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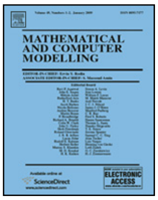

# Four-dimensional matrix transformation and rate of $A$-statistical convergence of periodic functions

Kamil Demirci *, Fadime Dirik

*Sinop University, Faculty of Sciences and Arts, Department of Mathematics, 57000 Sinop, Turkey*

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ABSTRACT

In this paper, using the concept of $A$-statistical convergence for double real sequences, we obtain a Korovkin type-approximation theorem for double sequences of positive linear operators defined on the space of all $2\pi$-periodic and real valued continuous functions on the real two-dimensional space. Furthermore, we display an application which shows that our new result is stronger than its classical version. Also, we study rates of $A$-statistical convergence of a double sequence of positive linear operators acting on this space. Finally, displaying an example, it is shown that our statistical rates are more efficient than the classical aspects in the approximation theory.

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## 1. Introduction

For a sequence $\{L_n\}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset $X$ of real numbers, Korovkin [1] established first sufficient conditions for the uniform convergence of $L_n(f)$ to a function $f$ by using the test function $f_i$ defined by $f_i(x) = x^i$, $(i = 0, 1, 2)$. Later many researchers have investigated these conditions for various operators defined on different spaces. Furthermore, in recent years, with the help of the concept of uniform statistical convergence, which is stronger than uniform convergence, various statistical approximation results have been proved [2–8]. Also, a Korovkin type-approximation theorem has been studied via $A$-statistical convergence in the space $C^*$, which is the space of all $2\pi$-periodic and continuous functions on $\mathbb{R}$ in [9], and $C^*\left(\mathbb{R}^2\right)$, the space of all $2\pi$-periodic and real valued continuous functions on $\mathbb{R}^2$ in [10]. The main goal of this paper is to obtain a Korovkin type-approximation theorem for double sequences of positive linear operators defined on $C^*\left(\mathbb{R}^2\right)$. Also, we compute rates of $A$-statistical convergence of a double sequence of positive linear operators acting on $C^*\left(\mathbb{R}^2\right)$. Finally, displaying an example, it is shown that our statistical rates are more efficient than the classical ones in the approximation theory.

We now recall some basic definitions and notations used in the paper.

A double sequence $x = (x_{m,n})$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$, the set of all natural numbers, such that $\left|x_{m,n} - L\right| < \varepsilon$ whenever $m, n > N$, where $L$ is called the Pringsheim limit of $x$ and denoted by $\text{P-}\lim x = L$ (see [11]). We shall call such an $x$, briefly, "P-convergent". A double sequence is called bounded if there exists a positive number $M$ such that $\left|x_{m,n}\right| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded. A double sequence $x = \left\{x_{m,n}\right\}$ is said to be non-increasing in Pringsheim's sense if, for all $(m, n) \in \mathbb{N}^2$, $x_{m+1,n+1} \leq x_{m,n}$.

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* Corresponding author.
*E-mail addresses:* kamild@sinop.edu.tr (K. Demirci), fgezer@sinop.edu.tr (F. Dirik).

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Let $A = (a_{j,k,m,n})$ be a four-dimensional summability matrix. For a given double sequence $x = (x_{m,n})$, the $A$-transform of $x$, denoted by $Ax := ((Ax)_{j,k})$, is given by

$$(Ax)_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} x_{m,n}$$

provided the double series converges in Pringsheim's sense for every $(j, k) \in \mathbb{N}^2$.

A two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two-dimensional matrix transformations which are regular is known as Silverman–Toeplitz conditions (see, for instance, [12]). In 1926, Robison [13] presented a four-dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four-dimensional matrices is known as Robison–Hamilton conditions, or briefly, RH-regularity (see [14,13]).

Recall that a four-dimensional matrix $A = (a_{j,k,m,n})$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison–Hamilton conditions state that a four-dimensional matrix $A = (a_{j,k,m,n})$ is RH-regular if and only if

(i) $\text{P-}\lim_{j,k} a_{j,k,m,n} = 0$ for each $(m, n) \in \mathbb{N}^2$,
(ii) $\text{P-}\lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} = 1$,
(iii) $\text{P-}\lim_{j,k} \sum_{m\in\mathbb{N}} \left|a_{j,k,m,n}\right| = 0$ for each $n \in \mathbb{N}$,
(iv) $\text{P-}\lim_{j,k} \sum_{n\in\mathbb{N}} \left|a_{j,k,m,n}\right| = 0$ for each $m \in \mathbb{N}$,
(v) $\sum_{(m,n)\in\mathbb{N}^2} \left|a_{j,k,m,n}\right|$ is P-convergent for each $(j, k) \in \mathbb{N}^2$,
(vi) there exist finite positive integers $A$ and $B$ such that $\sum_{m,n>B} \left|a_{j,k,m,n}\right| < A$ holds for every $(j, k) \in \mathbb{N}^2$.

Now let $A = (a_{j,k,m,n})$ be a non-negative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then the $A$-density of $K$ is given by

$$\delta_A^{(2)}\{K\} := \text{P-}\lim_{j,k} \sum_{(m,n)\in K} a_{j,k,m,n}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. A real double sequence $x = (x_{m,n})$ is said to be $A$-statistically convergent to a number $L$ if, for every $\varepsilon > 0$,

$$\delta_A^{(2)}\{(m, n) \in \mathbb{N}^2 : \left|x_{m,n} - L\right| \geq \varepsilon\} = 0.$$

In this case, we write $\text{st}_A^{(2)} - \lim x = L$. Clearly, a P-convergent double sequence is $A$-statistically convergent to the same value but its converse is not always true. Also, note that an $A$-statistically convergent double sequence need not to be bounded. For example, consider the double sequence $x = (x_{m,n})$ given by

$$x_{m,n} = \begin{cases} mn, & \text{if } m \text{ and } n \text{ are squares,} \\ 1, & \text{otherwise,} \end{cases}$$

and $A = C(1, 1) := \left(c_{j,k,m,n}\right)$, the double Cesáro matrix, defined by

$$c_{j,k,m,n} = \begin{cases} \dfrac{1}{jk}, & \text{if } 1 \leq m \leq j \text{ and } 1 \leq n \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\delta_{C(1,1)}^{(2)}\{(m, n) \in \mathbb{N}^2 : \left|x_{m,n} - 1\right| \geq \varepsilon\} = 0$ for every $\varepsilon > 0$, $\text{st}_{C(1,1)}^{(2)} - \lim x = 1$. But $x$ is neither P-convergent nor bounded. We should note that if we take $A = C(1, 1)$, then $C(1, 1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [15,16]. Finally, if we replace the matrix $A$ by the identity matrix for four-dimensional matrices, then $A$-statistical convergence reduces to the Pringsheim convergence.

## 2. A Korovkin-type approximation theorem

We denote by $C^*\left(\mathbb{R}^2\right)$ the space of all $2\pi$-periodic and real valued continuous functions on $\mathbb{R}^2$. If a function $f$ on $\mathbb{R}^2$ has a $2\pi$-period, then, for all $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = f(x + 2k\pi, y) = f(x, y + 2k\pi)$$

holds for $k = 0, \pm 1, \pm 2, \ldots$. This space is equipped with the supremum norm

$$\|f\|_{C^*(\mathbb{R}^2)} = \sup_{(x,y)\in\mathbb{R}^2} |f(x, y)|, \quad \left(f \in C^*\left(\mathbb{R}^2\right)\right).$$

**Theorem 1.** *Let $A = (a_{j,k,m,n})$ be a non-negative* RH-*regular summability matrix and let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C^*(\mathbb{R}^2)$ into $C^*(\mathbb{R}^2)$. Then, for all $f \in C^*(\mathbb{R}^2)$*

$$\mathrm{st}_A^{(2)} - \lim \left\| L_{m,n}(f) - f \right\|_{C^*(\mathbb{R}^2)} = 0 \tag{1}$$

*if and only if the following statements hold:*

$$\mathrm{st}_A^{(2)} - \lim \left\| L_{m,n}(f_i) - f_i \right\|_{C^*(\mathbb{R}^2)} = 0, \tag{2}$$

*where $f_0(x, y) = 1, f_1(x, y) = \sin x, f_2(x, y) = \sin y, f_3(x, y) = \cos x$ and $f_4(x, y) = \cos y$.*

**Proof.** Under the hypotheses, since $1, \sin x, \sin y, \cos x$ and $\cos y$ belong to $C^*(\mathbb{R}^2)$, the necessity is clear. Assume now that (2) holds. Let $f \in C^*(\mathbb{R}^2)$ and $I, J$ be closed subinterval of length $2\pi$ of $\mathbb{R}$. Fix $(x, y) \in I \times J$. As in the proof of Theorem 2.1 in [10], it follows from the continuity of $f$ that

$$|f(u, v) - f(x, y)| < \varepsilon + \frac{2M_f}{\sin^2 \frac{\delta}{2}} \varphi(u, v)$$

where $M_f = \|f\|_{C^*(\mathbb{R}^2)}$, $\varphi(u, v) = \sin^2 \frac{u-x}{2} + \sin^2 \frac{v-y}{2}$. Then, we have

$$\begin{aligned}
\left| L_{m,n}(f; x, y) - f(x, y) \right| &\le L_{m,n}(|f(u, v) - f(x, y)|; x, y) + |f(x, y)| \left| L_{m,n}(f_0; x) - f_0(x, y) \right| \\
&\le \left| L_{m,n}\left( \varepsilon + \frac{2M_f}{\sin^2 \frac{\delta}{2}} \varphi(u, v); x, y \right) \right| + M_f \left| L_{m,n}(f_0; x) - f_0(x, y) \right| \\
&\le \left( \varepsilon + M_f \right) \left| L_{m,n}(f_0; x) - f_0(x, y) \right| + \frac{M_f}{\sin^2 \frac{\delta}{2}} \Big\{ 2 \left| L_{m,n}(f_0; x) - f_0(x, y) \right| \\
&\quad + |\sin x| \left| L_{m,n}(f_1; x, y) - f_1(x, y) \right| + |\sin y| \left| L_{m,n}(f_2; x, y) - f_2(x, y) \right| \\
&\quad + |\cos x| \left| L_{m,n}(f_3; x, y) - f_3(x, y) \right| + |\cos y| \left| L_{m,n}(f_4; x, y) - f_4(x, y) \right| \Big\} + \varepsilon \\
&< \varepsilon + N \sum_{i=0}^{4} \left| L_{m,n}(f_i; x) - f_i(x, y) \right|,
\end{aligned}$$

where $N := \varepsilon + M_f + \frac{2M_f}{\sin^2 \frac{\delta}{2}}$. Then, taking supremum over $(x, y) \in \mathbb{R}^2$, we obtain

$$\left\| L_{m,n}(f) - f \right\|_{C^*(\mathbb{R}^2)} < \varepsilon + N \sum_{i=0}^{4} \left\| L_{m,n}(f_i) - f_i \right\|_{C^*(\mathbb{R}^2)}. \tag{3}$$

Now given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$, and define

$$\begin{aligned}
D &:= \left\{ (m, n) : \left\| L_{m,n}(f) - f \right\|_{C^*(\mathbb{R}^2)} \ge r \right\}, \\
D_i &:= \left\{ (m, n) : \left\| L_{m,n}(f_i) - f_i \right\|_{C^*(\mathbb{R}^2)} \ge \frac{r - \varepsilon}{5N} \right\}, \quad i = 0, 1, 2, 3, 4.
\end{aligned}$$

By (3) it is easy to that

$$D \subseteq \bigcup_{i=0}^{4} D_i.$$

Hence, we may write

$$\sum_{(m,n) \in D} a_{j,k,m,n} \le \sum_{i=0}^{4} \sum_{k \in D_i} a_{j,k,m,n}.$$

Now taking the limit $j, k \to \infty$ (in any manner), (2) yield the result. □

**Remark 2.** If we replace the matrix $A$ by the identity matrix for four-dimensional matrices in Theorem 1, then we immediately get the following result in Pringsheim's sense, which corresponds to the Theorem 2.1 in [10].

**Corollary 3.** *Let $\left\{L_{m,n}\right\}$ be a double sequence of positive linear operators acting from $C^{*}\left(\mathbb{R}^{2}\right)$ into $C^{*}\left(\mathbb{R}^{2}\right)$. Then, for all $f \in C^{*}\left(\mathbb{R}^{2}\right)$,*

$$\text{P-}\lim \left\|L_{m,n}(f)-f\right\|_{C^{*}\left(\mathbb{R}^{2}\right)}=0$$

*if and only if*

$$\text{P-}\lim \left\|L_{m,n}\left(f_{i}\right)-f_{i}\right\|_{C^{*}\left(\mathbb{R}^{2}\right)}=0,$$

*where $f_0(x,y)=1, f_1(x,y)=\sin x, f_2(x,y)=\sin y, f_3(x,y)=\cos x$ and $f_4(x,y)=\cos y$.*

**Example 4.** Now we present an example of double sequences of positive linear operators, showing that Corollary 3 does not work but our approximation theorem works. We note that the double sequence of Fejer operators on $C^{*}\left(\mathbb{R}^{2}\right)$ where

$$\sigma_{m,n}(f;x,y)=\frac{1}{(m\pi)(n\pi)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\,F_m(u)F_n(v)\mathrm{d}u\mathrm{d}v \tag{4}$$

where $F_m(u)=\frac{\sin^2\frac{m(u-x)}{2}}{2\sin^2\frac{u-x}{2}}$ and $\frac{1}{\pi}\int_{-\pi}^{\pi}F_m(u)\mathrm{d}u=1$. Observe that

$$\begin{aligned}
&\sigma_{m,n}(f_0;x,y)=f_0(x,y), \qquad \sigma_{m,n}(f_1;x,y)=\frac{m-1}{m}f_1(x,y),\\
&\sigma_{m,n}(f_2;x,y)=\frac{n-1}{n}f_2(x,y), \qquad \sigma_{m,n}(f_3;x,y)=\frac{m-1}{m}f_3(x,y),\\
&\sigma_{m,n}(f_4;x,y)=\frac{n-1}{n}f_4(x,y).
\end{aligned} \tag{5}$$

Now take $A=C(1,1)$ and define a double sequence $\{u_{m,n}\}$ by

$$u_{m,n}=\begin{cases}1, & \text{if } m \text{ and } n \text{ are squares,}\\ 0, & \text{otherwise.}\end{cases} \tag{6}$$

In this case, observe that

$$\mathrm{st}_{C(1,1)}^{(2)}-\lim u_{m,n}=0. \tag{7}$$

However, the sequence $\{u_{m,n}\}$ is not P-convergent. Now using (4) and (6), we define the following double positive linear operators on $C^{*}\left(\mathbb{R}^{2}\right)$ as follows:

$$L_{m,n}(f;x,y)=\left(1+u_{m,n}\right)\sigma_{m,n}(f;x,y). \tag{8}$$

Then, observe that the double sequence of positive linear operators $\left\{L_{m,n}\right\}$ defined by (8) satisfies all hypotheses of Theorem 1. Hence, by (5) and (7), we have, for all $f \in C^{*}\left(\mathbb{R}^{2}\right)$,

$$\mathrm{st}_{A}^{(2)}-\lim \left\|L_{m,n}(f)-f\right\|_{C^{*}\left(\mathbb{R}^{2}\right)}=0.$$

Since $\{u_{m,n}\}$ is not P-convergent, the sequence $\{L_{m,n}\}$ given by (8) does not converge uniformly to the function $f \in C^{*}\left(\mathbb{R}^{2}\right)$. So, we conclude that Corollary 3 does not work for the operators $L_{m,n}$ in (8) while our Theorem 1 still works.

## 3. Rate of $A$-statistical convergence

Various ways of defining rates of convergence in the $A$-statistical sense for four-dimensional summability matrices were introduced in [17]. In this section, we compute the rates $A$-statistical convergence in Theorem 1.

**Definition 5** (*[17]*)**.** Let $A=\left(a_{j,k,m,n}\right)$ be a non-negative RH-regular summability matrix and let $\left\{\alpha_{m,n}\right\}$ be a positive non-increasing double sequence in Pringsheim's sense. A double sequence $x=\left\{x_{m,n}\right\}$ is $A$-statistically convergent to a number $L$ with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon>0$,

$$\text{P-}\lim_{j,k\to\infty}\frac{1}{\alpha_{j,k}}\sum_{(m,n)\in K(\varepsilon)}a_{j,k,m,n}=0,$$

where

$$K(\varepsilon):=\left\{(m,n)\in\mathbb{N}^2:\left|x_{m,n}-L\right|\geq\varepsilon\right\}.$$

In this case, we write

$$x_{m,n} - L = \text{st}_A^{(2)} - o(\alpha_{m,n}) \quad \text{as } m, n \to \infty.$$

**Definition 6** (*[17]*). Let $A = \left(a_{j,k,m,n}\right)$ and $\left\{\alpha_{m,n}\right\}$ be the same as in Definition 5. Then, a double sequence $x = \left\{x_{m,n}\right\}$ is $A$-statistically bounded with the rate of $O(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$\sup_{j,k} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in L(\varepsilon)} a_{j,k,m,n} < \infty,$$

where

$$L(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : \left|x_{m,n}\right| \geq \varepsilon\right\}.$$

In this case, we write

$$x_{m,n} = \text{st}_A^{(2)} - O(\alpha_{m,n}) \quad \text{as } m, n \to \infty.$$

**Definition 7** (*[17]*). Let $A = \left(a_{j,k,m,n}\right)$ and $\left\{\alpha_{m,n}\right\}$ be the same as in Definition 5. Then, a double sequence $x = \left\{x_{m,n}\right\}$ is $A$-statistically convergent to a number $L$ with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$\text{P-}\lim_{j,k\to\infty} \sum_{(m,n)\in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : \left|x_{m,n} - L\right| \geq \varepsilon\alpha_{m,n}\right\}.$$

In this case, we write

$$x_{m,n} - L = \text{st}_A^{(2)} - o_{m,n}(\alpha_{m,n}) \quad \text{as } m, n \to \infty.$$

**Definition 8** (*[17]*). Let $A = \left(a_{j,k,m,n}\right)$ and $\left\{\alpha_{m,n}\right\}$ be the same as in Definition 5. Then, a double sequence $x = \left\{x_{m,n}\right\}$ is $A$-statistically bounded with the rate of $O_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$\text{P-}\lim_{j,k} \sum_{(m,n)\in N(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$N(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : \left|x_{m,n}\right| \geq \varepsilon\alpha_{m,n}\right\}.$$

In this case, we write

$$x_{m,n} = \text{st}_A^{(2)} - O_{m,n}(\alpha_{m,n}) \quad \text{as } m, n \to \infty.$$

As a tool, we use the modulus of continuity $\omega(f;\delta)$ defined as follows:

$$\omega(f;\delta) := \sup\left\{|f(u,v) - f(x,y)| : (u,v), (x,y) \in \mathbb{R}^2, \sqrt{(u-x)^2 + (v-y)^2} \leq \delta\right\}$$

where $f \in C^*\left(\mathbb{R}^2\right)$ and $\delta > 0$. In order to obtain our result, we will make use of the elementary inequality, for all $f \in C^*\left(\mathbb{R}^2\right)$ and for $\lambda, \delta > 0$,

$$\omega(f;\lambda\delta) \leq (1 + [\lambda])\,\omega(f;\delta) \tag{9}$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to $\lambda$.

Then we have the following result.

**Theorem 9.** *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C^*\left(\mathbb{R}^2\right)$ into itself and let $A = \left(a_{j,k,m,n}\right)$ be a non-negative* RH*-regular summability matrix. Let $\left\{\alpha_{m,n}\right\}$ and $\left\{\beta_{m,n}\right\}$ be a positive non-increasing double sequence in Pringsheim's sense. Then, for all $f \in C^*\left(\mathbb{R}^2\right)$,*

$$\left\|L_{m,n}(f) - f\right\|_{C^*\left(\mathbb{R}^2\right)} = \text{st}_A^{(2)} - o(\gamma_{m,n}) \quad \text{as } m, n \to \infty, \text{ with } \gamma_{m,n} := \max\left\{\alpha_{m,n}, \beta_{m,n}\right\}$$

*provided that the following conditions hold:*

(i) $\left\| L_{m,n}(f_0) - f_0 \right\|_{C^*(\mathbb{R}^2)} = \mathrm{st}_A^{(2)} - o(\alpha_{m,n})$ *as* $m, n \to \infty$, *with* $f_0(u, v) = 1$,

(ii) $\omega\left(f; \delta_{m,n}\right) = \mathrm{st}_A^{(2)} - o(\beta_{m,n})$ *as* $m, n \to \infty$,*where* $\delta_{m,n} := \sqrt{\left\| L_{m,n}(\Psi) \right\|_{C^*(\mathbb{R}^2)}}$ *with* $\Psi(u, v) = \sin^2 \frac{u-x}{2} + \sin^2 \frac{v-y}{2}$ *for each* $(x, y), (u, v) \in \mathbb{R}^2$. *Furthermore, similar results holds when the symbol "o" is replaced by "O".*

**Proof.** To see this, we first assume that $(x, y) \in [-\pi, \pi] \times [-\pi, \pi]$ and $f \in C^*\left(\mathbb{R}^2\right)$ be fixed, and that (i) and (ii) hold. Let $\delta$ be a positive number.

*Case* I. If $\delta < |u - x| \le \pi$ and $\delta < |v - y| \le \pi$, then $|u - x| \le \pi \left|\sin \frac{u-x}{2}\right|$ and $\delta < |v - y| \le \pi \left|\sin \frac{v-y}{2}\right|$ and therefore

$$\begin{aligned} |f(u, v) - f(x, y)| &\le \omega\left(f; \sqrt{(u-x)^2 + (v-y)^2}\right) \\ &\le \left(1 + \frac{\sqrt{(u-x)^2 + (v-y)^2}}{\delta}\right) \omega(f; \delta) \\ &\le \left(1 + \pi^2 \frac{\sin^2 \frac{u-x}{2} + \sin^2 \frac{v-y}{2}}{\delta^2}\right) \omega(f; \delta). \end{aligned} \tag{10}$$

If $|u - x| \le \delta$ and $|v - y| \le \delta$, then this inequality holds.

*Case* II. If $|u - x| > \pi$ and $|v - y| \le \pi$, let $k$ be an integer such that $|u + 2k\pi - x| \le \pi$; then

$$\begin{aligned} |f(u, v) - f(x, y)| &= |f(u + 2k\pi, v) - f(x, y)| \\ &\le \left(1 + \pi^2 \frac{\sin^2 \frac{u+2k\pi-x}{2} + \sin^2 \frac{v-y}{2}}{\delta^2}\right) \omega(f; \delta) \\ &= \left(1 + \pi^2 \frac{\sin^2 \frac{u-x}{2} + \sin^2 \frac{v-y}{2}}{\delta^2}\right) \omega(f; \delta). \end{aligned}$$

*Case* III. Let $|u - x| \le \pi$ and $|v - y| > \pi$. As in Case II, let $l$ be an integer such that $|v + 2l\pi - y| \le \pi$; then we have

$$\begin{aligned} |f(u, v) - f(x, y)| &= |f(u, v + 2l\pi) - f(x, y)| \\ &\le \left(1 + \pi^2 \frac{\sin^2 \frac{u-x}{2} + \sin^2 \frac{v-y}{2}}{\delta^2}\right) \omega(f; \delta). \end{aligned}$$

*Case* IV. Let $|u - x| > \pi$ and $|v - y| > \pi$. As in Case II and III, this situation is obtained.

Thus, (10) always holds. Using the definition of modulus of continuity and the linearity and the positivity of the operators $L_{m,n}$, for all $(m, n) \in \mathbb{N}^2$, we have

$$\begin{aligned} \left|L_{m,n}(f; x, y) - f(x, y)\right| &\le L_{m,n}\left(|f(u, v) - f(x, y)|; x, y\right) + |f(x, y)| \left|L_{m,n}(f_0; x) - f_0(x, y)\right| \\ &\le \omega(f; \delta) L_{m,n}(f_0, x, y) + \pi^2 \frac{\omega(f; \delta)}{\delta^2} L_{m,n}(\Psi; x, y) + |f(x, y)| \left|L_{m,n}(f_0, x, y) - f_0(x, y)\right|. \end{aligned}$$

Taking supremum over $(x, y)$ on the both sides of the above inequality and $\delta := \delta_{m,n} := \sqrt{\left\| L_{m,n}(\Psi) \right\|_{C^*(\mathbb{R}^2)}}$, then we obtain

$$\left\| L_{m,n}(f) - f \right\|_{C^*(\mathbb{R}^2)} \le \omega\left(f; \delta_{m,n}\right) \left\| L_{m,n}(f_0) - f_0 \right\|_{C^*(\mathbb{R}^2)} + \left(1 + \pi^2\right) \omega\left(f; \delta_{m,n}\right) + M \left\| L_{m,n}(f_0) - f_0 \right\|_{C^*(\mathbb{R}^2)} \tag{11}$$

where the quantity $M := \|f\|_{C^*(\mathbb{R}^2)}$ is a finite number since $f \in C^*\left(\mathbb{R}^2\right)$. Now, given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} D &:= \left\{(m, n) : \left\| L_{m,n}(f) - f \right\|_{C^*(\mathbb{R}^2)} \ge \varepsilon\right\}, \\ D_1 &:= \left\{(m, n) : \omega\left(f; \delta_{m,n}\right) \left\| L_{m,n}(f_0) - f_0 \right\|_{C^*(\mathbb{R}^2)} \ge \frac{\varepsilon}{3}\right\}, \\ D_2 &:= \left\{(m, n) : \omega\left(f; \delta_{m,n}\right) \ge \frac{\varepsilon}{3\left(1 + \pi^2\right)}\right\}, \\ D_3 &:= \left\{(m, n) : \left\| L_{m,n}(f_0) - f_0 \right\|_{C^*(\mathbb{R}^2)} \ge \frac{\varepsilon}{3M}\right\}. \end{aligned}$$

Then, it follows from (11) that $D \subset D_1 \cup D_2 \cup D_3$. Also, defining

$$D_4 := \left\{(m, n) \in \mathbb{N}^2 : \omega\left(f; \delta_{m,n}\right) \ge \sqrt{\frac{\varepsilon}{3}}\right\},$$

$$D_5 := \left\{ (m, n) \in \mathbb{N}^2 : \left\| L_{m,n}(f_0) - f_0 \right\|_{C^*(\mathbb{R}^2)} \geq \sqrt{\frac{\varepsilon}{3}} \right\},$$

we have $D_1 \subset D_4 \cup D_5$, which yields

$$D \subseteq \bigcup_{i=2}^{5} D_i.$$

Therefore, since $\gamma_{m,n} = \max\left\{\alpha_{m,n}, \beta_{m,n}\right\}$, we conclude that, for all $(j, k) \in \mathbb{N}^2$,

$$\frac{1}{\gamma_{j,k}} \sum_{(m,n)\in D} a_{j,k,m,n} \leq \frac{1}{\beta_{j,k}} \sum_{(m,n)\in D_2} a_{j,k,m,n} + \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in D_3} a_{j,k,m,n} + \frac{1}{\beta_{j,k}} \sum_{(m,n)\in D_4} a_{j,k,m,n} + \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in D_5} a_{j,k,m,n}. \tag{12}$$

Letting $j, k \to \infty$ (in any manner) on both sides of (12), we get

$$\text{P-}\lim_{j,k\to\infty} \frac{1}{\gamma_{j,k}} \sum_{(m,n)\in D} a_{j,k,m,n} = 0.$$

Therefore, the proof is completed. □

Now, specializing Theorem 9, we can give the ordinary rates of convergence of a sequence of positive linear operators defined on the space $C^*\left(\mathbb{R}^2\right)$. We first note that, if we choose $\alpha_{m,n} = \beta_{m,n} = 1$ for all $m, n \in \mathbb{N}$, then Theorem 1 is obtained from Theorem 9 at once. So our theorem gives us the rate of $A$-statistical convergence in Theorem 1. Furthermore, if one replaces the matrix $A = \left(a_{j,k,m,n}\right)$ by the double identity matrix, then Theorem 9 immediately gives the following result in Pringsheim's sense, which corresponds to Theorem 2.1 in [18].

**Corollary 10.** *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C^*\left(\mathbb{R}^2\right)$ into itself. Then, for all $f \in C^*\left(\mathbb{R}^2\right)$,*

$$\text{P-}\lim_{m,n} \left\| L_{m,n}(f) - f \right\|_{C^*(\mathbb{R}^2)} = 0,$$

*provided that the following conditions hold:*

(i) P-$\lim_{m,n} \left\| L_{m,n}(f_0) - f_0 \right\|_{C^*(\mathbb{R}^2)} = 0$,

(ii) P-$\lim_{m,n} \omega\left(f; \delta_{m,n}\right) = 0$, *where $f_0$ and $\left\{\delta_{m,n}\right\}$ are the same as in Theorem 9.*

One can immediately obtain the next result using a similar technique to that used in the proof of Theorem 9.

**Theorem 11.** *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C^*\left(\mathbb{R}^2\right)$ into itself and let $A = \left(a_{j,k,m,n}\right)$ be a non-negative* RH*-regular summability matrix. Let $\left\{\alpha_{m,n}\right\}$ and $\left\{\beta_{m,n}\right\}$ be a positive non-increasing double sequence in Pringsheim's sense. Then, for all $f \in C(\mathbb{R}^2)$,*

$$\left\| L_{m,n}(f) - f \right\|_{C^*(\mathbb{R}^2)} = \mathrm{st}_A^{(2)} - o_{m,n}(\gamma_{m,n}) \quad \text{as } m, n \to \infty,$$

*with $\gamma_{m,n} := \max\left\{\alpha_{m,n}, \beta_{m,n}, \alpha_{m,n}\beta_{m,n}\right\}$, provided that the following conditions hold:*

(i) $\left\| L_{m,n}(f_0) - f_0 \right\|_{C^*(\mathbb{R}^2)} = \mathrm{st}_A^{(2)} - o_{m,n}(\alpha_{m,n})$ *as $m, n \to \infty$, with $f_0(u, v) = 1$,*

(ii) $\omega\left(f; \delta_{m,n}\right) = \mathrm{st}_A^{(2)} - o_{m,n}(\beta_{m,n})$ *as $m, n \to \infty$, where $\delta_{m,n} := \sqrt{\left\| L_{m,n}(\Psi) \right\|_{C^*(\mathbb{R}^2)}}$ with $\Psi(u, v) = \sin^2 \frac{u-x}{2} + \sin^2 \frac{v-y}{2}$ for each $(x, y), (u, v) \in \mathbb{R}^2$.*

*Similar results hold when little "$o_{m,n}$" is replaced by big "$O_{m,n}$".*

## 4. An application to Theorem 9

In this section, we display an example of positive linear operators, which satisfies Theorem 9 but not Corollary 10.

Let $A = \left(a_{j,k,m,n}\right)$ be a non-negative RH-regular summability matrix. We know that a P-convergent double sequence is $A$-statistically convergent to the same value but the converse does not hold true. So, we can choose a non-negative double sequence $\left\{u_{m,n}\right\}$ that converges $A$-statistically to 0 but is not P-convergent. Then, we consider the following operators defined by (8) on $C^*\left(\mathbb{R}^2\right)$:

$$L_{m,n}(f; x, y) = \left(1 + u_{m,n}\right) \sigma_{m,n}(f; x, y). \tag{13}$$

Now, we take $A = C(1,1)$ and also replace the double sequence $\{u_{m,n}\}$ by

$$u_{m,n} = \begin{cases} \sqrt{mn}, & \text{if } m \text{ and } n \text{ are squares,} \\ 0, & \text{otherwise.} \end{cases}$$

Now, setting $\left\{\alpha_{m,n}\right\} = \left\{\frac{1}{\sqrt[4]{mn}}\right\}$, we have, for any $\varepsilon > 0$,

$$\frac{1}{\alpha_{j,k}} \sum_{(m,n):\left|u_{m,n}\right| \geq \varepsilon} c_{j,k,m,n} = \sqrt[4]{jk} \sum_{(m,n):\left|u_{m,n}\right| \geq \varepsilon} \frac{1}{jk} \leq \frac{\sqrt[4]{jk}\sqrt{jk}}{jk} = \frac{1}{\sqrt[4]{jk}}. \tag{14}$$

Taking the limit as $j, k \to \infty$ (in any manner) in (14), we get, for any $\varepsilon > 0$,

$$\text{P-}\lim_{j,k} \frac{1}{\alpha_{j,k}} \sum_{(m,n):\left|u_{m,n}\right| \geq \varepsilon} c_{j,k,m,n} = 0$$

which gives,

$$u_{m,n} = \mathrm{st}^{(2)}_{C(1,1)} - o\left(\frac{1}{\sqrt[4]{mn}}\right) \quad \text{as } m, n \to \infty. \tag{15}$$

Also, observe that

$$\begin{aligned} L_{m,n}(f_0; x, y) &= 1 + u_{m,n}, \\ L_{m,n}(f_1; x, y) &= \left(1 + u_{m,n}\right) \frac{m-1}{m} f_1(x, y), \\ L_{m,n}(f_2; x, y) &= \left(1 + u_{m,n}\right) \frac{n-1}{n} f_2(x, y), \\ L_{m,n}(f_3; x, y) &= \left(1 + u_{m,n}\right) \frac{m-1}{m} f_3(x, y), \\ L_{m,n}(f_4; x, y) &= \left(1 + u_{m,n}\right) \frac{n-1}{n} f_4(x, y), \end{aligned}$$

where $f_0(x, y) = 1, f_1(x, y) = \sin x, f_2(x, y) = \sin y, f_3(x, y) = \cos x$ and $f_4(x, y) = \cos y$. Since $\left\|L_{m,n}(f_0) - f_0\right\|_{C^*\left(\mathbb{R}^2\right)} = u_{m,n}$, we obtain from (15)

$$\left\|L_{m,n}(f_0) - f_0\right\|_{C(I^2)} = \mathrm{st}^2_{(C(1,1))} - o(\alpha_{m,n}) \quad \text{as } m, n \to \infty. \tag{16}$$

Now, we compute the quantity $L_{m,n}(\Psi; x, y)$, where $\Psi(u, v) = \sin^2 \frac{u-x}{2} + \sin^2 \frac{v-y}{2}$. After some calculations, we get

$$L_{m,n}(\Psi; x, y) = \frac{1 + u_{m,n}}{2} \left(\frac{1}{m} + \frac{1}{n}\right).$$

Then, we obtain $\delta_{m,n} := \sqrt{\left\|L_{m,n}(\Psi)\right\|_{C^*\left(\mathbb{R}^2\right)}} = \sqrt{\frac{1+u_{m,n}}{2}\left(\frac{1}{m} + \frac{1}{n}\right)}$. In this case, setting $\left\{\beta_{m,n}\right\} = \left\{\frac{1}{\sqrt[8]{mn}}\right\}$, we have, for any $\varepsilon > 0$,

$$\frac{1}{\beta_{j,k}} \sum_{(m,n):\left|\delta_{m,n}\right| \geq \varepsilon} c_{j,k,m,n} = \sqrt[3]{jk} \sum_{(m,n):\left|\delta_{m,n}\right| \geq \varepsilon} \frac{1}{jk} \leq \frac{\sqrt[8]{jk}\sqrt{jk}}{jk} = \frac{1}{\sqrt[8]{(jk)^3}}$$

which gives that

$$\text{P-}\lim_{j,k} \frac{1}{\beta_{j,k}} \sum_{(m,n):\left|\delta_{m,n}\right| \geq \varepsilon} c_{j,k,m,n} = 0.$$

Hence, we obtain $\delta_{m,n} = \mathrm{st}^{(2)}_{C(1,1)} - o\left(\frac{1}{\sqrt[8]{mn}}\right)$ as $m, n \to \infty$. By the uniform continuity of $f$ on $\mathbb{R}^2$, we write that

$$\omega\left(f; \delta_{m,n}\right) = \mathrm{st}^{(2)}_{C(1,1)} - o\left(\frac{1}{\sqrt[8]{mn}}\right) \quad \text{as } m, n \to \infty. \tag{17}$$

Then, the sequence of positive linear operators $\left\{L_{m,n}\right\}$ satisfy all hypotheses of Theorem 9 from (16) and (17). So, we have, for all $f \in C^*\left(\mathbb{R}^2\right)$,

$$\left\|L_{m,n}(f) - f\right\|_{C^*\left(\mathbb{R}^2\right)} = \mathrm{st}^2_{(C(1,1))} - o\left(\frac{1}{\sqrt[8]{mn}}\right) \quad \text{as } m, n \to \infty.$$

However, since $\left\{u_{m,n}\right\}$ is not P-convergent, the sequence $\{L_{m,n}\}$ given by (13) does not converge uniformly to the function $f \in C^*\left(\mathbb{R}^2\right)$.

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