DEMONSTRATIO MATHEMATICA
Vol. XLVI No 1 2013

Kamil Demirci and Sevda Karakuş

FOUR-DIMENSIONAL MATRIX TRANSFORMATION AND A-STATISTICAL FUZZY KOROVKIN TYPE APPROXIMATION

Abstract. In this paper, we prove a fuzzy Korovkin-type approximation theorem for fuzzy positive linear operators by using A-statistical convergence for four-dimensional summability matrices. Also, we obtain rates of A-statistical convergence of a double sequence of fuzzy positive linear operators for four-dimensional summability matrices.

1. Introduction

Anastassiou [3] first introduced the fuzzy analogue of the classical Korovkin theory (see also [1], [2], [4], [10]). Recently, some statistical fuzzy approximation theorems have been obtain by using the concept of statistical convergence (see, [5], [8]). In this paper, we prove a fuzzy Korovkin-type approximation theorem for fuzzy positive linear operators by using A-statistical convergence for four-dimensional summability matrices. Then, we construct an example such that our new approximation result works but its classical case does not work. Also we obtain rates of A-statistical convergence of a double sequence of fuzzy positive linear operators for four-dimensional summability matrices.

We now recall some basic definitions and notations used in the paper.

A fuzzy number is a function $\mu : \mathbb{R} \to [0,1]$, which is normal, convex, upper semi-continuous and the closure of the set $supp(\mu)$ is compact, where $supp(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$. The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathcal{F}}$. Let

$$[\mu]^0 = \overline{\{x \in \mathbb{R} : \mu(x) > 0\}} \text{ and } [\mu]^r = \{x \in \mathbb{R} : \mu(x) \geq r\}, \ (0 < r \leq 1).$$

Then, it is well-known [11] that, for each $r \in [0,1]$, the set $[\mu]^r$ is a closed

2000 *Mathematics Subject Classification*: 26E50, 41A25, 41A36, 40G15.
Key words and phrases: A-statistical convergence for double sequences, fuzzy positive linear operators, fuzzy Korovkin theory, rates of A-statistical convergence for double sequences, regularity for double sequences.

and bounded interval of $\mathbb{R}$. For any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, it is possible to define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ as follows:

$$[u \oplus v]^r = [u]^r + [v]^r \text{ and } [\lambda \odot u]^r = \lambda [u]^r, \ (0 \le r \le 1).$$

Now denote the interval $[u]^r$ by $\left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \le u_+^{(r)}$ and $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$ for $r \in [0,1]$. Then, for $u, v \in \mathbb{R}_{\mathcal{F}}$, define

$$u \preceq v \Leftrightarrow u_-^{(r)} \le v_-^{(r)} \text{ and } u_+^{(r)} \le v_+^{(r)} \text{ for all } 0 \le r \le 1.$$

Define also the following metric $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u,v) = \sup_{r\in[0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\}.$$

Hence, $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space [18].

A double sequence $x = \{x_{m,n}\}$, $m, n \in \mathbb{N}$, is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. Then, L is called the Pringsheim limit of x and is denoted by $P - \lim_{m,n} x_{m,n} = L$ (see [16]). In this case, we say that $x = \{x_{m,n}\}$ is "P-convergent to L". Also, if there exists a positive number M such that $|x_{m,n}| \le M$ for all $(m,n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = \{x_{m,n}\}$ is said to be bounded. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded. A double sequence $x = \{x_{m,n}\}$ is said to be non-increasing in Pringsheim's sense if, for all $(m,n) \in \mathbb{N}^2$, $x_{m+1,n+1} \le x_{m,n}$.

Now let $A = [a_{j,k,m,n}]$, $j, k, m, n \in \mathbb{N}$, be a four-dimensional summability matrix. For a given double sequence $x = \{x_{m,n}\}$, the A-transform of x, denoted by $Ax := \{(Ax)_{j,k}\}$, is given by

$$(Ax)_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \quad j, k \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(j,k) \in \mathbb{N}^2$. In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations is known as Silverman–Toeplitz conditions (see, for instance, [13]). In 1926, Robison [17] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison–Hamilton conditions, or briefly, RH-regularity (see, [12], [17]).

Recall that a four dimensional matrix $A = [a_{j,k,m,n}]$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison–Hamilton conditions state that a four dimensional matrix $A = [a_{j,k,m,n}]$ is RH-regular if and only if

(i) $P - \lim\limits_{j,k} a_{j,k,m,n} = 0$ for each $(m,n) \in \mathbb{N}^2$,

(ii) $P - \lim\limits_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} = 1$,

(iii) $P - \lim\limits_{j,k} \sum_{m\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,

(iv) $P - \lim\limits_{j,k} \sum_{n\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{j,k,m,n}|$ is P-convergent for each $(j,k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that $\sum_{m,n>B} |a_{j,k,m,n}| < A$ holds for every $(j,k) \in \mathbb{N}^2$.

Now let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then, a double sequence $\{x_{m,n}\}$ of fuzzy numbers is said to be A-statistically convergent to a fuzzy number $L \in \mathbb{R}_{\mathcal{F}}$ if, for every $\varepsilon > 0$,

$$P - \lim_{j,k} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon\}.$$

In this case we write $st^2_{(A)} - \lim\limits_{m,n} x_{m,n} = L$.

We should note that if we take $A = C(1;1) := [c_{j,k,m,n}]$, the double Cesáro matrix, defined by

$$c_{j,k,m,n} = \begin{cases} \frac{1}{jk}, & \text{if } 1 \leq m \leq j \text{ and } 1 \leq n \leq k, \\ 0, & \text{otherwise,} \end{cases}$$

then $C(1;1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [14], [15]. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence [16].

2. A-statistical fuzzy Korovkin type approximation

Let us choose the real numbers $a; b; c; d$ so that $a < b, c < d$, and $U := [a;b] \times [c;d]$. Let $C(U)$ denote the space of all real valued continuous functions on U endowed with the supremum norm

$$\|f\| = \sup_{(x,y)\in U} |f(x,y)|, \ (f \in C(U)).$$

Assume that $f : U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy number valued function. Then f is said to be fuzzy continuous at $x^0 := (x_0, y_0) \in U$ whenever $P - \lim_{m,n} x_{m,n} = x^0$, then $P-\lim D(f(x_{m,n}), f(x^0)) = 0$. If it is fuzzy continuous at every point $(x,y) \in U$, we say that f is fuzzy continuous on U. The set of all fuzzy continuous functions on U is denoted by $C_{\mathcal{F}}(U)$. Note that $C_{\mathcal{F}}(U)$ is a vector space. Now let $L : C_{\mathcal{F}}(U) \to C_{\mathcal{F}}(U)$ be an operator. Then L is said to be fuzzy linear if, for every $\lambda_1, \lambda_2 \in \mathbb{R}$ having the same sing and for every $f_1, f_2 \in C_{\mathcal{F}}(U)$, and $(x,y) \in U$,

$$L(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2; x, y) = \lambda_1 \odot L(f_1; x, y) \oplus \lambda_2 \odot L(f_2; x, y)$$

holds. Also L is called fuzzy positive linear operator if it is fuzzy linear and, the condition $L(f;x,y) \preceq L(g;x,y)$ is satisfied for any $f, g \in C_{\mathcal{F}}(U)$ and all $(x,y) \in U$ with $f(x,y) \preceq g(x,y)$. Also, if $f, g : U \to \mathbb{R}_{\mathcal{F}}$ are fuzzy number valued functions, then the distance between f and g is given by

$$D^*(f,g) = \sup_{(x,y)\in U} \sup_{r\in[0,1]} \max\left\{\left|f_-^{(r)} - g_-^{(r)}\right|, \left|f_+^{(r)} - g_+^{(r)}\right|\right\}$$

(see for details, [1], [2], [3], [4], [9], [10]). Throughout the paper we use the test functions given by

$$f_0(x,y) = 1, \;\; f_1(x,y) = x, \; f_2(x,y) = y, \;\; f_3(x,y) = x^2 + y^2.$$

Theorem 2.1. *Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{L_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ be a double sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(U)$ into itself. Assume that there exists a corresponding sequence $\{\widetilde{L}_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ of positive linear operators from $C(U)$ into itself with the property*

$$\{L_{m,n}(f;x,y)\}_{\pm}^{(r)} = \widetilde{L}_{m,n}\left(f_{\pm}^{(r)}; x, y\right) \tag{2.1}$$

for all $(x,y) \in U$, $r \in [0,1]$, $(m,n) \in \mathbb{N}^2$ and $f \in C_{\mathcal{F}}(U)$. Assume further that

$$st^2_{(A)} - \lim_{m,n\to\infty} \left\|\widetilde{L}_{m,n}(f_i) - f_i\right\| = 0 \;\; \text{for each } i = 0,1,2,3. \tag{2.2}$$

Then, for all $f \in C_{\mathcal{F}}(U)$, we have

$$st^2_{(A)} - \lim_{m,n\to\infty} D^*(L_{m,n}(f), f) = 0.$$

Proof. Let $f \in C_{\mathcal{F}}(U)$, $(x,y) \in U$ and $r \in [0,1]$. By the hypothesis, since $f_{\pm}^{(r)} \in C(U)$, we can write, for every $\varepsilon > 0$, that there exists a number $\delta > 0$ such that $\left|f_{\pm}^{(r)}(u,v) - f_{\pm}^{(r)}(x,y)\right| < \varepsilon$ holds for every $(u,v) \in U$ satisfying $|u-x| < \delta$ and $|v-y| < \delta$. Then we immediately get for all $(u,v) \in U$, that

$$\left|f_{\pm}^{(r)}(u,v) - f_{\pm}^{(r)}(x,y)\right| \le \varepsilon + \frac{2M_{\pm}^{(r)}}{\delta^2}\left\{(u-x)^2 + (v-y)^2\right\},$$

where $M_{\pm}^{(r)} := \left\| f_{\pm}^{(r)} \right\|$. Now, using the linearity and the positivity of the operators $\widetilde{L}_{m,n}$, we have, for each $(m,n) \in \mathbb{N}^2$, that

$$\begin{aligned}
&\left|\widetilde{L}_{m,n}\left(f_{\pm}^{(r)};x,y\right)-f_{\pm}^{(r)}(x,y)\right| \\
&\leq \widetilde{L}_{m,n}\left(\left|f_{\pm}^{(r)}(u,v)-f_{\pm}^{(r)}(x,y)\right|;x,y\right)+M_{\pm}^{(r)}\left|\widetilde{L}_{m,n}(f_0;x,y)-f_0(x,y)\right| \\
&\leq \widetilde{L}_{m,n}\left(\varepsilon+\frac{2M_{\pm}^{(r)}}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\};x,y\right)+M_{\pm}^{(r)}\left|\widetilde{L}_{m,n}(f_0;x,y)-f_0(x,y)\right| \\
&\leq \varepsilon+\left(\varepsilon+M_{\pm}^{(r)}\right)\left|\widetilde{L}_{m,n}(f_0;x,y)-f_0(x,y)\right|+\frac{2M_{\pm}^{(r)}}{\delta^2}\widetilde{L}_{m,n}\left((u-x)^2+(v-y)^2;x,y\right) \\
&\leq \varepsilon+\left(\varepsilon+M_{\pm}^{(r)}\right)\left|\widetilde{L}_{m,n}(f_0;x,y)-f_0(x,y)\right|+\frac{2M_{\pm}^{(r)}}{\delta^2}\left\{\left|\widetilde{L}_{m,n}(f_3;x,y)-f_3(x,y)\right|\right. \\
&\quad+2|x|\left|\widetilde{L}_{m,n}(f_1;x,y)-f_1(x,y)\right|+2|y|\left|\widetilde{L}_{m,n}(f_2;x,y)-f_2(x,y)\right| \\
&\quad\left.+\left(x^2+y^2\right)\left|\widetilde{L}_{m,n}(f_0;x,y)-f_0(x,y)\right|\right\} \\
&\leq \varepsilon+\left(\varepsilon+M_{\pm}^{(r)}+\frac{2M_{\pm}^{(r)}}{\delta^2}\left(x^2+y^2\right)\right)\left|\widetilde{L}_{m,n}(f_0;x,y)-f_0(x,y)\right| \\
&\quad+\frac{4M_{\pm}^{(r)}}{\delta^2}|x|\left|\widetilde{L}_{m,n}(f_1;x,y)-f_1(x,y)\right|+\frac{4M_{\pm}^{(r)}}{\delta^2}|y|\left|\widetilde{L}_{m,n}(f_2;x,y)-f_2(x,y)\right| \\
&\quad+\frac{2M_{\pm}^{(r)}}{\delta^2}\left|\widetilde{L}_{m,n}(f_3;x,y)-f_3(x,y)\right| \\
&\leq \varepsilon+K_{\pm}^{(r)}(\varepsilon)\left\{\left|\widetilde{L}_{m,n}(f_0;x,y)-f_0(x,y)\right|+\left|\widetilde{L}_{m,n}(f_1;x,y)-f_1(x,y)\right|\right. \\
&\quad\left.+\left|\widetilde{L}_{m,n}(f_2;x,y)-f_2(x,y)\right|+\left|\widetilde{L}_{m,n}(f_3;x,y)-f_3(x,y)\right|\right\}
\end{aligned}$$

where $K_{\pm}^{(r)}(\varepsilon) := \max\left\{\varepsilon+M_{\pm}^{(r)}+\frac{2M_{\pm}^{(r)}}{\delta^2}\left(A^2+B^2\right), \frac{4M_{\pm}^{(r)}}{\delta^2}A, \frac{4M_{\pm}^{(r)}}{\delta^2}B, \frac{2M_{\pm}^{(r)}}{\delta^2}\right\}$, $A := \max\{|a|,|b|\}$, $B := \max\{|c|,|d|\}$. Also taking supremum over $(x,y) \in U$, the above inequality implies that

$$\begin{aligned}
\text{(2.3)}\qquad &\left\|\widetilde{L}_{m,n}\left(f_{\pm}^{(r)}\right)-f_{\pm}^{(r)}\right\| \\
&\leq \varepsilon+K_{\pm}^{(r)}(\varepsilon)\left\{\left\|\widetilde{L}_{m,n}(f_0)-f_0\right\|+\left\|\widetilde{L}_{m,n}(f_1)-f_1\right\|\right. \\
&\qquad\left.+\left\|\widetilde{L}_{m,n}(f_2)-f_2\right\|+\left\|\widetilde{L}_{m,n}(f_3)-f_3\right\|\right\}.
\end{aligned}$$

Now, it follows from (2.1) that

$$\begin{aligned} D^{*}\left(L_{m,n}\left(f\right),f\right) &= \sup_{(x,y)\in U} D\left(L_{m,n}\left(f;x,y\right),f\left(x,y\right)\right) \\ &= \sup_{(x,y)\in U} \sup_{r\in[0,1]} \max\Big\{\big|\widetilde{L}_{m,n}\big(f_{-}^{(r)};x,y\big)-f_{-}^{(r)}\left(x,y\right)\big|, \\ &\qquad\qquad \big|\widetilde{L}_{m,n}\big(f_{+}^{(r)};x,y\big)-f_{+}^{(r)}\left(x,y\right)\big|\Big\} \\ &= \sup_{r\in[0,1]} \max\Big\{\big\|\widetilde{L}_{m,n}\big(f_{-}^{(r)}\big)-f_{-}^{(r)}\big\|,\big\|\widetilde{L}_{m,n}\big(f_{+}^{(r)}\big)-f_{+}^{(r)}\big\|\Big\}. \end{aligned}$$

Combining the above equality with (2.3), we have

$$\begin{aligned} D^{*}\left(L_{m,n}\left(f\right),f\right) \leq \varepsilon + K\left(\varepsilon\right)\Big\{&\big\|\widetilde{L}_{m,n}\left(f_0\right)-f_0\big\| + \big\|\widetilde{L}_{m,n}\left(f_1\right)-f_1\big\| \\ &+ \big\|\widetilde{L}_{m,n}\left(f_2\right)-f_2\big\| + \big\|\widetilde{L}_{m,n}\left(f_3\right)-f_3\big\|\Big\} \end{aligned} \tag{2.4}$$

where $K\left(\varepsilon\right) := \sup_{r\in[0,1]} \max\big\{K_{-}^{(r)}\left(\varepsilon\right),K_{+}^{(r)}\left(\varepsilon\right)\big\}$.

Now, for a given $r>0$, choose $\varepsilon>0$ such that $0<\varepsilon<r$, and also define the following sets:

$$\begin{aligned} G &:= \big\{(m,n)\in\mathbb{N}^2 : D^{*}\left(L_{m,n}\left(f\right),f\right)\geq r\big\}, \\ G_0 &:= \left\{(m,n)\in\mathbb{N}^2 : \big\|\widetilde{L}_{m,n}\left(f_0\right)-f_0\big\| \geq \frac{r-\varepsilon}{4K\left(\varepsilon\right)}\right\}, \\ G_1 &:= \left\{(m,n)\in\mathbb{N}^2 : \big\|\widetilde{L}_{m,n}\left(f_1\right)-f_1\big\| \geq \frac{r-\varepsilon}{4K\left(\varepsilon\right)}\right\}, \\ G_2 &:= \left\{(m,n)\in\mathbb{N}^2 : \big\|\widetilde{L}_{m,n}\left(f_2\right)-f_2\big\| \geq \frac{r-\varepsilon}{4K\left(\varepsilon\right)}\right\}, \\ G_3 &:= \left\{(m,n)\in\mathbb{N}^2 : \big\|\widetilde{L}_{m,n}\left(f_3\right)-f_3\big\| \geq \frac{r-\varepsilon}{4K\left(\varepsilon\right)}\right\}. \end{aligned}$$

Then inequality (2.4) gives

$$G \subset G_0 \cup G_1 \cup G_2 \cup G_3$$

which guarantees that, for each $(j,k)\in\mathbb{N}^2$

$$\begin{aligned} \sum_{(m,n)\in G} a_{j,k,m,n} \leq \sum_{(m,n)\in G_0} a_{j,k,m,n} &+ \sum_{(m,n)\in G_1} a_{j,k,m,n} \\ &+ \sum_{(m,n)\in G_2} a_{j,k,m,n} + \sum_{(m,n)\in G_3} a_{j,k,m,n}. \end{aligned} \tag{2.5}$$

If we take the limit as $j,k\to\infty$ on the both sides of inequality (2.5) and use the hypothesis (2.2), we immediately see that

$$\lim_{j,k} \sum_{(m,n)\in G} a_{j,k,m,n} = 0$$

whence the result. ∎

If $A = I$, the identity matrix, then we obtain the following new fuzzy Korovkin theorem in Pringsheim's sense.

Theorem 2.2. *Let $\{L_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ be a double sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(U)$ into itself. Assume that there exists a corresponding sequence $\{\widetilde{L}_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ of positive linear operators from $C(U)$ into itself with the property (2.1). Assume further that*

$$P-\lim_{m,n\to\infty} \left\| \widetilde{L}_{m,n}(f_i) - f_i \right\| = 0 \quad \textit{for each } i = 0,1,2,3.$$

Then, for all $f \in C_{\mathcal{F}}(U)$, we have

$$P-\lim_{m,n\to\infty} D^*\left(L_{m,n}(f), f\right) = 0.$$

We will now show that our result Theorem 2.1 is stronger than its classical (Theorem 2.2) version.

Example 2.3. Take $A = C(1,1) := [c_{j,k,m,n}]$, the double Cesáro matrix, and define the double sequence $\{u_{m,n}\}$ by

$$u_{m,n} = \begin{cases} \sqrt{mn}, & \text{if } m \text{ and } n \text{ are square,} \\ 0, & \text{otherwise.} \end{cases}$$

We observe that, $st^{(2)}_{C(1,1)} - \lim_{m,n\to\infty} u_{m,n} = 0$. But $\{u_{m,n}\}$ is neither P-convergent nor bounded. Then consider the fuzzy Bernstein-type polynomials as follows:

$$B^{(\mathcal{F})}_{m,n}(f;x,y) = (1+u_{m,n}) \odot \bigoplus_{s=0}^{m} \odot \bigoplus_{t=0}^{n} \binom{m}{s}\binom{n}{t} x^s y^t (1-x)^{m-s}(1-y)^{n-t} \odot f\left(\frac{s}{m},\frac{t}{n}\right), \tag{2.6}$$

where $f \in C_{\mathcal{F}}(U)$, $(x,y) \in U$, $(m,n) \in \mathbb{N}^2$. In this case, we write

$$\left\{B^{(\mathcal{F})}_{m,n}(f;x,y)\right\}^{(r)}_{\pm} = \widetilde{B}_{m,n}\left(f^{(r)}_{\pm};x,y\right) = (1+u_{m,n}) \sum_{s=0}^{m}\sum_{t=0}^{n} \binom{m}{s}\binom{n}{t} x^s y^t (1-x)^{m-s}(1-y)^{n-t} f^{(r)}_{\pm}\left(\frac{s}{m},\frac{t}{n}\right),$$

where $f_{\pm}^{(r)} \in C(U)$. Then, we get

$$\begin{aligned}
\widetilde{B}_{m,n}(f_0;x,y) &= (1+u_{m,n})\, f_0(x,y),\\
\widetilde{B}_{m,n}(f_1;x,y) &= (1+u_{m,n})\, f_1(x,y),\\
\widetilde{B}_{m,n}(f_2;x,y) &= (1+u_{m,n})\, f_2(x,y),\\
\widetilde{B}_{m,n}(f_3;x,y) &= (1+u_{m,n}) \left(f_3(x,y) + \frac{x-x^2}{m} + \frac{y-y^2}{n} \right).
\end{aligned}$$

So we conclude that

$$st^2_{C(1,1)} - \lim_{m,n\to\infty} \left\| \widetilde{B}_{m,n}(f_i) - f_i \right\| = 0 \quad \text{for each } i = 0,1,2,3.$$

By Theorem 2.1, we obtain for all $f \in C_{\mathcal{F}}(U)$, that

$$st^2_{C(1,1)} - \lim_{m,n\to\infty} D^* \left(B^{(\mathcal{F})}_{m,n}(f), f \right) = 0.$$

However, since the sequence $\{u_{m,n}\}$ is not convergent (in the Pringsheim's sense), we conclude that Theorem 2.2 do not work for the operators $\left\{B^{(\mathcal{F})}_{m,n}(f;x,y)\right\}$ in (2.6) while our Theorem 2.1 still works.

3. A-statistical fuzzy rates

Various ways of defining rates of convergence in the A-statistical sense for two-dimensional summability matrices were introduced in [7]. In a similar way, we obtain fuzzy approximation theorems based on A-statistical rates for four-dimensional summability matrices.

Definition 3.1. Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{\alpha_{m,n}\}$ be a non-increasing double sequence of positive real numbers. A double sequence $x = \{x_{m,n}\}$ of fuzzy numbers is A-statistically convergent to a fuzzy number L with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k\to\infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \left\{ (m,n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon \right\}.$$

In this case, we write

$$D(x_{m,n}, L) = st^2_{(A)} - o(\alpha_{m,n}) \quad \text{as} \quad m,n \to \infty.$$

Definition 3.2. Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 3.1. Then, a double sequence $x = \{x_{m,n}\}$ of fuzzy numbers is A-statistically convergent to a fuzzy number L with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k\to\infty} \sum_{(m,n)\in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \big\{(m,n) \in \mathbb{N}^2 : D(x_{m,n}, L) \geq \varepsilon\ \alpha_{m,n}\big\}.$$

In this case, we write

$$D(x_{m,n}, L) = st^2_{(A)} - o_{m,n}(\alpha_{m,n}) \ \text{ as } \ m,n \to \infty.$$

Note that the rate of convergence given by Definition 3.1 is more controlled by the entries of the summability matrices rather than the terms of the sequence $x = \{x_{m,n}\}$. However, according to the statistical rate given by Definition 3.2, the rate is mainly controlled by the terms of the fuzzy sequence $x = \{x_{m,n}\}$.

Also, we can give the corresponding A-statistical rates of real sequence $\{x_{m,n}\}$.

Definition 3.3. [6] Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{\alpha_{m,n}\}$ be a non-increasing double sequence of positive real numbers. A double sequence $x = \{x_{m,n}\}$ is A-statistically convergent to a number L with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k\to\infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \big\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\big\}.$$

In this case, we write

$$x_{m,n} - L = st^2_{(A)} - o(\alpha_{m,n}) \ \text{ as } \ m,n \to \infty.$$

Definition 3.4. [6] Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 3.3. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistically convergent to a number L with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,

$$P - \lim_{j,k\to\infty} \sum_{(m,n)\in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \big\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\ \alpha_{m,n}\big\}.$$

In this case, we write

$$x_{m,n} - L = st^2_{(A)} - o_{m,n}(\alpha_{m,n}) \ \text{ as } \ m,n \to \infty.$$

Then we have the following.

Theorem 3.5. *Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{L_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ be a double sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(U)$ into itself. Assume that there exists a corresponding sequence $\{\widetilde{L}_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ of positive linear operators from $C(U)$ into itself with the property (2.1). Assume further that $\{\alpha_{i,m,n}\}_{(m,n)\in\mathbb{N}^2}$, $i = 0,1,2,3$ are non-ingreasing sequences of positive real numbers. If, for each $i = 0,1,2,3$*

$$\left\|\widetilde{L}_{m,n}(f_i) - f_i\right\| = st^2_{(A)} - o(\alpha_{i,m,n}) \quad as \quad m,n\to\infty \tag{3.1}$$

then, for all $f \in C_{\mathcal{F}}(U)$, we have

$$D^*\left(L_{m,n}(f), f\right) = st^2_{(A)} - o(\gamma_{m,n}) \quad as \quad m,n\to\infty \tag{3.2}$$

where $\gamma_{m,n} := \max_{0\le i\le 3}\{\alpha_{i,m,n}\}$ for every $(m,n)\in\mathbb{N}^2$.

Proof. Let $f \in C_{\mathcal{F}}(U)$, $(x,y)\in U$ and $r\in[0,1]$. Then, we immediately see from Theorem 2.1's proof that, for every $\varepsilon > 0$,

$$\begin{aligned} D^*\left(L_{m,n}(f), f\right) \le \varepsilon + K(\varepsilon)\Big\{&\left\|\widetilde{L}_{m,n}(f_0) - f_0\right\| + \left\|\widetilde{L}_{m,n}(f_1) - f_1\right\| \\ &+ \left\|\widetilde{L}_{m,n}(f_2) - f_2\right\| + \left\|\widetilde{L}_{m,n}(f_3) - f_3\right\|\Big\} \end{aligned} \tag{3.3}$$

where $K(\varepsilon) := \sup_{r\in[0,1]} \max\left\{K_-^{(r)}(\varepsilon), K_+^{(r)}(\varepsilon)\right\}$.

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < r$, and also define the following sets:

$$G := \left\{(m,n)\in\mathbb{N}^2 : D^*\left(L_{m,n}(f), f\right) \ge r\right\},$$

$$G_0 := \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_0) - f_0\right\| \ge \frac{r-\varepsilon}{4K(\varepsilon)}\right\},$$

$$G_1 := \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_1) - f_1\right\| \ge \frac{r-\varepsilon}{4K(\varepsilon)}\right\},$$

$$G_2 := \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_2) - f_2\right\| \ge \frac{r-\varepsilon}{4K(\varepsilon)}\right\},$$

$$G_3 := \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_3) - f_3\right\| \ge \frac{r-\varepsilon}{4K(\varepsilon)}\right\}.$$

Then inequality (3.3) gives

$$G \subset G_0 \cup G_1 \cup G_2 \cup G_3$$

which guarantees that, for each $(j,k)\in\mathbb{N}^2$

$$\sum_{(m,n)\in G} a_{j,k,m,n} \le \sum_{i=0}^{3}\Big(\sum_{(m,n)\in G_i} a_{j,k,m,n}\Big).$$

Also, by the definition of $(\gamma_{m,n})_{(m,n)\in\mathbb{N}^2}$, we have

$$(3.4)\qquad \frac{1}{\gamma_{j,k}}\sum_{(m,n)\in G} a_{j,k,m,n} \le \sum_{i=0}^{3}\Bigg(\frac{1}{\alpha_{i,j,k}}\sum_{(m,n)\in G_i} a_{j,k,m,n}\Bigg).$$

If we take the limit as $j,k\to\infty$ on both sides of inequality (3.4) and use the hypothesis (3.1), we immediately see that

$$P-\lim_{j,k\to\infty}\frac{1}{\gamma_{j,k}}\sum_{(m,n)\in G} a_{j,k,m,n},$$

which gives (3.2). So, the proof is completed. ■

We also give the next result.

Theorem 3.6. *Let* $A=[a_{j,k,m,n}]$, $\{\alpha_{i,m,n}\}_{(m,n)\in\mathbb{N}^2}$ $(i=0,1,2,3)$, $\{\gamma_{m,n}\}_{(m,n)\in\mathbb{N}^2}$, $\{L_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ *and* $\{\widetilde{L}_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ *be the same as in Theorem* 3.5 *with the property* (2.1). *If, for each* $i=0,1,2,3$

$$(3.5)\qquad \left\|\widetilde{L}_{m,n}(f_i)-f_i\right\| = st^2_{(A)}-o_{m,n}(\alpha_{i,m,n}) \quad as \quad m,n\to\infty$$

then, for all $f\in C_{\mathcal{F}}(U)$, *we have*

$$(3.6)\qquad D^*\left(L_{m,n}(f),f\right) = st^2_{(A)}-o_{m,n}(\gamma_{m,n}) \quad as \quad m,n\to\infty.$$

Proof. By (3.3), it is clear that, for any $\varepsilon>0$,

$$(3.7)\qquad \begin{aligned} &D^*\left(L_{m,n}(f),f\right)\\ &\quad\le \varepsilon\gamma_{m,n}+B(\varepsilon)\left\{\left\|\widetilde{L}_{m,n}(f_0)-f_0\right\|+\left\|\widetilde{L}_{m,n}(f_1)-f_1\right\|\right.\\ &\qquad\qquad\left.+\left\|\widetilde{L}_{m,n}(f_2)-f_2\right\|+\left\|\widetilde{L}_{m,n}(f_3)-f_3\right\|\right\}\end{aligned}$$

holds for some $B(\varepsilon)>0$. Now, as in the proof of Theorem 3.5, for a given $\varepsilon'>0$, choosing $\varepsilon>0$ such that $\varepsilon<\varepsilon'$. Now we define the following sets:

$$\begin{aligned} E &:= \left\{(m,n)\in\mathbb{N}^2 : D^*\left(L_{m,n}(f),f\right)\ge\varepsilon'\gamma_{m,n}\right\},\\ E_0 &:= \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_0)-f_0\right\|\ge\left(\frac{\varepsilon'-\varepsilon}{4B(\varepsilon)}\right)\alpha_{0,m,n}\right\},\\ E_1 &:= \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_1)-f_1\right\|\ge\left(\frac{\varepsilon'-\varepsilon}{4B(\varepsilon)}\right)\alpha_{1,m,n}\right\},\\ E_2 &:= \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_2)-f_2\right\|\ge\left(\frac{\varepsilon'-\varepsilon}{4B(\varepsilon)}\right)\alpha_{2,m,n}\right\},\\ E_3 &:= \left\{(m,n)\in\mathbb{N}^2 : \left\|\widetilde{L}_{m,n}(f_3)-f_3\right\|\ge\left(\frac{\varepsilon'-\varepsilon}{4B(\varepsilon)}\right)\alpha_{3,m,n}\right\}.\end{aligned}$$

In this case, we claim that

$$E \subset E_0 \cup E_1 \cup E_2 \cup E_3. \tag{3.8}$$

Indeed, otherwise, there would be an element $(m,n) \in E$ but $(m,n) \notin E_0 \cup E_1 \cup E_2 \cup E_3$. So, we get

$$(m,n) \notin E_0 \Rightarrow \left\|\widetilde{L}_{m,n}(f_0) - f_0\right\| < \left(\frac{\varepsilon' - \varepsilon}{4B(\varepsilon)}\right)\alpha_{0,m,n},$$

$$(m,n) \notin E_1 \Rightarrow \left\|\widetilde{L}_{m,n}(f_1) - f_1\right\| < \left(\frac{\varepsilon' - \varepsilon}{4B(\varepsilon)}\right)\alpha_{1,m,n},$$

$$(m,n) \notin E_2 \Rightarrow \left\|\widetilde{L}_{m,n}(f_2) - f_2\right\| < \left(\frac{\varepsilon' - \varepsilon}{4B(\varepsilon)}\right)\alpha_{2,m,n},$$

$$(m,n) \notin E_3 \Rightarrow \left\|\widetilde{L}_{m,n}(f_3) - f_3\right\| < \left(\frac{\varepsilon' - \varepsilon}{4B(\varepsilon)}\right)\alpha_{3,m,n}.$$

By the definition of $\{\gamma_{m,n}\}_{(m,n)\in\mathbb{N}^2}$, we immediately see that

$$B(\varepsilon)\sum_{k=0}^{3}\left\|\widetilde{L}_{m,n}(f_k) - f_k\right\| < (\varepsilon' - \varepsilon)\,\gamma_{m,n}. \tag{3.9}$$

Since $(m,n) \in E$, we have $D^*(L_{m,n}(f), f) \geq \varepsilon'\gamma_{m,n}$, and hence, by (3.7),

$$B(\varepsilon)\sum_{k=0}^{3}\left\|\widetilde{L}_{m,n}(f_k) - f_k\right\| \geq (\varepsilon' - \varepsilon)\,\gamma_{m,n},$$

which contradicts with (3.9). So, our claim (3.8) holds true. Now, it follows from (3.8) that

$$\sum_{(m,n)\in E} a_{j,k,m,n} \leq \sum_{i=0}^{3}\Big(\sum_{(m,n)\in E_i} a_{j,k,m,n}\Big). \tag{3.10}$$

Letting $j,k \to \infty$ in (3.10) and using (3.5), we observe that

$$P - \lim_{j,k\to\infty}\sum_{(m,n)\in E} a_{j,k,m,n},$$

which means (3.6). The proof is completed. ∎

Remark 3.7. If $\alpha_{i,m,n} \equiv 1$ for each $i = 0,1,2,3$, then Theorem 3.6 reduced to Theorem 2.1. Also, if $A = I$, the identity matrix, $\alpha_{i,m,n} \equiv 1$ for each $i = 0,1,2,3$, then Theorem 3.6 reduced to Theorem 2.2.

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SINOP UNIVERSITY, FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS
57000, SINOP, TURKEY
E-mail: kamild@sinop.edu.tr (Kamil Demirci),
skarakus@sinop.edu.tr (Sevda Karakuş)

Received January 7, 2011.