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Research Article

Weighted Variable Sobolev Spaces and Capacity

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We define weighted variable Sobolev capacity and discuss properties of capacity in the space $W^{1,p(\cdot)}(\mathbb{R}^n, w)$. We investigate the role of capacity in the pointwise definition of functions in this space if the Hardy-Littlewood maximal operator is bounded on the space $W^{1,p(\cdot)}(\mathbb{R}^n, w)$. Also it is shown the relation between the Sobolev capacity and Bessel capacity.

1. Introduction

In 1991 Kováčik and Rákosník [1] introduced the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$ in higher dimensional Euclidean spaces. The spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ have many common properties. A crucial difference between $L^{p(\cdot)}(\mathbb{R}^n)$ and the classical Lebesgue spaces $L^p(\mathbb{R}^n)$ is that $L^{p(\cdot)}(\mathbb{R}^n)$ is not invariant under translation in general (Example 2.9 in [1] and Lemma 2.3 in [2]). The boundedness of the maximal operator was an open problem in $L^{p(\cdot)}(\mathbb{R}^n)$ for a long time. It was first proved by Diening [2] over bounded domains, under the assumption that $p(\cdot)$ is locally log-Hölder continuous, that is,

$$|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}, \quad x, y \in \Omega, \ |x-y| \le \frac{1}{2}. \tag{1.1}$$

He later extended the result to unbounded domains by supposing, in addition, that the exponent $p(\cdot)$ is constant outside a large ball. After this paper, many interesting and important papers appeared in nonweighted and weighted variable exponent spaces. For more details and historical background, see [1, 3–5]. Sobolev capacity for constant exponent spaces has found a great number of uses, see Maz'ja [6], Evans and Gariepy [7], and Heinonen et al. [8]. Also Kilpeläinen [9] introduced weighted Sobolev capacity and discussed the role of capacity in the pointwise definition of functions in Sobolev spaces involving weights of

Muckenhoupt's A_p-class. Variable Sobolev capacity was introduced in the spaces $W^{1,p(\cdot)}(\mathbb{R}^n)$ by Harjulehto et al. [10]. They generalized the Sobolev capacity to the variable exponent case. Our purpose is to generalize some results of [9–12] to the weighted variable exponent case.

2. Definition and Preliminary Results

We study weighted variable Lebesgue and Sobolev spaces in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$. Throughout this paper all sets and functions are Lebesgue measurable. The Lebesgue measure and the characteristic function of a subset $A \subset \mathbb{R}^n$ will be denoted by $\mu(A) = |A|$ and χ_A, respectively. The space $L^1_{\text{loc}}(\mathbb{R}^n)$ consists of all (classes of) measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^1(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$. It is a topological vector space with the family of seminorms $f \mapsto \|f\chi_K\|_{L^1}$. A Banach function space (shortly BF-space) on $\mathbb{R}^n$ is a Banach space $(B, \|\cdot\|_B)$ of measurable functions which is continuously embedded into $L^1_{\text{loc}}(\mathbb{R}^n)$, that is, for any compact subset $K \subset \mathbb{R}^n$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_{L^1} \leq C_K\|f\|_B$ for all $f \in B$. We denote it by $B \hookrightarrow L^1_{\text{loc}}(\mathbb{R}^n)$. The class $C_0^\infty(\mathbb{R}^n)$ is defined as set of infinitely differentiable functions with compact support in $\mathbb{R}^n$. For a measurable function $p : \mathbb{R}^n \to [1, \infty)$ (called a variable exponent on $\mathbb{R}^n$), we put

$$p^- = \operatorname*{essinf}_{x\in\mathbb{R}^n} p(x), \qquad p^+ = \operatorname*{esssup}_{x\in\mathbb{R}^n} p(x). \tag{2.1}$$

For every measurable functions f on $\mathbb{R}^n$ we define the function

$$\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx. \tag{2.2}$$

The function $\varrho_{p(\cdot)}$ is convex modular; that is, $\varrho_{p(\cdot)}(f) \geq 0$, $\varrho_{p(\cdot)}(f) = 0$ if and only if $f = 0$, $\varrho_{p(\cdot)}(-f) = \varrho_{p(\cdot)}(f)$ and $\varrho_{p(\cdot)}$ is convex. The variable exponent Lebesgue spaces (or generalized Lebesgue spaces) $L^{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \varrho_p\left(\frac{f}{\lambda}\right) \leq 1\right\}. \tag{2.3}$$

If $p^+ < \infty$, then $f \in L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $\varrho_{p(\cdot)}(f) < \infty$. The set $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{p(\cdot)}$. If $p(x) = p$ is a constant function, then the norm $\|\cdot\|_{p(\cdot)}$ coincides with the usual Lebesgue norm $\|\cdot\|_p$ [1]. In this paper we assume that $p^+ < \infty$.

A positive, measurable, and locally integrable function $w : \mathbb{R}^n \to (0, \infty)$ is called a weight function. The weighted modular is defined by

$$\varrho_{p(\cdot),w}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) dx. \tag{2.4}$$

The weighted variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n, w)$ consists of all measurable functions f on $\mathbb{R}^n$ for which $\|f\|_{p(\cdot),w} = \|fw^{1/p(\cdot)}\|_{p(\cdot)} < \infty$. The relations between the modular $\varrho_{p(\cdot),w}(\cdot)$ and $\|\cdot\|_{p(\cdot),w}$ as follows:

$$\min\left\{\varrho_{p(\cdot),w}(f)^{1/p^-},\ \varrho_{p(\cdot),w}(f)^{1/p^+}\right\} \le \|f\|_{p(\cdot),w} \le \max\left\{\varrho_{p(\cdot),w}(f)^{1/p^-},\ \varrho_{p(\cdot),w}(f)^{1/p^+}\right\}$$
$$\min\left\{\|f\|^{p^+}_{p(\cdot),w},\ \|f\|^{p^-}_{p(\cdot),w}\right\} \le \varrho_{p(\cdot),w}(f) \le \max\left\{\|f\|^{p^+}_{p(\cdot),w},\ \|f\|^{p^-}_{p(\cdot),w}\right\}, \tag{2.5}$$

see [13–15]. Moreover, if $0 < C \le w$, then we have $L^{p(\cdot)}(\mathbb{R}^n, w) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$, since one easily sees that

$$C\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \le \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) dx \tag{2.6}$$

and $C\|f\|_{p(\cdot)} \le \|f\|_{p(\cdot),w}$.

The Schwartz class $S = S(\mathbb{R}^n)$ consists of all infinitely differentiable and rapidly decreasing functions in $\mathbb{R}^n$. Then f and any derivative $D^\beta f$ die out faster than reciprocal of any polynomial at infinity. That is, $f \in S$ if and only if for any β and $k > 0$ there is a constant $C = C(\beta, k)$ such that

$$\left|D^\beta f(x)\right| \le \frac{C}{(1+|x|)^k}. \tag{2.7}$$

In particular, for $\beta = 0$,

$$|f(x)| \le \frac{C}{(1+|x|)^k}. \tag{2.8}$$

Also it is well known that $C_0^\infty(\mathbb{R}^n) \subset S$.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote an open ball with center x and radius r by $B(x, r)$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we denote the (centered) Hardy-Littlewood maximal operator Mf of f by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \tag{2.9}$$

where the supremum is taken over all balls $B(x, r)$.

Let $1 \le p < \infty$. A weight w satisfies Muckenhoupt's $A_p(\mathbb{R}^n) = A_p$ condition, or $w \in A_p$, if there exist positive constants C and c such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w\, dx\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)} dx\right)^{p-1} \le C, \quad 1 \le p < \infty, \tag{2.10}$$

or

$$\left(\frac{1}{|B|}\int_B w\,dx\right)\left(\operatorname*{esssup}_B \frac{1}{w}\right) \le c, \quad p = 1. \tag{2.11}$$

The infimum over the constants C and c is called the A_p and A_1, respectively. Also it is known that $A_\infty = \bigcup_{1\le p<\infty} A_p$. Let $1 < p < \infty$. Then Muckenhoupt proved that $w \in A_p$ if and only if the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbb{R}^n, w)$ [16]. Also Miller showed that the Schwartz class S is dense in $L^p(\mathbb{R}^n, w)$ for $1 < p < \infty$ and $w \in A_p$ [17, Lemma 2.1].

Hästö and Diening defined the class $A_{p(\cdot)}$ to consist of those weights w for which

$$\|w\|_{A_{p(\cdot)}} := \sup_{B\in\mathcal{B}} |B|^{-p_B} \|w\|_{L^1(B)} \left\|\frac{1}{w}\right\|_{L^{p'(\cdot)/p(\cdot)}(B)} < \infty, \tag{2.12}$$

where $\mathcal{B}$ denotes the set of all balls in $\mathbb{R}^n$, $p_B = \left((1/|B|)\int_B (1/p(x))dx\right)^{-1}$ and $1/p(\cdot) + 1/p'(\cdot) = 1$. Note that this class is ordinary Muckenhoupt class A_p if p is a constant function [13].

We say that $p(\cdot)$ satisfies the local log-Hölder continuity condition if

$$|p(x) - p(y)| \le \frac{C}{\log(e + 1/|x-y|)} \tag{2.13}$$

for all $x, y \in \mathbb{R}^n$. If

$$|p(x) - p_\infty| \le \frac{C}{\log(e + |x|)} \tag{2.14}$$

for some $p_\infty > 1$, $C > 0$ and all $x \in \mathbb{R}^n$, then we say $p(\cdot)$ satisfies the log-Hölder decay condition (at infinity). We denote by $P^{\log}(\mathbb{R}^n)$ the class of variable exponents which are log-Hölder continuous, that is, which satisfy the local log-Hölder continuity condition and the log-Hölder decay condition.

Let $p, q \in P^{\log}(\mathbb{R}^n)$, $1 < p^- \le p^+ < \infty$ and $1 < q^- \le q^+ < \infty$. If $q \le p$, then there exists a constant $C > 0$ depending on the characteristics of p and q such that $\|w\|_{A_{p(\cdot)}} \le C\|w\|_{A_{q(\cdot)}}$ [13, Lemma 3.1]. As a result of this Lemma we have

$$A_1 \subset A_{p^-} \subset A_{p(\cdot)} \subset A_{p^+} \subset A_\infty \tag{2.15}$$

for $p \in P^{\log}(\mathbb{R}^n)$ and $1 < p^- \le p^+ < \infty$.

Let $p \in P^{\log}(\mathbb{R}^n)$ and $1 < p^- \le p^+ < \infty$. Then $M : L^{p(\cdot)}(\mathbb{R}^n, w) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n, w)$ if and only if $w \in A_{p(\cdot)}$ [13, Theorem 1.1].

We use the notation

$$\mathcal{P}(\mathbb{R}^n) = \left\{p(\cdot) : 1 < p^- \le p(x) \le p^+ < \infty,\ \|Mf\|_{p(\cdot),w} \le C\|f\|_{p(\cdot),w}\right\}, \tag{2.16}$$

that is, the maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n, w)$. Hence we can find a sufficient condition for $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

Proposition 2.1. *Let w be a weight function and $1 < p^- \leq p(x) \leq p^+ < \infty$. If $w^{-1/(p(\cdot)-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $L^{p(\cdot)}(\mathbb{R}^n, w) \hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. Suppose that $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$, and let $K \subset \mathbb{R}^n$ be a compact set. For $1/p(\cdot) + 1/q(\cdot) = 1$, by using Hölder's inequality for variable exponent Lebesgue spaces [1], then there exists a $A_K > 0$ such that

$$\begin{aligned}\int_K |f(x)|dx &\leq A_K \left\| f w^{1/p(\cdot)} \right\|_{p(\cdot),K} \left\| w^{-1/p(\cdot)} \right\|_{q(\cdot),K} \\ &\leq A_K \left\| f w^{1/p(\cdot)} \right\|_{p(\cdot)} \left\| w^{-1/p(\cdot)} \right\|_{q(\cdot),K}.\end{aligned} \tag{2.17}$$

It is known that $\|w^{-1/p(\cdot)}\|_{q(\cdot),K} < \infty$ if and only if $\varrho_{q(\cdot),K}(w^{-1/p(\cdot)}) < \infty$ for $q^+ < \infty$. Since $w^{-1/(p(\cdot)-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then we have

$$\varrho_{q(\cdot),K}\left(w^{-1/p(\cdot)}\right) = \int_K w(x)^{-q(x)/p(x)} dx = \int_K w(x)^{-1/(p(x)-1)} dx = B_K < \infty. \tag{2.18}$$

If we use (2.17) and (2.18), then the proof is completed. □

Definition 2.2 (Mollifiers). Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative, radial, decreasing function belonging to $C_0^\infty(\mathbb{R}^n)$ and having the properties:

(i) $\varphi(x) = 0$ if $|x| \geq 1$,

(ii) $\int_{\mathbb{R}^n} \varphi(x)dx = 1$.

Let $\varepsilon > 0$. If the function $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ is nonnegative, belongs to $C_0^\infty(\mathbb{R}^n)$, and satisfies

(i) $\varphi_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$ and

(ii) $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)dx = 1$,

then φ_ε is called a mollifier and we define the convolution by

$$\varphi_\varepsilon * f(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) f(y) dy. \tag{2.19}$$

The following proposition was proved in [18, Proposition 2.7].

Proposition 2.3. *Let φ_ε be a mollifier and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\sup_{\varepsilon>0} \left| \varphi_\varepsilon * f(x) \right| \leq Mf(x). \tag{2.20}$$

Proposition 2.4. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$, then $\varphi_\varepsilon * f \to f$ in $L^{p(\cdot)}(\mathbb{R}^n, w)$ as $\varepsilon \to 0^+$.*

Proof. Let $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$ and $\varepsilon > 0$ be given. If f is continuous, then the assertion is trivial. By Proposition 2.3, we have

$$\|\varphi_\varepsilon * f\|_{p(\cdot),w} \le \|Mf\|_{p(\cdot),w} \le C\|f\|_{p(\cdot),w} \tag{2.21}$$

and we have $\varphi_\varepsilon * f \in L^{p(\cdot)}(\mathbb{R}^n, w)$ for all $\varepsilon > 0$. It can be proved that the class $C_0(\mathbb{R}^n)$ of continuous functions with compact support is dense in the space $L^{p(\cdot)}(\mathbb{R}^n, w)$. Then there is a function $g \in C_0(\mathbb{R}^n)$ such that

$$\|f - g\|_{p(\cdot),w} < \varepsilon. \tag{2.22}$$

Also it is well known that if $g \in C_0(\mathbb{R}^n)$, then $\varphi_\varepsilon * g \in C_0^\infty(\mathbb{R}^n)$ for all $\varepsilon > 0$. It is easily seen that $\varphi_\varepsilon * g \to g$ uniformly on compact sets as $\varepsilon \to 0^+$. Hence we have

$$\begin{aligned} &\left|\varphi_\varepsilon * g(x) - g(x)\right|^{p(x)} \longrightarrow 0, \\ \varrho_{p(\cdot),w}(\varphi_\varepsilon * g - g) &= \int_K \left|\varphi_\varepsilon * g(x) - g(x)\right|^{p(x)} w(x)\,dx \\ &\le \varepsilon^{p^-} \int_K w(x)\,dx, \end{aligned} \tag{2.23}$$

where $\operatorname{supp}(\varphi_\varepsilon * g) \cup \operatorname{supp} g \subset K$, $K \subset \mathbb{R}^n$ compact. Hence $\varrho_{p(\cdot),w}(\varphi_\varepsilon * g - g) \to 0$ as $\varepsilon \to 0^+$ and we write

$$\|\varphi_\varepsilon * g - g\|_{p(\cdot),w} < \varepsilon. \tag{2.24}$$

Finally by using (2.22) and (2.24),

$$\begin{aligned} \|f - \varphi_\varepsilon * f\|_{p(\cdot),w} &\le \|f - g\|_{p(\cdot),w} + \|g - \varphi_\varepsilon * g\|_{p(\cdot),w} + \|\varphi_\varepsilon * g - \varphi_\varepsilon * f\|_{p(\cdot),w} \\ &< (C + 2)\varepsilon. \end{aligned} \tag{2.25}$$

The proof is complete. □

As a direct consequence of Proposition 2.4 there follows.

Corollary 2.5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The class $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n, w)$.*

This result was proved without the assumption that the maximal operator is bounded in $L^{p(\cdot)}(\mathbb{R}^n, w)$ by Kokilashvili and Samko [19].

Remark 2.6. Let $1 < p^- \le p(x) \le p^+ < \infty$ and $w^{-1/(p(\cdot)-1)} \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then every function in $L^{p(\cdot)}(\mathbb{R}^n, w)$ has distributional derivatives by Proposition 2.1.

3. Weighted Variable Sobolev Spaces

Let $1 < p^- \le p(x) \le p^+ < \infty$, $w^{-1/(p(\cdot)-1)} \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. We define the weighted variable Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n, w)$ by

$$W^{k,p(\cdot)}(\mathbb{R}^n, w) = \left\{ f \in L^{p(\cdot)}(\mathbb{R}^n, w) : D^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n, w),\ 0 \le |\alpha| \le k \right\} \tag{3.1}$$

equipped with the norm

$$\|f\|_{k,p(\cdot),w} = \sum_{0 \le |\alpha| \le k} \|D^\alpha f\|_{p(\cdot),w} \tag{3.2}$$

where $\alpha \in \mathbb{N}_0^n$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $D^\alpha = \partial^{|\alpha|}/(\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n})$. It can be shown that $W^{k,p(\cdot)}(\mathbb{R}^n, w)$ is a reflexive Banach space. Throughout this paper, we will always assume that $1 < p^- \le p(x) \le p^+ < \infty$ and $w^{-1/(p(\cdot)-1)} \in L^1_{\text{loc}}(\mathbb{R}^n)$.

The space $W^{1,p(\cdot)}(\mathbb{R}^n, w)$ is defined by

$$W^{1,p(\cdot)}(\mathbb{R}^n, w) = \left\{ f \in L^{p(\cdot)}(\mathbb{R}^n, w) : |\nabla f| \in L^{p(\cdot)}(\mathbb{R}^n, w) \right\}. \tag{3.3}$$

The function $\varrho_{1,p(\cdot),w} : W^{1,p(\cdot)}(\mathbb{R}^n, w) \to [0, \infty)$ is defined as $\varrho_{1,p(\cdot),w}(f) = \varrho_{p(\cdot),w}(f) + \varrho_{p(\cdot),w}(\nabla f)$. The norm $\|f\|_{1,p(\cdot),w} = \|f\|_{p(\cdot),w} + \|\nabla f\|_{p(\cdot),w}$.

The Bessel kernel g_α order α, $\alpha > 0$, is defined by

$$g_\alpha(x) = \frac{\pi^{n/2}}{\Gamma(\alpha/2)} \int_0^\infty e^{-s-(\pi^2|x|^2)/s} s^{(\alpha-n)/2} \frac{ds}{s}, \quad x \in \mathbb{R}^n. \tag{3.4}$$

Let $\alpha \ge 0$. The weighted variable Bessel potential space $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n, w)$ is, for $\alpha > 0$, defined by

$$\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n, w) := \left\{ h = g_\alpha * f ; f \in L^{p(\cdot)}(\mathbb{R}^n, w) \right\}, \tag{3.5}$$

and is equipped with the norm

$$\|h\|_{\alpha;p(\cdot),w} = \|f\|_{p(\cdot),w}. \tag{3.6}$$

If $\alpha = 0$ we put $g_0 * f = f$ and $\mathcal{L}^{0,p(\cdot)}(\mathbb{R}^n, w) = L^{p(\cdot)}(\mathbb{R}^n, w)$.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$, then $g_\alpha * f \in L^{p(\cdot)}(\mathbb{R}^n, w)$. Indeed, since $g_\alpha \in L^1(\mathbb{R}^n)$ and g_α is radial, we have $(g_\alpha * f)(x) \le Mf(x)$, $x \in \mathbb{R}^n$ [20, page 62]. The assertion thus follows from boundedness of maximal function in $L^{p(\cdot)}(\mathbb{R}^n, w)$.

The unweighted variable Bessel potential space $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$ was firstly studied by Almeida and Samko in [21].

Lemma 3.1. *Let* $p(\cdot) \in P^{\log}(\mathbb{R}^n)$, $1 < p^- \leq p^+ < \infty$, *and* $w \in A_{p(\cdot)}$. *Then*

(i) $C_0^\infty(\mathbb{R}^n)$ *is dense in* $W^{k,p(\cdot)}(\mathbb{R}^n, w)$, $k \in \mathbb{N}$,

(ii) *The Schwartz class* S *is dense in* $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n, w)$, $\alpha \geq 0$.

Proof. (i) By Proposition 2.4 the proof is complete.

(ii) Let $\alpha = 0$. The class $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n, w)$ by Corollary 2.5. It remains only to show that $S \subset L^{p(\cdot)}(\mathbb{R}^n, w)$. Let $f \in S$. Then there exist $C = C(r) > 0$ and $r > 0$ such that

$$|f(x)| \leq \frac{C}{(1+|x|)^r}. \tag{3.7}$$

Also since $rp(x) \geq r$ and $(1+|x|)^r \geq 1$, then

$$\begin{aligned} \varrho_{p(\cdot),w}(f) &= \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x)\,dx \\ &\leq \max\left\{C^{p^-}, C^{p^+}\right\} \int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{rp(x)}}\,dx \\ &\leq \max\left\{C^{p^-}, C^{p^+}\right\} \int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{r}}\,dx. \end{aligned} \tag{3.8}$$

It is known that $A_{p(\cdot)} \subset A_{p^+}$ for $1 < p^+ < \infty$. Also the fact that the Muckenhoupt weights with constant p^+ are integrable with some power weight. Then

$$\int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^r}\,dx < \infty, \tag{3.9}$$

see [22, Lemma 1]. If we use (3.9) in (3.8), then the Schwartz class S is dense in $L^{p(\cdot)}(\mathbb{R}^n, w)$.

Let $\alpha > 0$ and $h \in \mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n, w)$. Then there is a function $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$ such that $h = g_\alpha * f$. By density of $C_0^\infty(\mathbb{R}^n)$ in $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$ we can find a sequence $(f_j)_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n) \subset S$ converging to f in $L^{p(\cdot)}(\mathbb{R}^n, w)$. Since the mapping $f \mapsto g_\alpha * f$ maps S onto S [20], the functions $h_j = g_\alpha * f_j$, $j \in \mathbb{N}$, belong to S. Moreover,

$$\|h - h_j\|_{\alpha;p(\cdot),w} = \|f - f_j\|_{p(\cdot),w} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty \tag{3.10}$$

and the assertion follows. □

The following Theorem can be proved similarly in [12, Theorem 3.1].

Theorem 3.2. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $k \in \mathbb{N}$. *Then* $\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n, w) = W^{k,p(\cdot)}(\mathbb{R}^n, w)$ *and the corresponding norms are equivalent.*

Remark 3.3. The equivalence of the spaces $\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n, w)$ and $W^{k,p(\cdot)}(\mathbb{R}^n, w)$ fails when $p = 1$ or $p = \infty$.

For $E \subset \mathbb{R}^n$, we denote

$$S_{p(\cdot),w}(E) = \left\{ f \in W^{1,p(\cdot)}(\mathbb{R}^n, w) : f \geq 1 \text{ in open set containing } E \right\}. \tag{3.11}$$

The Sobolev $(p(\cdot), w)$-capacity of E is defined by

$$C_{p(\cdot),w}(E) = \inf_{f \in S_{p(\cdot),w}(E)} \varrho_{1,p(\cdot),w}(f) = \inf_{f \in S_{p(\cdot),w}(E)} \int_{\mathbb{R}^n} \left(|f(x)|^{p(x)} + |\nabla f(x)|^{p(x)} \right) w(x) dx. \tag{3.12}$$

In case $S_{p(\cdot),w}(E) = \emptyset$, we set $C_{p(\cdot),w}(E) = \infty$. The $C_{p(\cdot),w}$-capacity has the following properties.

(i) $C_{p(\cdot),w}(\emptyset) = 0$.

(ii) If $E_1 \subset E_2$, then $C_{p(\cdot),w}(E_1) \leq C_{p(\cdot),w}(E_2)$.

(iii) If E is a subset of $\mathbb{R}^n$, then

$$C_{p(\cdot),w}(E) = \inf\left\{ C_{p(\cdot),w}(U) : E \subset U,\ U \text{ open} \right\}. \tag{3.13}$$

(iv) If E_1 and E_2 are subsets of $\mathbb{R}^n$, then

$$C_{p(\cdot),w}(E_1 \cup E_2) + C_{p(\cdot),w}(E_1 \cap E_2) \leq C_{p(\cdot),w}(E_1) + C_{p(\cdot),w}(E_2). \tag{3.14}$$

(v) If $K_1 \supset K_2 \supset \cdots$ are compact, then

$$\lim_{i \to \infty} C_{p(\cdot),w}(K_i) = C_{p(\cdot),w}\left(\bigcap_{i=1}^{\infty} K_i \right). \tag{3.15}$$

Note that the assertion (v) above is not true in general for noncompact sets [9].

(vi) If $E_1 \subset E_2 \subset \cdots$ are subsets of $\mathbb{R}^n$, then

$$\lim_{i \to \infty} C_{p(\cdot),w}(E_i) = C_{p(\cdot),w}\left(\bigcup_{i=1}^{\infty} E_i \right). \tag{3.16}$$

(vii) If $E_i \subset \mathbb{R}^n$ for $i = 1, 2, \ldots$, then

$$C_{p(\cdot),w}\left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} C_{p(\cdot),w}(E_i). \tag{3.17}$$

For the proof of these properties see [8, 10]. Hence the Sobolev $C_{p(\cdot),w}$ capacity is an outer measure. A set function which satisfies the capacity properties (i), (ii), (v), and (vi) is called Choquet capacity; see [23]. Therefore we have the following result.

Corollary 3.4. *The set function $E \mapsto C_{p(\cdot),w}(E)$, $E \subset \mathbb{R}^n$, is a Choquet capacity. In particular, all Suslin sets $E \subset \mathbb{R}^n$ are capacitable, that is,*

$$C_{p(\cdot),w}(E) = \inf_{\substack{E\subset U \\ U \text{ open}}} C_{p(\cdot),w}(U) = \sup_{\substack{K\subset E \\ K \text{ compact}}} C_{p(\cdot),w}(K). \tag{3.18}$$

Lemma 3.5. *Let $w(x) \geq 1$ for $x \in \mathbb{R}^n$. Then every measurable set $E \subset \mathbb{R}^n$ satisfies $|E| \leq C_{p(\cdot),w}(E)$.*

Proof. If $f \in S_{p(\cdot),w}(E)$, then there is an open set $E \subset U$ such that $f \geq 1$ in U and hence

$$|E| \leq |U| \leq \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x)dx \leq \int_{\mathbb{R}^n} \left(|f(x)|^{p(x)} + |\nabla f(x)|^{p(x)}\right) w(x)dx. \tag{3.19}$$

We obtain the claim by taking the infimum on $S_{p(\cdot),w}(E)$. □

Definition 3.6 (Bessel Capacity). Let $E \subset \mathbb{R}^n$, $\alpha > 0$. Define that the $(\alpha, p(\cdot), w)$-Bessel capacity in $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n, w)$ is the number

$$B_{\alpha,p(\cdot),w}(E) = \inf \varrho_{p(\cdot),w}(f), \tag{3.20}$$

where the infimum is taken over all $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$ such that $g_\alpha * f \geq 1$ on E. Since g_α is nonnegative we can assume that $f \geq 0$.

Theorem 3.7. *$B_{\alpha,p(\cdot),w}$ is an outer capacity defined on all subsets of $\mathbb{R}^n$.*

Proof. It is known that

(i) $B_{\alpha,p(\cdot),w}(\emptyset) = 0$;

(ii) if $E_1 \subset E_2$, then $B_{\alpha,p(\cdot),w}(E_1) \leq B_{\alpha,p(\cdot),w}(E_2)$;

(iii) if $E_i \subset \mathbb{R}^n$ for $i = 1, 2, \ldots$, then

$$B_{\alpha,p(\cdot),w}\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} B_{\alpha,p(\cdot),w}(E_i) \tag{3.21}$$

by [12, Lemma 4.1]. We will show that

$$B_{\alpha,p(\cdot),w}(E) = \inf_{\substack{E\subset G \\ G \text{ open}}} B_{\alpha,p(\cdot),w}(G). \tag{3.22}$$

for any $E \subset \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be arbitrary. Obviously $B_{\alpha,p(\cdot),w}(E) \leq \inf_{\substack{E\subset G \\ G \text{ open}}} B_{\alpha,p(\cdot),w}(G)$. We assume that $B_{\alpha,p(\cdot),w}(E) < \infty$. If $0 < \varepsilon < 1$ there must exist a test function (measurable and nonnegative) for $B_{\alpha,p(\cdot),w}(E)$, call it f, such that $g_\alpha * f \geq 1$ on E, and

$$\varrho_{p(\cdot),w}(f) < B_{\alpha,p(\cdot),w}(E) + \varepsilon. \tag{3.23}$$

Let $G = \{x \in \mathbb{R}^n : g_\alpha * f > 1 - \varepsilon\}$. Since $g_\alpha * f$ is lower semicontinuous in x, G is an open set and since $g_\alpha * f > 1 - \varepsilon$ on E, $G \supset E$. Therefore $(1 - \varepsilon)^{-1} f$ is a test function for $B_{\alpha,p(\cdot),w}(G)$ and we have

$$B_{\alpha,p(\cdot),w}(G) \leq \varrho_{p(\cdot),w}\left(\frac{f}{1-\varepsilon}\right) \leq (1-\varepsilon)^{-p^+} \varrho_{p(\cdot),w}(f) < (1-\varepsilon)^{-p^+} \left(B_{\alpha,p(\cdot),w}(E) + \varepsilon\right). \tag{3.24}$$

This proves the theorem as $\varepsilon \to 0^+$. □

Now we give relationship between the capacities $B_{\alpha,p(\cdot),w}$ and $C_{p(\cdot),w}$ [12].

Lemma 3.8. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$. Then*

$$\begin{aligned} B_{1,p(\cdot),w}(E) &\leq c \max\left\{C_{p(\cdot),w}(E)^{p^-/p^+}, C_{p(\cdot),w}(E)^{p^+/p^-}\right\}, \\ C_{p(\cdot),w}(E) &\leq C \max\left\{B_{1,p(\cdot),w}(E)^{p^-/p^+}, B_{1,p(\cdot),w}(E)^{p^+/p^-}\right\}. \end{aligned} \tag{3.25}$$

Here c and C are positive constants independent of E.

Proposition 3.9. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.*

(i) *If $f \in W^{1,p(\cdot)}(\mathbb{R}^n, w)$, then $Mf \in W^{1,p(\cdot)}(\mathbb{R}^n, w)$ and $|\nabla Mf(x)| \leq M|\nabla f(x)|$ for almost everywhere in $\mathbb{R}^n$.*

(ii) *Let $1 \leq s < \infty$. Then $sp(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and there exists a constant $C > 0$ such that the inequality*

$$\|Mf\|_{1,sp(\cdot),w} \leq C\|f\|_{1,sp(\cdot),w} \tag{3.26}$$

holds for all $f \in W^{1,sp(\cdot)}(\mathbb{R}^n, w)$.

Proof. (i) By Proposition 2.1 we have $L^{p(\cdot)}(\mathbb{R}^n, w) \hookrightarrow L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n, w) \hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $W^{1,p(\cdot)}(\mathbb{R}^n, w) \hookrightarrow W^{1,p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n, w) \hookrightarrow W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Since $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, then we have $|\nabla Mf(x)| \leq M|\nabla f(x)|$ for almost everywhere in $\mathbb{R}^n$ by [24]. Since $f, |\nabla f| \in L^{p(\cdot)}(\mathbb{R}^n, w)$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then Mf, $|\nabla Mf| \in L^{p(\cdot)}(\mathbb{R}^n, w)$. Hence $Mf \in W^{1,p(\cdot)}(\mathbb{R}^n, w)$.

(ii) Let $f \in L^{sp(\cdot)}(\mathbb{R}^n, w)$. By using definition of $\|\cdot\|_{p(\cdot),w}$, we have

$$\|f\|_{sp(\cdot),w} = \left\||f|^s\right\|^{1/s}_{p(\cdot),w} \tag{3.27}$$

and $|f|^s \in L^{p(\cdot)}(\mathbb{R}^n, w)$. Therefore we have

$$\|Mf\|_{sp(\cdot),w} = \left\|(Mf)^s\right\|^{1/s}_{p(\cdot),w} \leq \left\|M(|f|^s)\right\|^{1/s}_{p(\cdot),w} \leq c\left\||f|^s\right\|^{1/s}_{p(\cdot),w} = c\|f\|_{sp(\cdot),w} \tag{3.28}$$

and $sp(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Since $f \in W^{1,sp(\cdot)}(\mathbb{R}^n, w)$, then $f, |\nabla f| \in L^{sp(\cdot)}(\mathbb{R}^n, w)$. Hence we write

$$\begin{aligned}
\|Mf\|_{1,sp(\cdot),w} &= \|(Mf)^s\|_{p(\cdot),w}^{1/s} + \||\nabla Mf|^s\|_{p(\cdot),w}^{1/s} \\
&\le \|(Mf)^s\|_{p(\cdot),w}^{1/s} + \|(M|\nabla f|)^s\|_{p(\cdot),w}^{1/s} \\
&\le \|M(|f|^s)\|_{p(\cdot),w}^{1/s} + \|M(|\nabla f|^s)\|_{p(\cdot),w}^{1/s} \\
&\le C_1\||f|^s\|_{p(\cdot),w}^{1/s} + C_2\||\nabla f|^s\|_{p(\cdot),w}^{1/s}.
\end{aligned} \tag{3.29}$$

by (3.28). If we set $C = \max\{C_1, C_2\}$, then

$$\|Mf\|_{1,sp(\cdot),w} \le C\|f\|_{1,sp(\cdot),w}. \tag{3.30}$$

This completes the proof. □

Proposition 3.10. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then for every $\lambda > 0$ and every $f \in W^{1,p(\cdot)}(\mathbb{R}^n, w)$ we have*

$$C_{p(\cdot),w}(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \le c \max\left\{\left\|\frac{f}{\lambda}\right\|_{1,p(\cdot),w}^{p^+}, \left\|\frac{f}{\lambda}\right\|_{1,p(\cdot),w}^{p^-}\right\}. \tag{3.31}$$

Proof. Since Mf is lower semicontinuous, the set $\{x \in \mathbb{R}^n : Mf(x) > \lambda\}$ is open for every $\lambda > 0$. By Proposition 3.9 we can take $(Mf)/\lambda = M(f/\lambda)$ as a test function for the capacity. Then we have

$$\begin{aligned}
C_{p(\cdot),w}(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) &\le \varrho_{1,p(\cdot),w}\left(M\frac{f}{\lambda}\right) \\
&\le \max\left\{\left\|M\frac{f}{\lambda}\right\|_{1,p(\cdot),w}^{p^+}, \left\|M\frac{f}{\lambda}\right\|_{1,p(\cdot),w}^{p^-}\right\} \\
&\le c \max\left\{\left\|\frac{f}{\lambda}\right\|_{1,p(\cdot),w}^{p^+}, \left\|\frac{f}{\lambda}\right\|_{1,p(\cdot),w}^{p^-}\right\}.
\end{aligned} \tag{3.32}$$

□

We say that a property holds $(p(\cdot), w)$-quasi everywhere if it holds except in a set of capacity zero. A function f is $(p(\cdot), w)$-quasicontinuous in $\mathbb{R}^n$ if for each $\varepsilon > 0$ there exists an open set E with $C_{p(\cdot),w}(E) < \varepsilon$ such that f restricted to $\mathbb{R}^n \setminus E$ is continuous. The following proof of theorem is quite similar to Theorem 4.7 in [11].

Theorem 3.11. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in W^{1,p(\cdot)}(\mathbb{R}^n, w)$, then the limit*

$$f^*(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy \tag{3.33}$$

exists $(p(\cdot), w)$-quasi everywhere in $\mathbb{R}^n$. The function f^* is the $(p(\cdot), w)$-quasicontinuous representative of f.

Proof. Since the class $C_0^{\infty}(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n,w)$ by Lemma 3.1, then we can choose a sequence (f_i) such that

$$\|f-f_i\|_{1,p(\cdot),w} \le 2^{-2i}. \tag{3.34}$$

For $i=1,2,\dots$ we denote

$$A_i=\left\{x\in\mathbb{R}^n : M(f-f_i)(x)>2^{-i}\right\}, \qquad B_i=\bigcup_{j=i}^{\infty}A_j, \qquad E=\bigcap_{j=1}^{\infty}B_j. \tag{3.35}$$

By using Proposition 3.10 and the subadditivity of $C_{p(\cdot),w}$ we have

$$\begin{aligned} C_{p(\cdot),w}(A_i) &\le c\max\left\{\left\|\frac{M(f-f_i)}{2^{-i}}\right\|_{1,p(\cdot),w}^{p^+},\left\|\frac{M(f-f_i)}{2^{-i}}\right\|_{1,p(\cdot),w}^{p^-}\right\} \\ &= c\max\left\{\left(\frac{1}{2^{-i}}\right)^{p^+}\|M(f-f_i)\|_{1,p(\cdot),w}^{p^+},\left(\frac{1}{2^{-i}}\right)^{p^-}\|M(f-f_i)\|_{1,p(\cdot),w}^{p^-}\right\} \\ &\le c\max\left\{\left(\frac{1}{2^{-i}}\right)^{p^+}\left(2^{-2i}\right)^{p^+},\left(\frac{1}{2^{-i}}\right)^{p^-}\left(2^{-2i}\right)^{p^-}\right\}\le c2^{-i}, \end{aligned} \tag{3.36}$$

$C_{p(\cdot),w}(B_i)\le c2^{1-i}$ and $C_{p(\cdot),w}(E)=0$. If we follow the proof of Theorem 4.7 in [11], then this proves the theorem. □

Corollary 3.12. *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. If $f\in W^{1,p(\cdot)}(\mathbb{R}^n,w)$ and f is quasicontinuous, then we have*

$$f(x)=\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)}f(y)\,dy \tag{3.37}$$

$(p(\cdot),w)$-quasi everywhere in $\mathbb{R}^n$.

Proof. By using the Theorem in [25] the proof is completed. □

Now we show that every quasicontinuous function satisfies a weak type capacity inequality; the proofs follow the ideas by [10].

Lemma 3.13. *Let $p^+<\infty$ and $E\subset\mathbb{R}^n$. If $u\in W^{1,p(\cdot)}(\mathbb{R}^n,w)$ is a nonnegative $(p(\cdot),w)$-quasicontinuous function such that $u\ge 1$ on E. Then for every $\varepsilon>0$ there exists a function $h\in S_{p(\cdot),w}(E)$ such that $\varrho_{1,p(\cdot),w}(u-h)<\varepsilon$.*

Proof. Let $0<\delta<1$, and let $V\subset\mathbb{R}^n$ be an open set such that u is continuous in $\mathbb{R}^n\setminus V$ and $C_{p(\cdot),w}(V)<\delta$. By definition of $C_{p(\cdot),w}$ there exists a $v\in S_{p(\cdot),w}(E)$ such that $\varrho_{1,p(\cdot),w}(v)<\delta$. If we set $h=(1+\delta)u+|v|$, then it is easy to show that $h\in W^{1,p(\cdot)}(\mathbb{R}^n,w)$ by [10, Theorem 2.2]. Since the function u is continuous and the set V is open, then the set

$$G=\{x\in\mathbb{R}^n\setminus V : u(x)>1\}\cup V \tag{3.38}$$

is open, contains E, and $h \geq 1$ on G, thus $h \in S_{p(\cdot),w}(E)$. It is known that for $1 \leq p(\cdot) \leq p^+ < \infty$ and $a, b \geq 0$, $(a+b)^{p(\cdot)} \leq 2^{p^+-1}(a^{p(\cdot)} + b^{p(\cdot)})$ and $|\nabla|v|| = |\nabla v|$. Hence we obtain $||v| + \delta u|^{p(\cdot)} \leq 2^{p^+-1}(|v|^{p(\cdot)} + |\delta u|^{p(\cdot)})$ and

$$\begin{aligned} \varrho_{1,p(\cdot),w}(u-h) &= \int_{\mathbb{R}^n} \left(||v| + \delta u|^{p(x)} + |\nabla(|v| + \delta u)|^{p(x)} \right) w(x)dx \\ &\leq 2^{p^+-1} \int_{\mathbb{R}^n} \left(|v|^{p(\cdot)} + |\delta u|^{p(\cdot)} + |\nabla|v(x)||^{p(x)} + |\delta \nabla u(x)|^{p(x)} \right) w(x)dx \\ &\leq 2^{p^+-1} \left(\varrho_{1,p(\cdot),w}(v) + \delta^{p^-} \varrho_{1,p(\cdot),w}(u) \right) < 2^{p^+-1} \left(\delta + \delta^{p^-} \varrho_{1,p(\cdot),w}(u) \right). \end{aligned} \tag{3.39}$$

This completes the proof as $\delta \longrightarrow 0$. □

Theorem 3.14. *Let $p^+ < \infty$. If $u \in W^{1,p(\cdot)}(\mathbb{R}^n, w)$ is a $(p(\cdot), w)$-quasicontinuous function and $\lambda > 0$, then*

$$C_{p(\cdot),w}(\{x \in \mathbb{R}^n : |u(x)| > \lambda\}) \leq \int_{\mathbb{R}^n} \left(\left| \frac{u(x)}{\lambda} \right|^{p(x)} + \left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} \right) w(x)dx. \tag{3.40}$$

Proof. By [10, Theorem 2.2], $|u| \in W^{1,p(\cdot)}(\mathbb{R}^n, w)$ and $|\nabla|u|| = |\nabla u|$. By Lemma 3.13, there is a sequence $h_j \in S_{p(\cdot),w}(\{x \in \mathbb{R}^n : |u(x)|/\lambda > 1\})$ such that

$$\varrho_{1,p(\cdot),w}\left(\frac{|u|}{\lambda} - h_j \right) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{3.41}$$

Hence we have by [10, Lemma 2.6] that

$$\varrho_{1,p(\cdot),w}(h_j) \longrightarrow \varrho_{1,p(\cdot),w}\left(\frac{|u|}{\lambda} \right) \quad \text{as } j \longrightarrow \infty. \tag{3.42}$$

By definition of $C_{p(\cdot),w}$, we write

$$C_{p(\cdot),w}(\{x \in \mathbb{R}^n : |u(x)| > \lambda\}) \leq \varrho_{1,p(\cdot),w}(h_j). \tag{3.43}$$

Therefore

$$C_{p(\cdot),w}(\{x \in \mathbb{R}^n : |u(x)| > \lambda\}) \leq \varrho_{1,p(\cdot),w}\left(\frac{|u|}{\lambda} \right) \quad \text{as } j \longrightarrow \infty. \tag{3.44}$$

□

Proposition 3.15. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $u \in W^{1,p(\cdot)}(\mathbb{R}^n, w)$, then there is a $C > 0$ such that*

$$B_{1,p(\cdot),w}(\{x \in \mathbb{R}^n : Mu(x) \geq \lambda\}) \leq C \max\left\{ \left\| \frac{u}{\lambda} \right\|_{1,p(\cdot),w}^{p^+}, \left\| \frac{u}{\lambda} \right\|_{1,p(\cdot),w}^{p^-} \right\}. \tag{3.45}$$

Proof. For $r > 0$, we take $h = |B(0,r)|^{-1}\chi_{B(0,r)}$. Choose $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$ such that $u = g_1 * f$ and $\|f\|_{p(\cdot),w} \approx \|u\|_{1,p(\cdot),w}$ by Theorem 3.2. Then

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)}|u(y)|dy &= \frac{1}{|B(0,r)|}\int_{B(0,r)}\chi_{B(0,r)}(x-y)|u(y)|dy\\ &= (h*|u|)(x) \le (h*(g_1*|f|))(x)\\ &= (g_1*(h*|f|))(x) \le (g_1 * Mf)(x)\end{aligned} \tag{3.46}$$

and $Mu(x) \le (g_1 * Mf)(x)$. Also it is known that if $E_1 \subset E_2$, then $B_{1,p(\cdot),w}(E_1) \le B_{1,p(\cdot),w}(E_2)$ by [12, Lemma 4.1]. Therefore we have

$$\begin{aligned}B_{1,p(\cdot),w}(\{x\in\mathbb{R}^n : Mu(x) \ge \lambda\}) &\le B_{1,p(\cdot),w}(\{x\in\mathbb{R}^n : (g_1 * Mf)(x) \ge \lambda\})\\ &\le \varrho_{p(\cdot),w}\left(M\frac{f}{\lambda}\right)\\ &\le \max\left\{\left\|M\frac{f}{\lambda}\right\|_{p(\cdot),w}^{p^+}, \left\|M\frac{f}{\lambda}\right\|_{p(\cdot),w}^{p^-}\right\}\\ &\le c\max\left\{\left\|\frac{f}{\lambda}\right\|_{p(\cdot),w}^{p^+}, \left\|\frac{f}{\lambda}\right\|_{p(\cdot),w}^{p^-}\right\}\\ &\le C\max\left\{\left\|\frac{u}{\lambda}\right\|_{1,p(\cdot),w}^{p^+}, \left\|\frac{u}{\lambda}\right\|_{1,p(\cdot),w}^{p^-}\right\}.\end{aligned} \tag{3.47}$$

□

The following Theorem is obtained directly from Lemma 3.8 and Theorem 3.11.

Theorem 3.16. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $u \in W^{1,p(\cdot)}(\mathbb{R}^n, w)$ and u is quasicontinuous, then the limit*

$$u(x) = \lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)dy \tag{3.48}$$

exists $(1, p(\cdot), w)$-quasi everywhere in $\mathbb{R}^n$.

The following proposition can be proved similarly as in [12, Proposition 5.1].

Proposition 3.17. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Every $u \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n, w)$ is quasicontinuous. That is, for every $\varepsilon > 0$, there exists a set $F \subset \mathbb{R}^n$, $B_{1,p(\cdot),w}(F) \le \varepsilon$, so that u restricted to $\mathbb{R}^n \setminus F$ is continuous.*

Proposition 3.18. *Let $1 < p^- \le p^+ < \infty$. Then for all $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$ and $0 < \lambda < \infty$ we have*

$$B_{\alpha,p(\cdot),w}(\{x\in\mathbb{R}^n : (g_\alpha * f)(x) \ge \lambda\}) \le \max\left\{\left\|\frac{f}{\lambda}\right\|_{p(\cdot),w}^{p^+}, \left\|\frac{f}{\lambda}\right\|_{p(\cdot),w}^{p^-}\right\}. \tag{3.49}$$

Proof. We first note that by definition of $B_{1,p(\cdot),w}$-capacity, $\lambda^{-1}f$ is a test function for the Bessel capacity. Hence

$$\begin{aligned} B_{\alpha,p(\cdot),w}\left(\left\{x\in\mathbb{R}^n:\left(g_\alpha * f\right)(x)\geq\lambda\right\}\right) &\leq \varrho_{p(\cdot),w}\left(\frac{f}{\lambda}\right) \\ &\leq \max\left\{\left\|\frac{f}{\lambda}\right\|_{p(\cdot),w}^{p^+},\left\|\frac{f}{\lambda}\right\|_{p(\cdot),w}^{p^-}\right\}. \end{aligned} \tag{3.50}$$

□

Proposition 3.19. *Let $1<p^-\leq p^+<\infty$. If $f\in L^{p(\cdot)}(\mathbb{R}^n,w)$ and*

$$E=\left\{x\in\mathbb{R}^n:\left(g_\alpha * f\right)(x)=\infty\right\}, \tag{3.51}$$

then $B_{\alpha,p(\cdot),w}(E)=0$.

Proof. By Proposition 3.18, we write $B_{\alpha,p(\cdot),w}(E)=0$ as $\lambda\to\infty$. □

Proposition 3.20. *Let $1<p^-\leq p^+<\infty$. If $f\in L^{p(\cdot)}(\mathbb{R}^n,w)$, then*

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)}\left(g_\alpha * f\right)(y)dy=\left(g_\alpha * f\right)(x) \tag{3.52}$$

for $B_{\alpha,p(\cdot),w}$-q.e. $x\in\mathbb{R}^n$.

Proof. Let χ be the characteristic function for the unit ball $B(0,1)$, and define for $r>0$, $\chi_r(x)=(1/|B(0,1)|)\chi(x/r)$, $x\in\mathbb{R}^n$. Then

$$\begin{aligned} \frac{1}{|B(x,r)|}\int_{B(x,r)}\left(g_\alpha * f\right)(y)dy &= \chi_r * \left(g_\alpha * f\right)(x)=\left(\chi_r * g_\alpha\right) * f(x) \\ &= \int_{B(x,r)}\chi_r * g_\alpha(y)f(x-y)dy. \end{aligned} \tag{3.53}$$

As $r\to 0$, $\chi_r * g_\alpha(y)\to g_\alpha(y)$ for every $y\in\mathbb{R}^n$. This implies that, for fixed $x\in\mathbb{R}^n$, $\chi_r * g_\alpha(y)f(x-y)\to g_\alpha(y)f(x-y)$ for a.e. $y\in\mathbb{R}^n$. It was shown that $\chi_r * g_\alpha(y)\leq Cg_\alpha(y)$ for $0<r\leq 1$ and $y\in\mathbb{R}^n$ [26, page 161]. By Proposition 3.19, the integrand in (3.53) is dominated by a constant times $g_\alpha(y)|f(x-y)|$, which is $L^1(\mathbb{R}^n)$ function for $B_{\alpha,p(\cdot),w}$-q.e. $x\in\mathbb{R}^n$. If we use the Lebesgue's dominated convergence theorem, then the proof is completed. □

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