.991010E+00	.332117E-02	-.987354E+00	.334045E-03
-.125664E+01	-.951057E+00	-.954070E+00	.301384E-02	-.951009E+00	.476252E-04
-.109956E+01	-.891007E+00	-.894194E+00	.318768E-02	-.890833E+00	.173579E-03
-.942478E+00	-.809017E+00	-.811927E+00	.291043E-02	-.808591E+00	.425900E-03
-.785398E+00	-.707107E+00	-.709991E+00	.288390E-02	-.707135E+00	.285526E-04
-.628319E+00	-.587785E+00	-.590376E+00	.259122E-02	-.586921E+00	.864526E-03
-.471239E+00	-.453990E+00	-.456418E+00	.242748E-02	-.454227E+00	.236237E-03
-.314159E+00	-.309017E+00	-.311125E+00	.210820E-02	-.307721E+00	.129617E-02
-.157080E+00	-.156434E+00	-.158292E+00	.185783E-02	-.156852E+00	.417934E-03
.000000E+00	.000000E+00	-.152180E-02	.152180E-02	.166220E-02	.166220E-02
.157080E+00	.156434E+00	.155205E+00	.122941E-02	.155886E+00	.548042E-03
.314159E+00	.309017E+00	.308119E+00	.898460E-03	.310933E+00	.191594E-02
.471239E+00	.453990E+00	.453386E+00	.604871E-03	.453382E+00	.608263E-03
.628319E+00	.587785E+00	.587479E+00	.306284E-03	.589811E+00	.202547E-02
.785398E+00	.707107E+00	.707059E+00	.481999E-04	.706518E+00	.588612E-03
.942478E+00	.809017E+00	.809207E+00	.190301E-03	.810993E+00	.197563E-02
.109956E+01	.891007E+00	.891388E+00	.381596E-03	.890519E+00	.487828E-03
.125664E+01	.951057E+00	.951592E+00	.535616E-03	.952825E+00	.176875E-02
.141372E+01	.987688E+00	.988324E+00	.635698E-03	.987375E+00	.313036E-03
.157080E+01	.100000E+01	.100069E+01	.686574E-03	.100142E+01	.142411E-02
.172788E+01	.987688E+00	.988367E+00	.678469E-03	.987610E+00	.784910E-04
.188496E+01	.951057E+00	.951670E+00	.613645E-03	.952033E+00	.976132E-03
.204204E+01	.891007E+00	.891492E+00	.485934E-03	.891203E+00	.196925E-03
.219911E+01	.809017E+00	.809312E+00	.294613E-03	.809489E+00	.472495E-03
.235619E+01	.707107E+00	.707138E+00	.308381E-04	.707602E+00	.495305E-03
.251327E+01	.587785E+00	.587465E+00	.320492E-03	.587762E+00	.234950E-04
.267035E+01	.453990E+00	.453182E+00	.808603E-03	.454816E+00	.825687E-03
.282743E+01	.309017E+00	.307399E+00	.161841E-02	.308643E+00	.373926E-03
.298451E+01	.156434E+00	.152392E+00	.404226E-02	.157947E+00	.151280E-02

Table 3: The results of Eq. (5.8)

by using the Toeplitz matrix method and the product Nystrom method.

REFERENCES

1. Abdou M. A., Khamis I. M. and Ismal A. S.; Toeplitz matrix and product Nystrom methods for solving the singular integral equation; Estratto da “Le Matematiche” Vol. LVII 2002- Fasc. I. 21-37.

2. Amer S. M. and Nagdy A.S.; On the modified Newton’s approximation method for the solution of nonlinear singular integral equations, Hokkaido Mathematical Journal. Vol. 29 (2000). 59-72.

3. Amer S. M. and Dardery S.; On the theory of nonlinear singular integral equations with shift on a closed contour ;Forum Math. Vol.17 No.5(2005)753-780

4. Amer S.M.; On the approximate solution of nonlinear singular integral equations with positive index; Inernat, J. Math. & Math. Sci. Vol.19 No.2 (1996) 389-396.

5. Amer S. M. On solution of nonlinear singular integral equations with shift in generalized Holder space; Chaos, Solitons & Fractals ,Vol.12 / 7 (2001)1323-1334.

6. Amer S. M.; Existence results for a class of nonlinear singular integral equations with shift; Il Nuovo Cimento B Vol. 120 issue 03 (2005) 313-333.

7. Filomena Cianciaruso and Espedito De Pascale, Estimates of majorizing sequences in the Newton-Kantorovich method: A further improvement, J. Math. Anal. Appl. 322(2006)329-335.

8. Guseinov, A. I., and Mukhtarov, Kh. Sh., Introduction to the Theory of Nonlinear Singular Integral Equations, Nauka Moscow (in Russian) 1980.

9. Hiroaki Mukaidani, Newton’s method for solving cross-coupled sign-indefinite algebraic Riccati equations for weakly coupled large-scale systems, Applied Mathematics and Computation 188(2007)103-115.

10. Jinyuan Du.; The collocation methods and singular integral equations with Cauchy kernels; Acta. Math. Sci. 20 B (3)(2000), 289-302.

11. Junghanns P. and Weber U., On the solvability of nonlinear singular integral equations, ZAA.Vol. 12 (1993) pp. 683-698.

12. Kravchenko L. V. and Akilov G. P.; Functional Analysis; Pergamon Press. Oxford (1982).

13. Ladopoulos E. G. and Zisis V.A.; Nonlinear singular integral approximations in Banach spaces; Nonlinear Analysis, Theory, Methods & Applications, Vol. 26, No.7 (1996), 1293-1299.

14. Mikhlin S. G. and Prossdorf S.; Singular Integral Operator; Academy-Verlag , Berlin, (1986).

15. Mosaev B. I. and Salaev V. V.; On the convergence of quadratic process for singular integrals with Hilbert kernel; (in Russian), In Coll. Timely problems in theory of functions. AGU. (1980). 186-194.

16. Pogorzelski W.; Integral Equations and their Applications;Vol.1, Oxford Pergamon Press and Warszawa, PWN (1966).

17. Saleh M.H. and Amer S. M.; Approximate solution of a certain class of nonlinear singular equations; Collect. Math. Vol.38. (1987). 161-174.

18. Saleh M.H. and Amer S. M.; The approximate solution of nonlinear singular Integro-differential equations; Collect. Math. Vol.41. 2(1990). 175-188.

19. Saleh M.H.; Basis of quadrature method for nonlinear singular integral equations with Hilbert kernel in the space $H_{\varphi,k}$; (in Russian), In AZ. NIINTI, No.279 (1984). 1-40.

20. Wegert E.; Nonlinear Boundary Value Problems for Holomorphic Functions and Singular Integral Equations; Akademie Verlage (1992).

21. Wolfersdorf L. V.; On the theory of nonlinear singular integral equations of cauchy type; Math. Meth. In the Appl. Sci. 7(1985), 493-517.

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ABEL'S METHOD ON SUMMATION BY PARTS AND BILATERAL WELL-POISED ${}_4\psi_4$-SERIES IDENTITIES

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ABSTRACT. By means of Abel's lemma on summation by parts, we establish two new identities of nonterminating bilateral well-poised basic hypergeometric ${}_4\psi_4$-series. Their applications to q-Clausen formulae are also discussed.

1. INTRODUCTION AND ABEL'S LEMMA ON SUMMATION BY PARTS

For two complex x and q, the shifted-factorial of x with base q is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad \text{for} \quad n \in \mathbb{N}. \tag{1.1}$$

When $|q|<1$, we have two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad \text{and} \quad (x;q)_n = (x;q)_\infty / (xq^n;q)_\infty. \tag{1.2}$$

In particular, the shifted factorial with negative integer order can be written explicitly from the last fraction as

$$(x;q)_{-n} = \frac{(-1)^n q^{\binom{1+n}{2}} x^{-n}}{(q/x;q)_n} \quad \text{for} \quad n \in \mathbb{N}. \tag{1.3}$$

The product and fraction of shifted factorials are abbreviated respectively to

$$[\alpha, \beta, \cdots, \gamma; q]_n = (\alpha;q)_n (\beta;q)_n \cdots (\gamma;q)_n, \tag{1.4}$$

$$\left[\begin{matrix}\alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C\end{matrix} \,\middle|\, q\right]_n = \frac{(\alpha;q)_n (\beta;q)_n \cdots (\gamma;q)_n}{(A;q)_n (B;q)_n \cdots (C;q)_n}. \tag{1.5}$$

Following Bailey [2] and Slater [13], the basic hypergeometric series and the corresponding bilateral series are defined, respectively, by

$${}_{1+r}\phi_s\left[\begin{matrix}a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s\end{matrix} \,\middle|\, q;\, z\right] = \sum_{n=0}^{+\infty} z^n \left[\begin{matrix}a_0, a_1, \cdots, a_r \\ q, b_1, \cdots, b_s\end{matrix} \,\middle|\, q\right]_n \tag{1.6}$$

$${}_r\psi_s\left[\begin{matrix}a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s\end{matrix} \,\middle|\, q;\, z\right] = \sum_{n=-\infty}^{+\infty} z^n \left[\begin{matrix}a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s\end{matrix} \,\middle|\, q\right]_n \tag{1.7}$$

where the base q will be restricted to $|q|<1$ for non-terminating q-series.

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One of the most important and useful identities in the theory of basic hypergeometric series is Bailey's summation formula [3] for a nonterminating bilateral very-well-poised ${}_6\psi_6$-series (see also [10, II-33]):

$$
{}_6\psi_6\left[\begin{matrix} qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e \\ a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\,\middle|\, q;\ \frac{qa^2}{bcde}\right] \tag{1.8a}
$$

$$
= \left[\begin{matrix} q,\ qa,\ q/a,\ qa/bc,\ qa/bd,\ qa/be,\ qa/cd,\ qa/ce,\ qa/de \\ qa/b,\ qa/c,\ qa/d,\ qa/e,\ q/b,\ q/c,\ q/d,\ q/e,\ qa^2/bcde \end{matrix}\,\middle|\, q\right]_\infty \tag{1.8b}
$$

where we assume $\left|qa^2/bcde\right| < 1$ for convergence. It has recently been provided a completely new and simple proof by Chu [8] through Abel's lemma on summation by parts. By employing this approach further, this paper will prove two nonterminating bilateral well-poised ${}_4\psi_4$-series identities. Their finite forms are closely related to q-Clausen formulae.

Abel's lemma on summation by parts has been shown very useful and important in classical analysis. For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and $\triangle$, respectively, by

$$
\nabla\tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \triangle\tau_k = \tau_k - \tau_{k+1} \tag{1.9}
$$

where $\triangle$ is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign.

Then **Abel's lemma** on summation by parts may be reformulated as

$$
\sum_{k=-\infty}^{+\infty} B_k \nabla A_k \;=\; [AB]_{+\infty} - [AB]_{-\infty} + \sum_{k=-\infty}^{+\infty} A_k \triangle B_k \tag{1.10}
$$

provided that the two limits $[AB]_{\pm\infty} := \lim_{n\to\pm\infty} A_n B_{n+1}$ exist and one of both series just displayed is convergent.

Proof. Let n be a natural number. According to the definition of the backward difference, we have

$$
\sum_{k=-n}^{n} B_k \nabla A_k \;=\; \sum_{k=-n}^{n} B_k\big\{A_k - A_{k-1}\big\} \;=\; \sum_{k=-n}^{n} A_k B_k \;-\; \sum_{k=-n}^{n} A_{k-1} B_k.
$$

Replacing k by $k+1$ for the last sum, we get the following expression:

$$
\begin{aligned}
\sum_{k=-n}^{n} B_k \nabla A_k \quad &= \quad A_n B_{n+1} - A_{-n-1} B_{-n} + \sum_{k=-n}^{n} A_k\big\{B_k - B_{k+1}\big\} \\
&= \quad A_n B_{n+1} - A_{-n-1} B_{-n} + \sum_{k=-n}^{n} A_k \triangle B_k.
\end{aligned}
$$

Letting $n \to +\infty$, we get the identity stated in the lemma. □

2. The Bilateral ${}_4\psi_4$-Series Identities

In this section, we state two main summation theorems (see Theorem 1 and Theorem 3 below) on nonterminating bilateral well-poised ${}_4\psi_4$-series and discuss their implications. The proofs of the theorems will be given in the next section. We also remark that the q-series identities presented here are not consequences of Bailey's celebrated ${}_6\psi_6$-series identity (1.8a-1.8b), even it has been expected by the author before starting to write this work.

§2.1. **The first well-poised ${}_4\psi_4$-series identity.** The first nonterminating bilateral well-poised ${}_4\psi_4$-series identity is given by the following surprising theorem, which expresses bilateral well-poised ${}_4\psi_4$-series in terms of unilateral well-poised series.

Theorem 1 (Bilateral well-poised ${}_4\psi_4$-series identity)**.** *For three indeterminate $\{b,c,d\}$ satisfying the condition $|q^{1/2}/bd|<1$, there holds the transformation:*

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} b, & c, & d, & q^{1/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\middle|\, q;q\right] = {}_4\psi_4\left[\begin{matrix} b, & c, & d, & q^{1/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\middle|\, q;q^2\right] && (2.1a)\\
&= \left[\begin{matrix} q,\ q/bc,\ q/bd,\ q/cd,\ q^{1/2}/b,\ q^{1/2}/c,\ q^{1/2}/d,\ q^{1/2}/bcd \\ q^{1/2},\ q^{1/2}/bc,\ q^{1/2}/bd,\ q^{1/2}/cd,\ q/b,\ q/c,\ q/d,\ q/bcd \end{matrix}\,\middle|\, q\right]_\infty && (2.1b)\\
&+ \frac{(q^{1/2}/bcd)(1-q^{3/2}/bc^2d)}{(1-q^{1/2}/bc)(1-q^{1/2}/cd)(1-q/bcd)}\left[\begin{matrix} b, & c, & d, & q^{3/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\middle|\, q\right]_\infty && (2.1c)\\
&\times \sum_{k=0}^{+\infty}\frac{1-q^{2k+3/2}/bc^2d}{1-q^{3/2}/bc^2d}\left[\begin{matrix} q/c, & q^{1/2}/c, & q/bc, & q/cd \\ q^{3/2}/bc, & q^{3/2}/cd, & q^2/bcd, & q^{3/2}/bcd \end{matrix}\,\middle|\, q\right]_k\Big(\frac{q^{1/2}}{bd}\Big)^k. && (2.1d)
\end{aligned}
$$

Both series ${}_4\psi_4$ displayed in (2.1a) are reversal each other. In view of the fact that

$$
\frac{(qw;q)_k}{(w;q)_k} = \frac{1-wq^k}{1-w} = \frac{1}{1-w} - \frac{wq^k}{1-w} \tag{2.2}
$$

the linear combination of the two series in (2.1a) leads us to the following bilateral series identity with an extra w-parameter.

Proposition 2 (Bilateral well-poised ${}_5\psi_5$-series identity)**.** *For three indeterminate $\{b,c,d\}$ satisfying the condition $|q^{1/2}/bd|<1$, there holds the following identity:*

$$
\begin{aligned}
&{}_5\psi_5\left[\begin{matrix} qw, & b, & c, & d, & q^{1/2}/bcd \\ w, & q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\middle|\, q;\ q\right] && (2.3a)\\
&= \left[\begin{matrix} q,\ q/bc,\ q/bd,\ q/cd,\ q^{1/2}/b,\ q^{1/2}/c,\ q^{1/2}/d,\ q^{1/2}/bcd \\ q^{1/2},\ q^{1/2}/bc,\ q^{1/2}/bd,\ q^{1/2}/cd,\ q/b,\ q/c,\ q/d,\ q/bcd \end{matrix}\,\middle|\, q\right]_\infty && (2.3b)\\
&+ \frac{(q^{1/2}/bcd)(1-q^{3/2}/bc^2d)}{(1-q^{1/2}/bc)(1-q^{1/2}/cd)(1-q/bcd)}\left[\begin{matrix} b, & c, & d, & q^{3/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\middle|\, q\right]_\infty && (2.3c)\\
&\times \sum_{k=0}^{+\infty}\frac{1-q^{2k+3/2}/bc^2d}{1-q^{3/2}/bc^2d}\left[\begin{matrix} q/c, & q^{1/2}/c, & q/bc, & q/cd \\ q^{3/2}/bc, & q^{3/2}/cd, & q^2/bcd, & q^{3/2}/bcd \end{matrix}\,\middle|\, q\right]_k\Big(\frac{q^{1/2}}{bd}\Big)^k. && (2.3d)
\end{aligned}
$$

§2.2. **The second well-poised ${}_4\psi_4$-series identity.** The second nonterminating bilateral well-poised ${}_4\psi_4$-series identity is given by the following theorem, which expresses another bilateral well-poised ${}_4\psi_4$-series in terms of unilateral well-poised series.

Theorem 3 (Bilateral well-poised ${}_4\psi_4$-series identity). *For three indeterminate $\{b, c, d\}$ satisfying the condition $|q^{3/2}/bd| < 1$, there holds the transformation:*

$$
{}_4\psi_4\left[\begin{matrix} b, & c, & d, & q^{5/2}/bcd \\ q^2/b, & q^2/c, & q^2/d, & bcd/q^{1/2}\end{matrix}\,\middle|\, q; q\right] = q^{\frac12}\,{}_4\psi_4\left[\begin{matrix} b, & c, & d, & q^{5/2}/bcd \\ q^2/b, & q^2/c, & q^2/d, & bcd/q^{1/2}\end{matrix}\,\middle|\, q; q^2\right] \tag{2.4a}
$$
$$
= \left[\begin{matrix} q,\ q^{3/2}/b,\ q^{3/2}/c,\ q^{3/2}/d,\ q^2/bc,\ q^2/bd,\ q^2/cd,\ q^{3/2}/bcd \\ q^{1/2},\ q^2/b,\ q^2/c,\ q^2/d,\ q^{3/2}/bc,\ q^{3/2}/bd,\ q^{3/2}/cd,\ q^2/bcd\end{matrix}\,\middle|\, q\right]_\infty \tag{2.4b}
$$
$$
+ \frac{(q^{3/2}/bcd)(1-q^{7/2}/bc^2d)}{(1-q^{3/2}/bc)(1-q^{3/2}/cd)(1-q^2/bcd)}\left[\begin{matrix} b, & c, & d, & q^{7/2}/bcd \\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd\end{matrix}\,\middle|\, q\right]_\infty \tag{2.4c}
$$
$$
\times \sum_{k=0}^{+\infty} \frac{1-q^{2k+7/2}/bc^2d}{1-q^{7/2}/bc^2d}\left[\begin{matrix} q/c, & q^{3/2}/c, & q^2/bc, & q^2/cd \\ q^{5/2}/bc, & q^{5/2}/cd, & q^3/bcd, & q^{7/2}/bcd\end{matrix}\,\middle|\, q\right]_k \Big(\frac{q^{3/2}}{bd}\Big)^k. \tag{2.4d}
$$

The first equation (2.4a) follows from inverting the summation index for the bilateral series. According to (2.2), we have similarly the linear combination of the two series displayed in (2.4a) with an extra w-parameter.

Proposition 4 (Bilateral well-poised ${}_5\psi_5$-series identity). *For three indeterminate $\{b, c, d\}$ satisfying the condition $|q^{3/2}/bd| < 1$, there holds the following identity:*

$$
{}_5\psi_5\left[\begin{matrix} qw, & b, & c, & d, & q^{5/2}/bcd \\ w, & q^2/b, & q^2/c, & q^2/d, & bcd/q^{1/2}\end{matrix}\,\middle|\, q;\ q\right] \times \frac{1-w}{1-w/q^{\frac12}} \tag{2.5a}
$$
$$
= \left[\begin{matrix} q,\ q^{3/2}/b,\ q^{3/2}/c,\ q^{3/2}/d,\ q^2/bc,\ q^2/bd,\ q^2/cd,\ q^{3/2}/bcd \\ q^{1/2},\ q^2/b,\ q^2/c,\ q^2/d,\ q^{3/2}/bc,\ q^{3/2}/bd,\ q^{3/2}/cd,\ q^2/bcd\end{matrix}\,\middle|\, q\right]_\infty \tag{2.5b}
$$
$$
+ \frac{(q^{3/2}/bcd)(1-q^{7/2}/bc^2d)}{(1-q^{3/2}/bc)(1-q^{3/2}/cd)(1-q^2/bcd)}\left[\begin{matrix} b, & c, & d, & q^{7/2}/bcd \\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd\end{matrix}\,\middle|\, q\right]_\infty \tag{2.5c}
$$
$$
\times \sum_{k=0}^{+\infty} \frac{1-q^{2k+7/2}/bc^2d}{1-q^{7/2}/bc^2d}\left[\begin{matrix} q/c, & q^{3/2}/c, & q^2/bc, & q^2/cd \\ q^{5/2}/bc, & q^{5/2}/cd, & q^3/bcd, & q^{7/2}/bcd\end{matrix}\,\middle|\, q\right]_k \Big(\frac{q^{3/2}}{bd}\Big)^k. \tag{2.5d}
$$

§2.3. **Well-poised terminating series.** Letting $c = q^{-n}$ in Propositions 2 and 4, it is trivial to see that the two extra terms displayed in (2.3c-2.3d) and (2.5c-2.5d) vanish. Reformulating the corresponding terminating sums and then relabeling the parameters, we can unify the resulting identities to the following single one.

Corollary 5 (Well-poised terminating ${}_5\phi_4$-series identity). *For a natural number m, there holds the following terminating series identity:*

$$
\begin{aligned}
&{}_5\phi_4\left[\begin{matrix} q^{-m}, & qw, & U, & V, & q^{1/2-m}/UV \\ & w, & q^{1-m}/U, & q^{1-m}/V, & q^{1/2}UV\end{matrix}\,\middle|\, q; q\right] \\
&= \frac{w-q^{-m/2}}{w-1}\left[\begin{matrix} q, UV \\ U, V\end{matrix}\,\middle|\, q\right]_m \left[\begin{matrix} U, V \\ q^{1/2}, UV\end{matrix}\,\middle|\, q^{1/2}\right]_m.
\end{aligned}
$$

The last identity is essentially discovered by Jackson [12, Eq 3]. Special cases of this identity have been investigated by Bailey [4, Eq 4], Carlitz [6, Eq 1.3], Guo [11, Eq 3.9] and Jain-Verma [15, Eqs 4.13 & 4.18]. It will systematically be utilized in the fourth section of this paper to treat the q-Clausen product formulae.

§2.4. **Classical hypergeometric series.** We write down also the classical hypergeometric series identity corresponding to the limiting case $q \to 1$. For the notation of classical hypergeometric series, we refer the reader to Slater [13, Chapter 6].

Recall the q-Gamma function [10, §1.10]

$$\Gamma_q(x) \;=\; (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} \quad \text{and} \quad \lim_{q\to 1^-}\Gamma_q(x) \;=\; \Gamma(x). \tag{2.6}$$

Performing replacements $b \to q^b$, $c \to q^c$, $d \to q^d$ and $w \to q^w$ in Proposition 2 and then letting $q \to 1$, we derive the following classical hypergeometric series identity.

Theorem 6 (Bilateral well-poised ${}_5H_5$-series identity)**.** *For three indeterminate $\{b,c,d\}$ satisfying the condition $\Re(b+d) < 1/2$, there holds the following identity:*

$$\begin{aligned}
&{}_5H_5\left[\begin{matrix}1+w, & b, & c, & d, & 1/2-b-c-d\\ w, & 1-b, & 1-c, & 1-d, & 1/2+b+c+d\end{matrix}\,\middle|\,1\right] && \text{(2.7a)}\\
=\;&\sqrt{\pi}\,\Gamma\left[\begin{matrix}1/2-b-c,1/2-b-d,1/2-c-d,1-b,1-c,1-d,1-b-c-d\\ 1-b-c,1-b-d,1-c-d,1/2-b,1/2-c,1/2-d,1/2-b-c-d\end{matrix}\right] && \text{(2.7b)}\\
+\;&\frac{b+2c+d-3/2}{(b+c-1/2)(c+d-1/2)(b+c+d-1)}\Gamma\left[\begin{matrix}1-b, & 1-c, & 1-d, & 1/2+b+c+d\\ b, & c, & d, & 3/2-b-c-d\end{matrix}\right] && \text{(2.7c)}\\
\times\;&\sum_{k=0}^{+\infty}\frac{3/2-b-2c-d+2k}{3/2-b-2c-d}\left[\begin{matrix}1-b-c, & 1-c-d\\ 3/2-b-c, & 3/2-c-d\end{matrix}\right]_k\frac{(1-2c)_{2k}}{(3-2b-2c-2d)_{2k}}. && \text{(2.7d)}
\end{aligned}$$

Similarly, we find from Proposition 4 another bilateral well-poised series identity.

Theorem 7 (Bilateral well-poised ${}_5H_5$-series identity)**.** *For three indeterminate $\{b,c,d\}$ satisfying the condition $\Re(b+d) < 3/2$, there holds the following identity:*

$$\begin{aligned}
&{}_5H_5\left[\begin{matrix}1+w, & b, & c, & d, & 5/2-b-c-d\\ w, & 2-b, & 2-c, & 2-d, & b+c+d-1/2\end{matrix}\,\middle|\,1\right]\times\frac{2w}{2w-1} && \text{(2.8a)}\\
=\;&\sqrt{\pi}\,\Gamma\left[\begin{matrix}3/2-b-c,3/2-b-d,3/2-c-d,2-b,2-c,2-d,2-b-c-d\\ 2-b-c,2-b-d,2-c-d,3/2-b,3/2-c,3/2-d,3/2-b-c-d\end{matrix}\right] && \text{(2.8b)}\\
+\;&\frac{b+2c+d-7/2}{(b+c-3/2)(c+d-3/2)(b+c+d-2)}\Gamma\left[\begin{matrix}2-b, & 2-c, & 2-d, & b+c+d-1/2\\ b, & c, & d, & 7/2-b-c-d\end{matrix}\right] && \text{(2.8c)}\\
\times\;&\sum_{k=0}^{+\infty}\frac{7/2-b-2c-d+2k}{7/2-b-2c-d}\left[\begin{matrix}2-b-c, & 2-c-d\\ 5/2-b-c, & 5/2-c-d\end{matrix}\right]_k\frac{(2-2c)_{2k}}{(6-2b-2c-2d)_{2k}}. && \text{(2.8d)}
\end{aligned}$$

It seems that the both identities just displayed have not appeared previously in the literature on classical hypergeometric series.

3. Proofs via Abel's method on summation by parts

This section will be devoted to the proofs of the identities displayed in the last section, essentially Theorem 1 and Theorem 3. It is interesting to observe that the two bilateral well-poised ${}_4\psi_4$-series from both theorems will be tied together by two crossing recurrence relations and then be confirmed through functional equation iteration and limiting process.

Let us first define two functions by

$$\mathfrak{U}(b,c,d) \;:=\; {}_4\psi_4\left[\begin{matrix}b, & c, & d, & q^{1/2}/bcd\\ q/b, & q/c, & q/d, & q^{1/2}bcd\end{matrix}\,\middle|\,q;q\right], \tag{3.1a}$$

$$\mathfrak{V}(b,c,d) \;:=\; {}_4\psi_4\left[\begin{matrix}b, & c, & d, & q^{5/2}/bcd\\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd\end{matrix}\,\middle|\,q;q\right]. \tag{3.1b}$$

Their reversals read respectively as

$$\mathfrak{U}(b,c,d) := \quad {}_4\psi_4\left[\begin{matrix} b, & c, & d, & q^{1/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q; q^2\right], \tag{3.2a}$$

$$\mathfrak{V}(b,c,d) := q^{1/2}{}_4\psi_4\left[\begin{matrix} b, & c, & d, & q^{5/2}/bcd \\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd \end{matrix}\,\Big|\, q; q^2\right]. \tag{3.2b}$$

§3.1. For two sequences defined by

$$A_k = \left[\begin{matrix} qb, \ q^{1/2}cd, \ q^{3/2}/bcd \\ q/b, \ q^{3/2}/cd, \ q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_k \quad \text{and} \quad B_k = \left[\begin{matrix} c, \ d, \ q^{3/2}/cd \\ q/c, \ q/d, \ q^{-1/2}cd \end{matrix}\,\Big|\, q\right]_k$$

it is almost trivial to compute the limits

$$[AB]_{+\infty} = -[AB]_{-\infty} = \frac{1}{1-q^{-1/2}cd}\left[\begin{matrix} qb, & c, & d, & q^{3/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_\infty$$

and the following difference relations:

$$\nabla\mathcal{A}_k = (1+q^k)\left[\begin{matrix} b, \ q^{-1/2}cd, \ q^{1/2}/bcd \\ q/b, \ q^{3/2}/cd, \ q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_k q^k,$$

$$\triangle\mathcal{B}_k = (1+q^{k+1/2})\left[\begin{matrix} c, \ d, \ q^{3/2}/cd \\ q^2/c, \ q^2/d, \ q^{1/2}cd \end{matrix}\,\Big|\, q\right]_k q^k \times \frac{(1-q^{1/2}/c)(1-q^{1/2}/d)(1-q/cd)}{(1-q/c)(1-q/d)(1-q^{1/2}/cd)}.$$

On account of (3.1a) and (3.2a), we can apply the Abel method on summation by parts to manipulate the well-poised ${}_4\psi_4$-series as follows:

$$\begin{aligned} 2\mathfrak{U}(b,c,d) &= \sum_k (1+q^k)\left[\begin{matrix} b, & c, & d, & q^{1/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_k q^k \\ &= \sum_k B_k\nabla A_k \ = \ 2[AB]_{+\infty} + \sum_k A_k\triangle B_k \\ &= \frac{2}{1-q^{-1/2}cd}\left[\begin{matrix} qb, & c, & d, & q^{3/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_\infty \\ &+ \frac{(1-q^{1/2}/c)(1-q^{1/2}/d)(1-q/cd)}{(1-q/c)(1-q/d)(1-q^{1/2}/cd)} \\ &\times \sum_k (1+q^{k+1/2})\left[\begin{matrix} qb, & c, & d, & q^{3/2}/bcd \\ q/b, & q^2/c, & q^2/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_k q^k. \end{aligned}$$

By means of (3.1b) and (3.2b), the last sum with respect to k reduces to $2\mathfrak{V}(qb,c,d)$. We therefore have established the following crossing relation:

$$\mathfrak{U}(b,c,d) \quad = \quad \mathfrak{V}(qb,c,d) \times \frac{(1-q^{1/2}/c)(1-q^{1/2}/d)(1-q/cd)}{(1-q/c)(1-q/d)(1-q^{1/2}/cd)} \tag{3.3a}$$

$$+ \quad \frac{1}{1-q^{-1/2}cd}\left[\begin{matrix} qb, & c, & d, & q^{3/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_\infty. \tag{3.3b}$$

§3.2. Similarly, for two sequences defined by

$$\mathcal{A}_k = \left[\begin{matrix} qb, \ qc, \ q/bc \\ q/b, \ q/c, \ qbc \end{matrix}\,\Big|\, q\right]_k \quad \text{and} \quad \mathcal{B}_k = \left[\begin{matrix} qbc, \ d, \ q^{1/2}/bcd \\ 1/bc, \ q/d, \ q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_k$$

it is not difficult to compute the limits

$$[\mathcal{AB}]_{+\infty} = -[\mathcal{AB}]_{-\infty} = \frac{1}{1-1/bc}\left[\begin{matrix} qb, & qc, & d, & q^{1/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_\infty$$

and the following difference relations:

$$\begin{aligned}\nabla\mathcal{A}_k &= (1+q^k)\left[\begin{matrix} b,\ c,\ 1/bc \\ q/b,\ q/c,\ qbc \end{matrix}\,\Big|\, q\right]_k q^k,\\ \triangle\mathcal{B}_k &= (1+q^{k+1/2})\left[\begin{matrix} qbc,\ d,\ q^{1/2}/bcd \\ q/bc,\ q^2/d,\ q^{3/2}bcd \end{matrix}\,\Big|\, q\right]_k q^k\\ &\quad\times\frac{(1-q^{1/2}bc)(1-q^{1/2}/d)(1-bcd)}{(1-bc)(1-q/d)(1-q^{1/2}bcd)}.\end{aligned}$$

In view of (3.1a) and (3.2a), we can again apply the Abel method on summation by parts to manipulate the well-poised ${}_4\psi_4$-series as follows:

$$\begin{aligned}2\mathfrak{U}(b,c,d) &= \sum_k(1+q^k)\left[\begin{matrix} b, & c, & d, & q^{1/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_k q^k\\ &= \sum_k \mathcal{B}_k\nabla\mathcal{A}_k \;=\; 2[\mathcal{AB}]_{+\infty} + \sum_k \mathcal{A}_k\triangle\mathcal{B}_k\\ &= \frac{2}{1-1/bc}\left[\begin{matrix} qb, & qc, & d, & q^{1/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_\infty\\ &+ \frac{(1-q^{1/2}bc)(1-q^{1/2}/d)(1-bcd)}{(1-bc)(1-q/d)(1-q^{1/2}bcd)}\\ &\times\sum_k(1+q^{k+1/2})\left[\begin{matrix} qb, & qc, & d, & q^{1/2}/bcd \\ q/b, & q/c, & q^2/d, & q^{3/2}bcd \end{matrix}\,\Big|\, q\right]_k q^k.\end{aligned}$$

Recalling (3.1b) and (3.2b), we see that the last sum with respect to k equals $2\mathfrak{V}(qb,qc,d)$. We therefore have established another crossing relation:

$$\mathfrak{U}(b,c,d) \quad=\quad \mathfrak{V}(qb,qc,d)\times\frac{(1-q^{1/2}bc)(1-q^{1/2}/d)(1-bcd)}{(1-bc)(1-q/d)(1-q^{1/2}bcd)} \tag{3.4a}$$

$$+\quad\frac{1}{1-1/bc}\left[\begin{matrix} qb, & qc, & d, & q^{1/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_\infty. \tag{3.4b}$$

§3.3. Now combining (3.3a-3.3b) with (3.4a-3.4b) under the replacement $c\to c/q$ and then canceling $\mathfrak{V}(b,c,d)$, we derive the following independent relation for $\mathfrak{U}(b,c,d)$:

$$\mathfrak{U}(b,c,d) = \mathfrak{U}(b,c/q,d)\times\tfrac{(1-q^{1/2}/c)(1-q/bc)(1-q/cd)(1-q^{1/2}/bcd)}{(1-q/c)(1-q^{1/2}/bc)(1-q^{1/2}/cd)(1-q/bcd)} \tag{3.5a}$$

$$+\tfrac{(q^{1/2}/bcd)(1-q^{3/2}/bc^2d)}{(1-q^{1/2}/bc)(1-q^{1/2}/cd)(1-q/bcd)}\left[\begin{matrix} b, & c, & d, & q^{3/2}/bcd \\ q/b, & q/c, & q/d, & q^{1/2}bcd \end{matrix}\,\Big|\, q\right]_\infty. \tag{3.5b}$$

Observing that the last relation results from shifting parameter c by q. Iterating it m-times, we derive the following relation with a natural number parameter m:

$$\begin{aligned}\mathfrak{U}(b,c,d) &= \mathfrak{U}(b,c/q^m,d)\left[\begin{matrix}q^{1/2}/c, & q/bc, & q/cd, & q^{1/2}/bcd\\ q/c, & q^{1/2}/bc, & q^{1/2}/cd, & q/bcd\end{matrix}\,\Big|\,q\right]_m\\ &+ \sum_{k=0}^{m-1}\frac{(q^{k+1/2}/bcd)(1-q^{2k+3/2}/bc^2d)}{(1-q^{k+1/2}/bc)(1-q^{k+1/2}/cd)(1-q^{k+1}/bcd)}\\ &\quad\times\left[\begin{matrix}b, & q^{-k}c, & d, & q^{k+3/2}/bcd\\ q/b, & q^{k+1}/c, & q/d, & q^{1/2-k}bcd\end{matrix}\,\Big|\,q\right]_\infty\\ &\quad\times\left[\begin{matrix}q^{1/2}/c, & q/bc, & q/cd, & q^{1/2}/bcd\\ q/c, & q^{1/2}/bc, & q^{1/2}/cd, & q/bcd\end{matrix}\,\Big|\,q\right]_k\end{aligned}$$

which can be further simplified as

$$\mathfrak{U}(b,c,d) = \mathfrak{U}(b,c/q^m,d)\times\left[\begin{matrix}q^{1/2}/c, & q/bc, & q/cd, & q^{1/2}/bcd\\ q/c, & q^{1/2}/bc, & q^{1/2}/cd, & q/bcd\end{matrix}\,\Big|\,q\right]_m \tag{3.6a}$$

$$+\frac{(q^{1/2}/bcd)(1-q^{3/2}/bc^2d)}{(1-q^{1/2}/bc)(1-q^{1/2}/cd)(1-q/bcd)}\left[\begin{matrix}b, & c, & d, & q^{3/2}/bcd\\ q/b, & q/c, & q/d, & q^{1/2}bcd\end{matrix}\,\Big|\,q\right]_\infty \tag{3.6b}$$

$$\times\sum_{k=0}^{m-1}\frac{1-q^{2k+3/2}/bc^2d}{1-q^{3/2}/bc^2d}\left[\begin{matrix}q/c, & q^{1/2}/c, & q/bc, & q/cd\\ q^{3/2}/bc, & q^{3/2}/cd, & q^2/bcd, & q^{3/2}/bcd\end{matrix}\,\Big|\,q\right]_k\Big(\frac{q^{1/2}}{bd}\Big)^k. \tag{3.6c}$$

When $|q^{1/2}/bd|<1$, we can compute, by means of the Weierstrass M-test on uniformly convergent series (cf. Stromberg [14, P 141]), the following limit

$$\lim_{m\to\infty}\mathfrak{U}(b,c/q^m,d) = {}_2\psi_2\left[\begin{matrix}b, & d\\ q/b, & q/d\end{matrix}\,\Big|\,q;\frac{q^{1/2}}{bd}\right] = \left[\begin{matrix}q, q^{1/2}/b, q^{1/2}/d, q/bd\\ q^{1/2}, q/b, q/d, q^{1/2}/bd\end{matrix}\,\Big|\,q\right]_\infty$$

where we have appealed the well-poised bilateral series identity due to Bailey [5, Eq 2.2] (cf. [15, Eq 5.5] also):

$${}_3\psi_3\left[\begin{matrix}b, & c, & d\\ q/b, & q/c, & q/d\end{matrix}\,\Big|\,q;\frac{q}{bcd}\right] = \left[\begin{matrix}q, q/bc, q/bd, q/cd\\ q/b, q/c, q/d, q/bcd\end{matrix}\,\Big|\,q\right]_\infty, \quad (|q/bcd|<1).$$

Letting $m\to\infty$ in (3.6a-3.6b-3.6c), we find the following transformation formula:

$$\mathfrak{U}(b,c,d) = \left[\begin{matrix}q, q^{1/2}/b, q^{1/2}/c, q^{1/2}/d, q/bc, q/bd, q/cd, q^{1/2}/bcd\\ q^{1/2}, q/b, q/c, q/d, q^{1/2}/bc, q^{1/2}/bd, q^{1/2}/cd, q/bcd\end{matrix}\,\Big|\,q\right]_\infty \tag{3.7a}$$

$$+\frac{(q^{1/2}/bcd)(1-q^{3/2}/bc^2d)}{(1-q^{1/2}/bc)(1-q^{1/2}/cd)(1-q/bcd)}\left[\begin{matrix}b, & c, & d, & q^{3/2}/bcd\\ q/b, & q/c, & q/d, & q^{1/2}bcd\end{matrix}\,\Big|\,q\right]_\infty \tag{3.7b}$$

$$\times\sum_{k=0}^{+\infty}\frac{1-q^{2k+3/2}/bc^2d}{1-q^{3/2}/bc^2d}\left[\begin{matrix}q/c, & q^{1/2}/c, & q/bc, & q/cd\\ q^{3/2}/bc, & q^{3/2}/cd, & q^2/bcd, & q^{3/2}/bcd\end{matrix}\,\Big|\,q\right]_k\Big(\frac{q^{1/2}}{bd}\Big)^k. \tag{3.7c}$$

We remark that (3.7c) is the partial sum of the terms labeled with nonnegative integers of Bailey's very-well-poised bilateral ${}_6\psi_6$-series.

This completes the proof of Theorem 1 .

§3.4. Analogously, the difference between (3.3a-3.3b) and (3.4a-3.4b) under the replacements $b \to b/q$ and $c \to c/q$ leads us to another independent relation:

$$\mathfrak{V}(b,c,d) = \mathfrak{V}(b,c/q,d) \times \frac{(1-q^{3/2}/c)(1-q^2/bc)(1-q^2/cd)(1-q^{3/2}/bcd)}{(1-q^2/c)(1-q^{3/2}/bc)(1-q^{3/2}/cd)(1-q^2/bcd)} \tag{3.8a}$$

$$+ \frac{(q^{3/2}/bcd)(1-q^{7/2}/bc^2d)}{(1-q^{3/2}/bc)(1-q^{3/2}/cd)(1-q^2/bcd)} \left[\begin{matrix} b, & c, & d, & q^{7/2}/bcd \\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd \end{matrix} \,\Big|\, q\right]_\infty . \tag{3.8b}$$

Iterating the last relation m-times, we derive the recurrence relation for $\mathfrak{V}(b,c,d)$:

$$\begin{aligned} \mathfrak{V}(b,c,d) &= \mathfrak{V}(b,c/q^m,d) \left[\begin{matrix} q^{3/2}/c, & q^2/bc, & q^2/cd, & q^{3/2}/bcd \\ q^2/c, & q^{3/2}/bc, & q^{3/2}/cd, & q^2/bcd \end{matrix} \,\Big|\, q\right]_m \\ &+ \sum_{k=0}^{m-1} \frac{(q^{k+3/2}/bcd)(1-q^{2k+7/2}/bc^2d)}{(1-q^{k+3/2}/bc)(1-q^{k+3/2}/cd)(1-q^{k+2}/bcd)} \\ &\times \left[\begin{matrix} b, & q^{-k}c, & d, & q^{k+7/2}/bcd \\ q^2/b, & q^{k+2}/c, & q^2/d, & q^{-k-1/2}bcd \end{matrix} \,\Big|\, q\right]_\infty \\ &\times \left[\begin{matrix} q^{3/2}/c, & q^2/bc, & q^2/cd, & q^{3/2}/bcd \\ q^2/c, & q^{3/2}/bc, & q^{3/2}/cd, & q^2/bcd \end{matrix} \,\Big|\, q\right]_k \end{aligned}$$

which can be further simplified as

$$\mathfrak{V}(b,c,d) = \mathfrak{V}(b,c/q^m,d) \times \left[\begin{matrix} q^{3/2}/c, & q^2/bc, & q^2/cd, & q^{3/2}/bcd \\ q^2/c, & q^{3/2}/bc, & q^{3/2}/cd, & q^2/bcd \end{matrix} \,\Big|\, q\right]_m \tag{3.9a}$$

$$+ \frac{(q^{3/2}/bcd)(1-q^{7/2}/bc^2d)}{(1-q^{3/2}/bc)(1-q^{3/2}/cd)(1-q^2/bcd)} \left[\begin{matrix} b, & c, & d, & q^{7/2}/bcd \\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd \end{matrix} \,\Big|\, q\right]_\infty \tag{3.9b}$$

$$\times \sum_{k=0}^{m-1} \frac{1-q^{2k+7/2}/bc^2d}{1-q^{7/2}/bc^2d} \left[\begin{matrix} q/c, & q^{3/2}/c, & q^2/bc, & q^2/cd \\ q^{5/2}/bc, & q^{5/2}/cd, & q^3/bcd, & q^{7/2}/bcd \end{matrix} \,\Big|\, q\right]_k \Big(\frac{q^{3/2}}{bd}\Big)^k . \tag{3.9c}$$

When $|q^{3/2}/bd| < 1$, we can compute, by means of the Weierstrass M-test on uniformly convergent series (cf. Stromberg [14, P 141]), the following limit

$$\lim_{m\to\infty} \mathfrak{V}(b,c/q^m,d) = {}_2\psi_2 \left[\begin{matrix} b, & d \\ q^2/b, & q^2/d \end{matrix} \,\Big|\, q; \frac{q^{3/2}}{bd}\right] = \left[\begin{matrix} q, q^{3/2}/b, q^{3/2}/d, q^2/bd \\ q^{1/2}, q^2/b, q^2/d, q^{3/2}/bd \end{matrix} \,\Big|\, q\right]_\infty$$

where we have invoked the well-poised series identity due to Bailey [5, Eq 2.3]:

$${}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ q^2/b, & q^2/c, & q^2/d \end{matrix} \,\Big|\, q; \frac{q^2}{bcd}\right] = \left[\begin{matrix} q, q^2/bc, q^2/bd, q^2/cd \\ q^2/b, q^2/c, q^2/d, q^2/bcd \end{matrix} \,\Big|\, q\right]_\infty, \quad (|q^2/bcd| < 1).$$

Letting $m \to \infty$ in (3.9a-3.9b-3.9c), we find the following transformation formula:

$$\mathfrak{V}(b,c,d) = \left[\begin{matrix} q, q^{3/2}/b, q^{3/2}/c, q^{3/2}/d, q^2/bc, q^2/bd, q^2/cd, q^{3/2}/bcd \\ q^{1/2}, q^2/b, q^2/c, q^2/d, q^{3/2}/bc, q^{3/2}/bd, q^{3/2}/cd, q^2/bcd \end{matrix} \,\Big|\, q\right]_\infty \tag{3.10a}$$

$$+ \frac{(q^{3/2}/bcd)(1-q^{7/2}/bc^2d)}{(1-q^{3/2}/bc)(1-q^{3/2}/cd)(1-q^2/bcd)} \left[\begin{matrix} b, & c, & d, & q^{7/2}/bcd \\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd \end{matrix} \,\Big|\, q\right]_\infty \tag{3.10b}$$

$$\times \sum_{k=0}^{+\infty} \frac{1-q^{2k+7/2}/bc^2d}{1-q^{7/2}/bc^2d} \left[\begin{matrix} q/c, & q^{3/2}/c, & q^2/bc, & q^2/cd \\ q^{5/2}/bc, & q^{5/2}/cd, & q^3/bcd, & q^{7/2}/bcd \end{matrix} \,\Big|\, q\right]_k \Big(\frac{q^{3/2}}{bd}\Big)^k \tag{3.10c}$$

which has been anticipated in Theorem 3.

§3.5. The transformation displayed in Theorem 3 can also be derived by substituting the identity in Theorem 1 into the crossing relation (3.3a-3.3b).

First, rewrite the crossing relation stated in (3.3a-3.3b) as

$$\begin{aligned}\mathfrak{V}(b,c,d) &= \frac{q^{1/2}/cd}{(1-q^{1/2}/c)(1-q^{1/2}/d)(1-q/cd)}\left[\begin{matrix}b, & c, & d, & q^{5/2}/bcd\\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd\end{matrix}\,\Big|\, q\right]_\infty\\ &+ \mathfrak{U}(b/q,c,d)\frac{(1-q/c)(1-q/d)(1-q^{1/2}/cd)}{(1-q^{1/2}/c)(1-q^{1/2}/d)(1-q/cd)}.\end{aligned}$$

Next replacing the last $\mathfrak{U}(b/q,c,d)$ according to Theorem 1 and then simplifying the result, we have the following equation:

$$\mathfrak{V}(b,c,d) = \left[\begin{matrix}q, q^{3/2}/b, q^{3/2}/c, q^{3/2}/d, q^2/bc, q^2/bd, q^2/cd, q^{3/2}/bcd\\ q^{1/2}, q^2/b, q^2/c, q^2/d, q^{3/2}/bc, q^{3/2}/bd, q^{3/2}/cd, q^2/bcd\end{matrix}\,\Big|\, q\right]_\infty \tag{3.11a}$$

$$+\frac{q^{1/2}/cd}{(1-q^{1/2}/c)(1-q^{1/2}/d)(1-q/cd)}\left[\begin{matrix}b, & c, & d, & q^{5/2}/bcd\\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd\end{matrix}\,\Big|\, q\right]_\infty \tag{3.11b}$$

$$+\frac{q^{3/2}/bcd}{(1-q^{3/2}/bc)(1-q^{1/2}/d)(1-q^2/bcd)}\left[\begin{matrix}b, & c, & d, & q^{5/2}/bcd\\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd\end{matrix}\,\Big|\, q\right]_\infty \tag{3.11c}$$

$$\times\frac{(1-b/q)(1-q^{5/2}/bc^2d)}{(1-q^{1/2}/c)(1-q/cd)}\sum_{k=0}^{+\infty}\frac{1-q^{2k+5/2}/bc^2d}{1-q^{5/2}/bc^2d} \tag{3.11d}$$

$$\times\left[\begin{matrix}q/c, & q^{1/2}/c, & q^2/bc, & q/cd\\ q^{5/2}/bc, & q^{3/2}/cd, & q^3/bcd, & q^{5/2}/bcd\end{matrix}\,\Big|\, q\right]_k\left(\frac{q^{3/2}}{bd}\right)^k. \tag{3.11e}$$

This seems far from the transformation displayed in Theorem 3. However their equivalence can be shown through the following slightly modified **Abel's lemma** on summation by parts for nonterminating unilateral series:

$$\sum_{k=0}^{+\infty} B_k\nabla A_k = [AB]_{+\infty} - A_{-1}B_0 + \sum_{k=0}^{+\infty} A_k\triangle B_k \tag{3.12}$$

provided that the limit $[AB]_\infty := \lim_{n\to\infty} A_nB_{n+1}$ exists and one of both series just displayed is convergent.

For $|q^{3/2}/bd| < 1$, define two sequences $\{C_k, D_k\}$ by

$$C_k = \left[\begin{matrix}q^{3/2}/c, & q^2/cd\\ q^3/bcd, & q^{5/2}/bc\end{matrix}\,\Big|\, q\right]_k\left(\frac{q}{b}\right)^k \quad\text{and}\quad D_k = \left[\begin{matrix}q^2/bc, & q/c\\ q^{3/2}/cd, & q^{5/2}/bcd\end{matrix}\,\Big|\, q\right]_k\left(\frac{q^{1/2}}{d}\right)^k.$$

We have no difficulty to check

$$[CD]_{+\infty} = 0 \quad\text{and}\quad C_{-1}D_0 = \frac{b}{q}\frac{(1-q^{3/2}/bc)(1-q^2/bcd)}{(1-q^{1/2}/c)(1-q/cd)}$$

and the finite differences

$$\nabla C_k = \frac{(1-b/q)(1-q^{5/2}/bc^2d)}{(1-q^{1/2}/c)(1-q/cd)}\left[\begin{matrix}q^{1/2}/c, & q/cd\\ q^3/bcd, & q^{5/2}/bc\end{matrix}\,\Big|\, q\right]_k\frac{1-q^{2k+5/2}/bc^2d}{1-q^{5/2}/bc^2d}\left(\frac{q}{b}\right)^k,$$

$$\triangle D_k = \frac{(1-q^{1/2}/d)(1-q^{7/2}/bc^2d)}{(1-q^{3/2}/cd)(1-q^{5/2}/bcd)}\left[\begin{matrix}q^2/bc, & q/c\\ q^{5/2}/cd, & q^{7/2}/bcd\end{matrix}\,\Big|\, q\right]_k\frac{1-q^{2k+7/2}/bc^2d}{1-q^{7/2}/bc^2d}\left(\frac{q^{1/2}}{d}\right)^k.$$

Then we can reformulate (3.11d-3.11e) through (3.12) as follows:

$$\begin{aligned}\sum_{k\geq 0} D_k \nabla C_k &= -C_{-1}D_0 + \sum_{k\geq 0} C_k \triangle D_k = -\frac{b}{q}\frac{(1-q^{3/2}/bc)(1-q^2/bcd)}{(1-q^{1/2}/c)(1-q/cd)}\\ &+ \frac{(1-q^{1/2}/d)(1-q^{7/2}/bc^2d)}{(1-q^{3/2}/cd)(1-q^{5/2}/bcd)}\sum_{k\geq 0}\frac{1-q^{2k+7/2}/bc^2d}{1-q^{7/2}/bc^2d}\\ &\times \left[\begin{matrix} q/c, & q^{3/2}/c, & q^2/bc, & q^2/cd \\ q^{5/2}/bc, & q^{5/2}/cd, & q^3/bcd, & q^{7/2}/bcd\end{matrix} \,\Big|\, q\right]_k \Big(\frac{q^{3/2}}{bd}\Big)^k.\end{aligned}$$

Substituting the last equation into (3.11) and then canceling the extra term displayed in (3.11b), we get finally again the transformation stated in Theorem 3.

$$\begin{aligned}&\mathfrak{V}(b,c,d) = \left[\begin{matrix} q, q^{3/2}/b, q^{3/2}/c, q^{3/2}/d, q^2/bc, q^2/bd, q^2/cd, q^{3/2}/bcd \\ q^{1/2}, q^2/b, q^2/c, q^2/d, q^{3/2}/bc, q^{3/2}/bd, q^{3/2}/cd, q^2/bcd\end{matrix} \,\Big|\, q\right]_\infty\\ &+ \frac{(q^{3/2}/bcd)(1-q^{7/2}/bc^2d)}{(1-q^{3/2}/bc)(1-q^{3/2}/cd)(1-q^2/bcd)}\left[\begin{matrix} b, & c, & d, & q^{7/2}/bcd \\ q^2/b, & q^2/c, & q^2/d, & q^{-1/2}bcd\end{matrix} \,\Big|\, q\right]_\infty\\ &\times \sum_{k=0}^{+\infty}\frac{1-q^{2k+7/2}/bc^2d}{1-q^{7/2}/bc^2d}\left[\begin{matrix} q/c, & q^{3/2}/c, & q^2/bc, & q^2/cd \\ q^{5/2}/bc, & q^{5/2}/cd, & q^3/bcd, & q^{7/2}/bcd\end{matrix} \,\Big|\, q\right]_k \Big(\frac{q^{3/2}}{bd}\Big)^k.\end{aligned}$$

4. The q-Clausen Product Formulae

Following Jackson [12], consider the product of two basic hypergeometric series

$$\sum_{n\geq 0}\mathcal{W}_n\, z^n = {}_2\phi_1\left[\begin{matrix} au^2,\ bv^2 \\ cu^2v^2\end{matrix} \,\Big|\, q^2; z\right] \times {}_2\phi_1\left[\begin{matrix} \alpha u^2,\ \beta v^2 \\ \gamma u^2v^2\end{matrix} \,\Big|\, q^2; qz\right] \tag{4.1}$$

where the $\mathcal{W}_n$-coefficients are determined by the q-binomial convolution

$$\begin{aligned}\mathcal{W}_n &= \sum_{k=0}^{n}\left[\begin{matrix} au^2,\ bv^2 \\ q^2,\ cu^2v^2\end{matrix} \,\Big|\, q^2\right]_k \left[\begin{matrix} u^2\alpha,\ v^2\beta \\ q^2,\ u^2v^2\gamma\end{matrix} \,\Big|\, q^2\right]_{n-k} q^{n-k} &\text{(4.2a)}\\ &= \left[\begin{matrix} au^2,\ bv^2 \\ q^2,\ cu^2v^2\end{matrix} \,\Big|\, q^2\right]_n {}_4\phi_3\left[\begin{matrix} q^{-2n},\ \alpha u^2,\ \beta v^2,\ q^{2-2n}/cu^2v^2 \\ q^{2-2n}/au^2,\ q^{2-2n}/bv^2,\ \gamma u^2v^2\end{matrix} \,\Big|\, q^2; \frac{q^3c}{ab}\right] &\text{(4.2b)}\\ &= q^n\left[\begin{matrix} \alpha u^2, \beta v^2 \\ q^2, \gamma u^2v^2\end{matrix} \,\Big|\, q^2\right]_n {}_4\phi_3\left[\begin{matrix} q^{-2n},\ au^2,\ bv^2,\ q^{2-2n}/\gamma u^2v^2 \\ q^{2-2n}/\alpha u^2,\ q^{2-2n}/\beta v^2,\ cu^2v^2\end{matrix} \,\Big|\, q^2; \frac{q\gamma}{\alpha\beta}\right]. &\text{(4.2c)}\end{aligned}$$

When the $\mathcal{W}_n$-coefficients have closed form in terms of q-factorial fraction, the q-series product formulae (including the q-Clausen formula) can be established consequently.

For an alternative q-analogue of the Clausen theorem and other proofs of Jackson's q-Clausen formula, refer to Gasper [9].

§4.1. For the terminating series (4.2b) and (4.2c), letting

$$a = b = 1,\ c = q \quad\text{and}\quad \alpha = \beta = 1,\ \gamma = q$$

we may evaluate them by means of Corollary 5 as follows:

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, & U, & V, & q^{1/2-n}/UV \\ & q^{1-n}/U, & q^{1-n}/V, & q^{1/2}UV \end{matrix} \,\Big|\, q; q^{\epsilon}\right] \tag{4.3a}
$$

$$
= \; q^{n(\epsilon-2)/2}\left[\begin{matrix} q,\, UV \\ U,\, V \end{matrix} \,\Big|\, q\right]_n \left[\begin{matrix} U,\, V \\ q^{1/2},\, UV \end{matrix} \,\Big|\, q^{1/2}\right]_n, \quad \epsilon = 1,\, 2. \tag{4.3b}
$$

The corresponding product (4.1) yields the q-Clausen formula due to Jackson [12, §5] (cf. also Chu [7, Example 5.1], Gasper [9, Eq 2.1] and Guo [11, Eq 3.1])

$$
{}_2\phi_1\left[\begin{matrix} u^2,\, v^2 \\ qu^2v^2 \end{matrix} \,\Big|\, q^2; z\right] \times {}_2\phi_1\left[\begin{matrix} u^2,\, v^2 \\ qu^2v^2 \end{matrix} \,\Big|\, q^2; qz\right] \tag{4.4a}
$$

$$
= \; {}_4\phi_3\left[\begin{matrix} u^2, & v^2, & uv, & -uv \\ & u^2v^2, & q^{1/2}uv, & -q^{1/2}uv \end{matrix} \,\Big|\, q; z\right] \tag{4.4b}
$$

which is the q-analogue of the Clausen (1828) theorem (cf. Bailey [2, P 86: Eq 4], Askey [1, Eq 1.2] and Gasper [9, Eq 1.1]):

$$
{}_2F_1\left[\begin{matrix} U,\, V \\ U+V+1/2 \end{matrix} \,\Big|\, z\right] \times {}_2F_1\left[\begin{matrix} U,\, V \\ U+V+1/2 \end{matrix} \,\Big|\, z\right] = {}_3F_2\left[\begin{matrix} 2U,\, 2V,\, U+V \\ 2U+2V,\, U+V+1/2 \end{matrix} \,\Big|\, z\right].
$$

§4.2. For the terminating series (4.2b), putting

$$
a = b = 1,\; c = q^{-1} \quad \text{and} \quad \alpha = \beta = 1,\; \gamma = q
$$

we may evaluate it by means of Corollary 5 as (see also [16, Eq 4.3])

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, & U, & V, & q^{3/2-n}/UV \\ & q^{1-n}/U, & q^{1-n}/V, & q^{1/2}UV \end{matrix} \,\Big|\, q; q\right] \tag{4.5a}
$$

$$
= \left[\begin{matrix} q,\, UV,\, q^{-1/2}UV \\ U,\, V,\, q^{1/2}UV \end{matrix} \,\Big|\, q\right]_n \left[\begin{matrix} U,\, V \\ q^{1/2},\, q^{-1/2}UV \end{matrix} \,\Big|\, q^{1/2}\right]_n. \tag{4.5b}
$$

While the terminating series corresponding to (4.2c) is just the reversal of (4.5a):

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, & U, & V, & q^{1/2-n}/UV \\ & q^{1-n}/U, & q^{1-n}/V, & q^{-1/2}UV \end{matrix} \,\Big|\, q; q\right] \tag{4.6a}
$$

$$
= \; q^{-n/2}\left[\begin{matrix} q,\, UV \\ U,\, V \end{matrix} \,\Big|\, q\right]_n \left[\begin{matrix} U,\, V \\ q^{1/2},\, q^{-1/2}UV \end{matrix} \,\Big|\, q^{1/2}\right]_n. \tag{4.6b}
$$

The corresponding (4.1) yields the product formula (cf. Chu [7, Example 5.2])

$$
{}_2\phi_1\left[\begin{matrix} u^2,\, v^2 \\ q^{-1}u^2v^2 \end{matrix} \,\Big|\, q^2; z\right] \times {}_2\phi_1\left[\begin{matrix} u^2,\, v^2 \\ qu^2v^2 \end{matrix} \,\Big|\, q^2; qz\right] \tag{4.7a}
$$

$$
= \; {}_4\phi_3\left[\begin{matrix} u^2, & v^2, & uv, & -uv \\ & q^{-1}u^2v^2, & q^{1/2}uv, & -q^{1/2}uv \end{matrix} \,\Big|\, q; z\right] \tag{4.7b}
$$

which is a q-analogue of the Orr (1899) theorem (cf. Bailey [2, P 86: Eq 5]):

$$
{}_2F_1\left[\begin{matrix} U,\, V \\ U+V-1/2 \end{matrix} \,\Big|\, z\right] \times {}_2F_1\left[\begin{matrix} U,\, V \\ U+V+1/2 \end{matrix} \,\Big|\, z\right] = {}_3F_2\left[\begin{matrix} 2U,\, 2V,\, U+V \\ 2U+2V-1,\, U+V+1/2 \end{matrix} \,\Big|\, z\right].
$$

§4.3. For the terminating series (4.2c), setting

$$a = c = 1,\ b = q \quad \text{and} \quad \alpha = \gamma = 1,\ \beta = q^{-1}$$

we may evaluate it by means of Corollary 5 as

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, & U, & q^{1/2}V, & q^{1-n}/UV \\ & q^{1-n}/U, & q^{3/2-n}/V, & UV \end{matrix}\,\Big|\, q; q\right] && \text{(4.8a)}\\
&= q^{-n/2}\left[\begin{matrix} q, q^{-1/2}UV \\ U, q^{-1/2}V \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} U, \quad V \\ q^{1/2}, q^{-1/2}UV \end{matrix}\,\Big|\, q^{1/2}\right]_n. && \text{(4.8b)}
\end{aligned}
$$

While the terminating series corresponding to (4.2b) is just the reversal of summation (4.8a):

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, & U, & Vq^{-1/2}, & q^{1-n}/UV \\ & q^{1-n}/U, & q^{1/2-n}/V, & UV \end{matrix}\,\Big|\, q; q\right] && \text{(4.9a)}\\
&= \left[\begin{matrix} q,\ q^{-1/2}UV \\ U,\ q^{1/2}V \end{matrix}\,\Big|\, q\right]_n \times \left[\begin{matrix} U, \quad V, \\ q^{1/2},\ q^{-1/2}UV \end{matrix}\,\Big|\, q^{1/2}\right]_n. && \text{(4.9b)}
\end{aligned}
$$

The corresponding (4.1) yields the product formula (cf. Chu [7, Example 5.3])

$$
\begin{aligned}
&{}_2\phi_1\left[\begin{matrix} u^2,\ v^2q \\ u^2v^2 \end{matrix}\,\Big|\, q^2; z\right] \times {}_2\phi_1\left[\begin{matrix} u^2,\ v^2/q \\ u^2v^2 \end{matrix}\,\Big|\, q^2; qz\right] && \text{(4.10a)}\\
&= {}_4\phi_3\left[\begin{matrix} u^2, & v^2, & q^{-1/2}uv, & -q^{-1/2}uv \\ & q^{-1}u^2v^2, & uv, & -uv \end{matrix}\,\Big|\, q; z\right] && \text{(4.10b)}
\end{aligned}
$$

which is the q-analogue of the Orr (1899) theorem (cf. Bailey [2, P 86: Eq 6]):

$$
{}_2F_1\left[\begin{matrix} U,\ V+1/2 \\ U+V \end{matrix}\,\Big|\, z\right] \times {}_2F_1\left[\begin{matrix} U,\ V-1/2 \\ U+V \end{matrix}\,\Big|\, z\right] = {}_3F_2\left[\begin{matrix} 2U,\ 2V,\ U+V-1/2 \\ U+V,\ 2U+2V-1 \end{matrix}\,\Big|\, z\right].
$$

§4.4. For the terminating series (4.2c), taking

$$a = b = c = q \quad \text{and} \quad \alpha = \beta = \gamma = q^{-1}$$

we may evaluate it by means of Corollary 5 as

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{1/2}U, & q^{1/2}V, & q^{3/2-n}/UV \\ & q^{3/2-n}/U, & q^{3/2-n}/V, & q^{1/2}UV \end{matrix}\,\Big|\, q; q\right] && \text{(4.11a)}\\
&= q^{-n/2}\tfrac{1-q^{-1/2}UV}{1-q^{n-1/2}UV}\left[\begin{matrix} q,\ UV \\ q^{-1/2}U,\ q^{-1/2}V \end{matrix}\,\Big|\, q\right]_n \left[\begin{matrix} U,\ V \\ q^{1/2},\ q^{-1/2}UV \end{matrix}\,\Big|\, q^{1/2}\right]_n. && \text{(4.11b)}
\end{aligned}
$$

The terminating series corresponding (4.2b) is the reversal of summation (4.11a):

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{-1/2}U, & q^{-1/2}V, & q^{1/2-n}/UV \\ & q^{1/2-n}/U, & q^{1/2-n}/V, & q^{-1/2}UV \end{matrix}\,\Big|\, q; q\right] && \text{(4.12a)}\\
&= \left[\begin{matrix} q,\ UV \\ q^{1/2}U,\ q^{1/2}V \end{matrix}\,\Big|\, q\right]_n \times \left[\begin{matrix} U,\ V \\ q^{1/2},\ q^{-1/2}UV \end{matrix}\,\Big|\, q^{1/2}\right]_n. && \text{(4.12b)}
\end{aligned}
$$

The corresponding (4.1) yields the product formula (cf. Chu [7, Example 5.4])

$$
\begin{aligned}
&{}_2\phi_1\left[\begin{matrix} u^2q,\ v^2q \\ u^2v^2q \end{matrix}\,\Big|\, q^2;\ z\right] \times {}_2\phi_1\left[\begin{matrix} u^2/q,\ v^2/q \\ u^2v^2/q \end{matrix}\,\Big|\, q^2;\ qz\right] && \text{(4.13a)}\\
&= {}_4\phi_3\left[\begin{matrix} u^2, & v^2, & uv, & -uv \\ & q^{-1}u^2v^2, & q^{1/2}uv, & -q^{1/2}uv \end{matrix}\,\Big|\, q; z\right] && \text{(4.13b)}
\end{aligned}
$$

which is the q-analogue of the following hypergeometric product (Askey [1, Eq 4.2]):

$$ {}_2F_1\left[\begin{matrix} U+1/2, V+1/2 \\ U+V+1/2 \end{matrix}\,\middle|\, z\right] \times {}_2F_1\left[\begin{matrix} U-1/2, V-1/2 \\ U+V-1/2 \end{matrix}\,\middle|\, z\right] = {}_3F_2\left[\begin{matrix} 2U,\ 2V,\ U+V \\ 2U+2V-1, U+V+1/2 \end{matrix}\,\middle|\, z\right]. $$

References

[1] R. Askey, *Variants of Clausen's formula for the square of a special* ${}_2F_1$*,* Number Theory and Related Topics; Tata Institute of Fundamental Research, Bombay, January 1988; pp 1-12.

[2] W. N. Bailey, *Generalized Hypergeometric Series,* Cambridge University Press, Cambridge, 1935.

[3] W. N. Bailey, *Series of hypergeometric type which are infinite in both directions,* Quart. J. Math. [Oxford] 7 (1936), 105-115.

[4] W. N. Bailey, *A note on certain q-identities,* Quart. J. Math. [Oxford] 12 (1941), 173-175.

[5] W. N. Bailey, *On the analogue of Dixon's theorem for bilateral basic hypergeometric series,* Quart. J. Math. [Oxford, Series 2] 1 (1950), 318-320.

[6] L. Carlitz, *Some formulas of F. H. Jackson,* Monatshhefte für Math. 73 (1969), 193-198.

[7] W. Chu, *Basic Almost Poised Hypergeometric Series,* Memoirs of American Mathematical Society 135:642 (1998), pp 99+iv; **MR** 99b:33035.

[8] W. Chu, *Bailey's very well poised* ${}_6\psi_6$*-series identity,* Journal of Combinatorial Theory (Series A) 113:6 (2006), 966-979.

[9] G. Gasper, *q-extensions of Clausen's formula and of the inequalities used by De Branges in his proof of the Bieberbach, Robertson, and Milin conjectures,* SIAM J. Math. Anal. 20:4 (1989), 1019-1034.

[10] G. Gasper - M. Rahman, *Basic Hypergeometric Series* (2nd ed.)*,* Cambridge University Press, Cambridge, 2004.

[11] V. J. W. Guo, *Elementary proofs of some q-identities of Jackson and Andrews-Jain,* Discrete Math. 295 (2005), 63-74.

[12] F. H- Jackson, *Certain q-identities,* Quart. J. Math. [Oxford] 12 (1941), 167-172.

[13] L. J. Slater, *Generalized Hypergeoemtric Functions,* Cambridge University Press, Cambridge, 1966.

[14] Karl R. Stromberg, *An Introduction to Classical Real Analysis,* Wadsworth, INC. Belmont, California, 1981.

[15] A. Verma - V. K. Jain, *Certain summation formulae for q-series,* J. Indian Math. Soc. 47 (1983), 71-85.

[16] A. Verma - V. K. Jain, *Some summation formulae for nonterminating basic hypergeometric series,* SIAM J. Math. Anal. 16:3 (1985), 647-655.

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IDEAL STRUCTURE AND PURE STATE SPACE OF SOME OPERATOR ALGEBRAS

Adel M. Zaki

Abstract

In this work, we consider two examples of C^*-algebras. We find the ideal structure of each and we obtain a precise form of the pure state space of both C^*-algebras.

Keywords, C^*-algebras, primal ideals, pure states.

1 Introduction

Let A be a C^*-algebra of operators acting on an infinite dimensional Hilbert space H. If S is a subset of the dual space A^*, we denote by $\bar{S}$ the closure of S in the weak*-topology. Firstly, we consider the C^*-algebra E of all sequences of $2n \times 2n$ matrices with elements of the C^*-algebra A as entries and converges (in norm) to some matrix of the form

$$\begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix},$$

for some $A(x)$ and $B(x)$ in $M_n(A)$. We get the general form of primal ideals in E (see proposition 3.4) and the precise form of the pure state space of E is obtained (see proposition 3.6). For further work see [6, 7, 8].

A necessary and sufficient condition for the C^*-algebra E being antiliminal is proved (see Lemma 3.2) and an equivalent condition for A being prime is shown in proposition 3.3.

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On the other hand, let B be the C^*-algebra which consists of all continuous functions from the interval $[0,1]$ into $M_n(\mathbb{C})$ (the $n \times n$ complex matrices). Let $B(1)$ be the C^*-algebra of B consisting of functions of B which satisfies

$$f(\frac{1}{2^k}) = \begin{pmatrix} \lambda_{2k-(n-1)}(f) & 0 \\ 0 & \lambda_{2k}(f) \end{pmatrix}, k \geq 1$$

and

$f(0) = \lambda(f)I_n$, for some complex numbers, $\lambda_{2k}(f), \ldots, \lambda_{2k-(n-1)}$ and $\lambda(f)$. For $m \geq 2$, let $B(m)$ be the C^*-subalgebra of $B(1)$ defined by

$$B(m) = \{f \in B(1), \lambda_{2k}(f) = \lambda_{2k-r}(f), (1 \leq r \leq n-1)\}$$

The general form of primal ideals in $B(m)$ is obtained (see proposition 4.2) and the pure state $\overline{P(B(m))}$ has been obtained (see proposition 4.3)

2 Preliminaries

Let A be a C^*-algebra. The state space $S(A)$ is defined by

$$S(A) = \{\varphi \in A^* : \varphi \geq 0, \|\varphi\| = 1\}.$$

The set of extreme points of $S(A)$ will be denoted by $P(A)$. The pure state space of A is the space $\overline{P(A)}$. By an ideal we shall always mean a closed two sided ideal. An ideal I is called primal if whenever $n \geq 1$ and $I_1, \ldots, I_n$ are ideals of A such that $I_1 I_2 \ldots I_n = (0)$, then $I_k \subset I$ for at least one k.

If $\varphi \in \overline{P(A)}$, then φ belong to the factorial state space $\overline{F(A)}$ (the w^*-closure of the set of factorial states of A). Using [1], $\ker\pi_\varphi$ is a primal ideal of A and since $\ker\pi_\varphi \subseteq \ker\varphi$, then φ must factor through some primal ideal of A. Let $\mathrm{Prim}(A)$ be the set of primitive ideals of A.

Let T be a subset of $\mathrm{Prim}(A)$. Let $I(T)$ be the intersection of elements of T which is an ideal of A. Then

$\bar{T} = \{P \in \text{Prim}(A) : P \supseteq I(T)\}$ denotes the closure of T. This topology is called the Jacobson topology [2, 3.1]. An open neighborhood U, say, in Prim(A) has the form

$U = \{P \in \text{Prim}(A) : P \not\supseteq I\}$ for some ideal I of A.

3 Main results

We start by considering the first example in this work.

Example 3.1.

Let A be a C^*-algebra. Let $M_n(A)$ denotes the C^*-algebra of all $n \times n$ matrices with elements of A as entries. Let E be the C^*-algebra of all sequences $x = (x_n)$, where $x_n \in M_{2n}(A)$ and x_n converges (in norm) to a matrix of the form

$$\begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix},$$

for some $A(x)$ and $B(x)$ in $M_n(A)$.

We denote by

$$\pi_A(x) : E \to M_n(A) \text{ and } \pi_B(x) : E \to M_n(A)$$

where

$\pi_A(x) = A(x), \pi_B(x) = B(x)$, for all $x = (x_n)$ in E. For $n \in \mathbb{N}$, we write

$\pi_n : E \to M_{2n}(A)$, where

$\pi_n(x) = x_n$, for all $x = (x_n)$ in E.

Note that, π_A, π_B are $*$-homomorphisms of E onto $M_n(A)$ and π_n is a $*$-homomorphisms of E onto $M_{2n}(A)$.

Recall that, a C^*-algebra A is called antiliminal, if $\{0\}$ is the only liminal ideal of A.

Lemma 3.2.

A is antiliminal if, and only if, E in antiliminal.

Proof.

($\Rightarrow$) Suppose that A is antiliminal. By [4] we get $M_k(A)(\cong M_k \otimes A)$ is antiliminal for all positive integers k. Now suppose E is not antiliminal. Then there exists a non-zero liminal ideal I of E. In fact, there is some $n_0 \in \mathbb{N}$ so that $\pi_{n_0}(I) \neq (0)$, otherwise, $I = (0)$. Pick n_0, then $\pi_{n_0}(I)$ is a non-zero liminal ideal of $M_{2n}(A)$, which is a contradiction.

($\Rightarrow$) Notice that

$$I_1 = \{x = (x_m) : x_m = 0, \quad \text{for all } m \neq 1\}$$

is an ideal of E. Moreover, the map ψ defined by $\psi(x) = x_1$, for all $x = (x_m)$ in I_1 is a $*$-homomorphism of I_1 onto $M_{2n}(A)$. Suppose that A has a non-zero liminal ideal J. Then $M_{2n} \otimes J$ is a non-empty liminal ideal of $M_{2n}(A)$. Thus I_1, and hence E, has a non-zero liminal ideal.

As a direct result of the latter Lemma that A is antiliminal if, and only if, $\overline{P(E)} = \overline{F(E)}$. Moreover, if A is a prime C^*-algebra (that is, $\{0\}$ is the only prime ideal of A), then ker π_A, ker π_B and ker $\pi_n, m = 1, 2, \ldots$ are prime ideals of E. Indeed, since $E/$ ker $\pi_A \simeq A$, then $\{0\}$ is a prime ideal of $E/$ ker π_A and hence ker π_A is a prime ideal of E. Non fix n, since $E/$ ker $\pi_n \cong M_{2n}(A)$, and $M_{2n}(A)$ is prime, then ker π_n is a prime ideal of E.

The next proposition gives an equivalent condition for a C^*-algebra A keing prime.

Proposition 3.3.

A is a prime C^*-algebra if, and only if, ker $\pi_A \cap$ ker π_B and ker $\pi_n, n = 1, 2, \ldots$ are primal ideals of E.

Proof.

($\Rightarrow$) suppose that A is prime, then ker $\pi_n, n = 1, 2, \ldots,$ are prime and hence primal ideals of E. Now, since A is prime, then $M_{2n}(A)$ is prime.

Let $I_1, I_2, \ldots, I_k$ be ideals of E such that

$I_r \nsubseteq$ ker $\pi_A \cap$ ker π_B, for all $1 \leq r \leq k$. Since $\pi_n(I_1), \ldots, \pi_n(I_k)$ are non-zero ideals of $M_{2n}(A)$, eventually and $M_{2n}(A)$ is prime, thus

$$\pi_n(I_1) \ldots \pi_n(I_k) = \pi_n(I_1 \ldots I_k) \neq (0),$$

eventually. Thus

$$I_1 \ldots I_k \neq (0) \text{ and } \ker \pi_A \cap \ker \pi_B$$

is primal.

($\Rightarrow$) Fix n and suppose that ker π_n is primal we show that $M_{2n}(A)$ is prime. Let J_1, J_2 be two ideals of $M_{2n}(A)$ so that $J_1 J_2 = (0)$. Let

$$I_1 = \{y = (y_m) : y = (0, \ldots, 0, \underset{\text{n-th place}}{r}, 0, \ldots), R \in J_1\}$$

$$I_2 = \{z = (z_m) : z = (0, \ldots, 0, \underset{\text{n-th place}}{s}, 0, \ldots), S \in J_2\},$$

Note that I_1, I_2 are two ideals of E with $I_1 I_2 = (0)$. Since ker π_n is primal, then either $I_1 \subseteq$ ker π_n or $I_2 \subseteq$ ker π_n. Hence either

$$(0) = \pi_n(I_1) = J_1 \quad \text{or} \quad (0) = \pi_n(I_2) = J_2$$

Then $M_{2n}(A)$ is prime and thus A is prime.

Next, assume that ker $\pi_A \cap$ ker π_B is primal. Let J_1, J_2 be ideals of A such that $J_1 J_2 = (0)$. Let

$I_1 = \{y = (y_m) : y_m \in M_{2n}(J_1)$ and $y_m \to \begin{pmatrix} A(y) & 0 \\ 0 & 0 \end{pmatrix}$, for some $A(y)$ in $J_1\}$.

$I_2 = \{z = (z_m) : z_m \in M_{2n}(J_2)$ and $z_m \to \begin{pmatrix} 0 & 0 \\ 0 & B(z) \end{pmatrix}$, for some $B(z)$ in $J_2\}$.

I_1, I_2 are ideals of E with $I_1 I_2 = (0)$. Since ker $\pi_A \cap$ ker π_B is primal, then either.

$$I_1 \subseteq \ker \pi_A \cap \ker \pi_B \text{ or } I_2 \subseteq \ker \pi_A \cap \ker \pi_B$$

Thus

$$(0) = \pi_A(I_1) = J_1 \quad \text{or} \quad (0) = \pi_B(I_2) = J_2$$

and A is prime.

The general form of a primal ideal in the C^*-algebra E is given in the following.

Proposition 3.4.

Let I be a primal ideal of E. The either $I \supseteq$ ker π_n, for some n or $I \supseteq$ ker $\pi_A \cap$ ker π_B.

Proof.

Suppose $I \not\supseteq$ ker π_n for all n and notice that

$$I_n = \{x = (x_m) : x_m = 0, \text{ for all } m \neq n\}$$

is an ideal of E. Since I is primal and I_n ker $\pi_n = (0)$, then $I \supseteq I_n$ for all n. Hence $I \supseteq I_1 + \cdots + I_n$, for all n and since I is closed,

$$I \supseteq \{x = (x_m) : x_m \to 0\} = \ker \pi_A \cap \ker \pi_B$$

Corollary 3.5.

If $P \in \mathrm{Prim}(E)$, then either $P \supseteq$ ker π_n, for some n or $P \supseteq$ ker π_A or $P \supseteq$ ker π_B.

Now suppose that A is unital, prime and antiliminal. Then $M_{2n}(A)$ is prime and using [4, th. 8], we get $M_{2n}(A)$ is antiliminal. By [3] and [5]

$$\overline{P(A)} = S(A) \text{ and } \overline{P(M_{2n}(A))} = S(M_{2n}(A))$$

The next proposition describes precisely the pure state space of the given C^*-algebra E.

Proposition 3.6.

Let A be a unital, prime and antiliminal C^*-algebra, then

$$\overline{P(E)} = \{\psi \circ \pi_m : m = 1, 2, \ldots \text{ and } \psi \in S(M_{2n}(A)\}$$

$$U\{\lambda(\psi_1 \circ \pi_A) + (1-\lambda)(\psi_2 \circ \pi_B) : 0 \leq \lambda \leq 1$$

and

$$\psi_1, \psi_2 \in S(M_n(A))\}$$

Proof.

Since $\overline{P(M_{2n}(A))} = S(M_{2n}(A))$, then $\psi \circ \pi_m \in \overline{P(E)}$, for all $m = 1, 2, \ldots$ and all ψ in $S(M_n(A))$. Let $\psi_1, \psi_2 \in S(M_n(A))$. Since $\overline{P(M_n(A))} = S(M_n(A))$, then $\psi_1 \circ \pi_A, \psi_2 \circ \pi_B$ belong to $\overline{P(E)}$. Now we prove that ψ_1, ψ_2 can be approximated by equivalent pure states of $M_n(A)$. Let U_1, U_2 be open neighborhoods of ψ_1, ψ_2, respectively. Let $V_i = U_i \cap P_n(M_n(A)), i = 1, 2$. Then V_i are non-empty open subsets (relative to $P(M_n(A))$. The map

$$\theta : P(M_n(A)) \to \widehat{M_n(A)},$$

where $\theta(\varphi) = [\pi_\varphi]$, for φ in $P(M_n(A))$ is open (see [2, 3.4, 11]. Therefore $\theta(V_1), \theta(V_2)$ are non-empty open subsets of $\widehat{M_n(A)}$. Let

$$[\pi] \in \theta(V_1) \cap \theta(V_2) \neq \varphi.$$

Hence there exist $\varphi_1 \in V_1$ and $\varphi_2 \in V_2$ such that

$$[\pi_{\varphi_1}] = \theta(\varphi_1) = [\pi] \quad \text{and} \quad [\pi_{\varphi_2}] = \theta(\varphi_2) = [\pi].$$

Then $\varphi_1 \simeq \varphi_2$. We show that

$$\varphi_0 = \lambda(\varphi_1 \circ \pi_A) + (1-\lambda)(\varphi_2 \circ \pi_B) \in \overline{P(E)},$$

where $0 \leq \lambda \leq 1$. Since

$$\varphi_0 = \omega^* - \lim_m \omega_z \circ (id \otimes \gamma) \circ \pi_m,$$

where $\pi_{\varphi_1} \simeq \pi_{\varphi_2} \simeq \gamma$ for some irreducible representation γ of A, and $\varphi_1(a) =< \gamma(a)\xi, \xi >$, $\varphi_2(a) =< \gamma(a)\eta, \eta >$ for some unit vectors ξ, η in H_γ. By considering $\gamma = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ \xi \end{pmatrix}$ and $z = \sqrt{\lambda}\alpha + \sqrt{1-\lambda}\beta$ we obtain $\varphi_0 \in \overline{P(E)}$. On the other hand, if $\varphi \in \overline{P(E)}$, then φ must factor through a primal ideal of E. Using proposition 3.4, if φ factors through some ker π_m, for some m, then $\varphi = \psi \circ \psi_m$, for some ψ in $S(M_{2n}(A))$. Otherwise, φ factors through ker $\pi_A \cap$ ker π_B. Notice that,

$$E/\ker \pi_A \cap \ker \pi_B \simeq M_n(A) \oplus M_n(A)$$

via the mapping $x + (\ker \pi_A \cap \ker \pi_B) \longmapsto (\pi_A(x), \pi_B(x))$ for all x in E. Hence φ has the form

$$\varphi = \lambda(\psi_1 \circ \pi_A) + (1-\lambda)(\psi_2 \circ \pi_B)$$

for some $0 \leq \lambda \leq 1$ and ψ_1, ψ_2 in $S(M_n(A))$.

Next, we consider an example of C^*-algebras which are analogous to the C^*-algebra E.

Example 3.7.

Let A be a unital C^*-algebra. Let n be an even positive integer. Let D_n be the C^*-algebra defined by

$$D_n = \{x = (x_m) : x_m \in M_n(A), \sup_m \|x_m\| < \infty\}$$

Let F be the C^*-subalgebra of D_n, where

$$F = \{x = (x_m) \in D_n : x_m \to \begin{pmatrix} A(x) & 0 \\ 0 & A(x) \end{pmatrix}$$

for some $A(x)$ in $M_{\frac{n}{2}}(A)\}$.

Let G be the C^*-subalgebra of D_{2n}, defined by

$$G = \{x = (x_m) \in D_{2n} : x_m \to \begin{pmatrix} B(x) & 0 \\ 0 & B(x) \end{pmatrix}$$

for some $B(x)$ in $M_n(A)\}$.

Proposition 3.8.

Let F, G be the C^*-algebras considered in example 3.7.

Then $F \odot M_2(A) \cong G$.

Proof.

Let $x = (x_m)$ be F. Thus

$$x_m = [a_{ij}^{(m)}] \to \begin{pmatrix} A(x) & 0 \\ 0 & A(x) \end{pmatrix}, \text{ as } m \to \infty.$$

Define the map

$$\tau : F \odot M_2(A) \to G,$$

by

$$\tau([a_{ij}^{(m)}], [b_{ij}]) = (a_{ij}^{(m)} \otimes [b_{ij}]),$$

for $b = [b_{ij}] \in M_2(A)$ and

$$a_{ij}^{(m)} \otimes [b_{ij}] = \begin{bmatrix} a_{11}b \ldots a_{1n}b \\ a_{n1}b \ldots a_{nn}b \end{bmatrix}$$

Since τ is bilinear, then there exists a unique well-defined linear map

$$\theta : F \odot M_2(A) \to G,$$

where

$$\theta([a_{ij}^{(m)}] \otimes [b_{ij}]) = \left(a_{ij}^{(m)} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right)$$

θ is a $*$-homomorphism. Furthermore, pick $c = (c_m) = (c_{ij}^{(m)})$ in G, where

$$c_m \to \begin{pmatrix} B(x) & 0 \\ 0 & B(x) \end{pmatrix} \text{ as } m \to \infty$$

for some $B(x)$ in $M_n(A)$. Let

$$e_1 = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}, \text{ and } e_4 = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}.$$

Let

$$a = x \otimes e_1 + y \otimes e_2 + z \otimes e_3 + \omega \otimes e_4$$

where

$$x = (x_m), y = (y_m), z = (z_m) \,\text{and}\, \omega = (\omega_m)$$

are defined by

$$x_m = \begin{pmatrix} c_{11}^{(m)} & c_{13}^{(m)} & \cdots & c_{1,n-1}^{(m)} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n-1,1}^{(m)} & c_{n-1,3}^{(m)} & \cdots & c_{n-1,n-1}^{(m)} \end{pmatrix}$$

$$y_m = \begin{pmatrix} c_{12}^{(m)} & c_{14}^{(m)} & \cdots & c_{1,n}^{(m)} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n-1,2}^{(m)} & c_{n-1,4}^{(m)} & \cdots & c_{n-1,n}^{(m)} \end{pmatrix}$$

$$z_m = \begin{pmatrix} c_{21}^{(m)} & c_{23}^{(m)} & \cdots & c_{2,n-1}^{(m)} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n,1}^{(m)} & c_{n,3}^{(m)} & \cdots & c_{n,n-1}^{(m)} \end{pmatrix}$$

and

$$\omega_m = \begin{pmatrix} c_{22}^{(m)} & c_{24}^{(m)} & \cdots & c_{2,n}^{(m)} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n,2}^{(m)} & c_{n,4}^{(m)} & \cdots & c_{n,n}^{(m)} \end{pmatrix}$$

x, y, z and ω belong to F. Finally $\theta(a) = (c_m) = c$ and θ is onto.
Next, we consider an example of a C^*-algebra which was considered by D. Somernet in the litterature.

Example 3.9.

Let A be the C^*-algebra consisting of all continuous functions from the interval [0, 1] into the $n \times n$ complex matrices. Let $A(1)$ be the C^*-subalgebra of A which consists of those functions f in A which satisfies

$$f(\frac{1}{2^k}) = \begin{pmatrix} \lambda_{2k-(n-1)}(f) & 0 \\ 0 & \lambda_{2k}(f) \end{pmatrix}, \ k \geq 1$$

and $f(0) = \lambda(f)I_n$, for some complex numbers $\lambda(f)$ and $\lambda_{2k-r}(0 \leq r \leq n-1)$.
For $m \geq 2$, let $A(m)$ be the C^*-subalgebra of $A(1)$ defined by

$$A(m) = \{f \in A(1) : \lambda_{2k} = \lambda_{2k-r}(1 \leq r \leq n-1), 1 \leq k < m\}.$$

It is known that Prim (A) is a Hausdorff space homeomorphic to the interval [0,1]. Let

$$X = \{\frac{1}{2^n} : n \geq 1\} \cup \{0\} \text{ and let } Y = [0,1] \backslash X.$$

For $1 \leq m < \infty$, let

$$P_y(m) = \{f \in A(m) : f(y) = 0\}, \ y \in Y.$$

$$Q(m) = \{f \in A(m) : \lambda(f) = 0\},$$

and

$$R_i(m) = \{f \in A(m) : \lambda_i(f) = 0\}, \ 1 \leq i < \infty$$

Thus

$$\pi_y : A(m) \to B(\mathbb{C}_2) \simeq M_2(\mathbb{C}), \ y \in Y$$

$$P_y(m) = \ker \pi_y,$$

$$\pi_Q : A(m) \to B(\mathbb{C}_2),$$

and

$$Q(m) = \ker \pi_Q$$

$$\pi_{R_i} : A(m) \to B(\mathbb{C}_2),$$

$$R_i(m) = \ker \pi_{R_i}, (1 \leq i < \infty).$$

$$\text{Prim}\,(A(m)) = \cup\{P_y(m) : y \in Y\} \cup \{Q(m)\} \cup \bigcup_{i \geq 1}\{R_i(m)\}.$$

The points $P_y(m), (y \in Y)$ and $Q(m)$ are separated points in Prim $(A(m))$. We find the general form of primal ideals in the C^*-algebra $A(m)$.

Lemma 3.10.

Let I be a primal ideal of $A(m)$. Then either $I \supseteq \ker \pi_Q$ or $I \supseteq \ker \pi_{R_i}$, for some $i \geq 1$ or $I \supseteq \ker \pi_y$, for some $y \in Y$.

Proof.

Suppose that $I \not\supseteq \ker \pi_y$, for all $y \in Y$ and $I \not\supseteq \ker \pi_{R_i}$, for all $i \geq 1$. Let

$$I_{i_0} = \{f \in A(m) : \lambda_j(f) = 0, \forall\, j \neq i_0\}.$$

Thus I_0 is an ideal of $A(m)$ with $I_{i_0} \ker \pi_{R_{i_0}} \ker \pi_y = 0$, for all $y \in Y$. Since I is primal, then $I \supseteq I_i$, for all $i \geq 1$. Therefore $I \supseteq I_1 + I_2 + \cdots + I_r$, for all r. Hence $I \supseteq \ker \pi_Q$.

The next proposition gives the preise form of the pure state space of $A(m)$.

Proposition 3.11.

$\overline{P(A(m))} = \{\omega_\xi \circ \pi_Q : \xi \text{ is a unit vector } \mathbb{C}^2\}$
$\bigcup_{i \geq 1}\{\psi \circ \pi_{R_i} : \psi \in S(\mathbb{C}^2)\}$
$\cup\{\gamma \circ \pi_y : \gamma \in S(\mathbb{C}^2)\}$.

Proof.

Let $\varphi \in \overline{P(A(m))}$. Then φ must factor through some primal ideal of $A(m)$. The using Lemma 3.10, if φ factors through ker π_{R_i}, the $\varphi = \psi \circ \pi_{R_i}$, for some $1 \leq i < \infty$. We show that $\varphi \in \overline{P(A(m))}$ if, and only if, ψ is a pure state of

$M_2(\mathbb{C})$. Indeed, if ψ is pure, then $\varphi \in P(A(m))(\subseteq \overline{P(A(m))}$. Let $\varphi \in \overline{P(A(m))}$. Therefore $\varphi_\alpha \xrightarrow[w^*]{} \varphi$, for some (φ_α) in $P(A(m))$. Hence

$$\ker \pi_{\varphi_\alpha} \rightarrow \ker \pi_{R_i} \text{ in } \mathrm{Prim}\,(A(m)).$$

Suppose U is an open neighborhood of $\ker \pi_{R_i}$ for some $1 \leq i < \infty$. Then using [2, 3, 2.2],

$$U = \{Q \in Prim(A(m)) : Q \not\supseteq J\},$$

for some ideal J of $A(m)$. Therefore

$\ker \pi_{R_i} \not\supseteq J$, for some $1 \leq i < \infty$, then $\pi_{R_i}(J) \neq 0$. We prove that $\ker \pi_{\varphi_\alpha} \in U$, eventually. Suppose not, then $\pi_{\varphi_\alpha}(J) = 0$, frequently and hence $\varphi_\alpha(J) = 0$, frequently. We have

$$\varphi(J) = \psi(\pi_{R_i}(J)) = 0, \text{ for some } 1 \leq i < \infty.$$

However, $\pi_{R_i}(J)$ is a proper ideal of $M_2(\mathbb{C})$. Thus $\pi_{R_i}(J) = (0)$, for some $1 \leq i < \infty$, a contradiction. Hence $\varphi_\alpha \in P(A(m)/\mathrm{ker}\pi_{R_i})$, for some $1 \leq i < \infty$, eventually.

Define $\psi_\alpha \in S(\mathbb{C}^2)$, so that $\varphi_\alpha = \psi_\alpha \circ \pi_{R_i}$, eventually, for some $1 \leq 0 < \infty$. Since φ_α is pure, then ψ_α is a pure state of $\mathbb{C}^2$. Moreover,

$$\psi_\alpha \circ \pi_{R_i} \rightarrow \psi \circ \pi_{R_i}, \text{ for some}\, 1 \leq i < \infty.$$

Hence $\psi_\alpha \rightarrow \psi$ and $\psi \in \overline{P(M_2(\mathbb{C}))}$. Thus by looking at $M_2(\mathbb{C})$ as $Le((\mathbb{C}^\alpha)$, ψ is a multiple of a pure state and hence pure, since ψ is a state.

On the other hand, if φ factors through ker π_Q or ker π_y, then

$$\varphi = \psi \circ \pi_Q \text{ or } \varphi = \gamma \circ \pi_y,$$

for some ψ, γ in $S(M_2(\mathbb{C}))$.

References

[1] Archbold, R.J., "On factorial states of operator algebras III" J. Funct. Analys. 55 (1984), 26-38.

[2] Dixmier, J. (1997), "C^*-algebras", North Holand, Amestrdam.

[3] Glimm, J., "A stone weiersstrass theorem for C^*-algebras", Ann. of Math., 72 (1960), 216-244.

[4] Guichardet, A., "Tensor product of C^*-algebras I", Aarhus Univ. Lecture notes, 12 (1969).

[5] Tomiyama, J., Takessaki, M., "Applications of fibre bundles to the certain class of C^*-algebras", Tohoku Math. J. (2), 13 (1961), 498-523.

[6] Zeinab M. Abd-Elkader and Adel M. Zaki "The pure state space of certain C^*-algebras III", J. Egyp. Math. Soc. Vol.9(2) (2001), 143-149.

[7] Zaki, A.M., "The pure state space of certain C^*-algebras II", Far East J. Math. Sci. 1 (1993), 1-12.

[8] Zaki, A.M., "The pure state space of certain C^*-algebras", Bull. Cal. Math. Soc. 86 (1994), 267-270.

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