

## APPROXIMATION FOR PERIODIC FUNCTIONS VIA STATISTICAL $A$ -SUMMABILITY

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ABSTRACT. In this paper, using the concept of statistical  $A$ -summability which is stronger than the  $A$ -statistical convergence, we prove a Korovkin type approximation theorem for sequences of positive linear operator defined on  $C^*(\mathbb{R})$  which is the space of all  $2\pi$ -periodic and continuous functions on  $\mathbb{R}$ , the set of all real numbers. We also compute the rates of statistical  $A$ -summability of sequence of positive linear operators.

### 1. INTRODUCTION

The idea of statistical convergence was introduced by Fast [5], which is closely related to the concept of natural density or asymptotic density of subsets of the set of natural numbers  $\mathbb{N}$ . Let  $K$  be a subset of  $\mathbb{N}$ . The natural density of  $K$  is the nonnegative real number given by  $\delta(K) := \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in K\}|$  provided that the limit exists, where  $|B|$  denotes the cardinality of the set  $B$  (see [14] for details). Then, a sequence  $x = \{x_k\}$  is called statistically convergent to a number  $L$  if for every  $\varepsilon > 0$ ,

$$\delta(\{k : |x_k - L| \geq \varepsilon\}) = 0.$$

This is denoted by  $st - \lim_{k \rightarrow \infty} x_k = L$  (see [5], [7]). It is easy to see that every convergent sequence is statistically convergent, but not conversely.

If  $x = \{x_k\}$  is a number sequence and  $A = \{a_{jk}\}$  is an infinite matrix, then  $Ax$  is the sequence whose  $j$ -th term is given by

$$A_j(x) := \sum_{k=1}^{\infty} a_{jk} x_k$$

provided that the series converges for each  $j \in \mathbb{N}$ . Thus we say that  $x$  is  $A$ -summable to  $L$  if

$$\lim_{j \rightarrow \infty} A_j(x) = L.$$

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We say that  $A$  is regular if  $\lim_{j \rightarrow \infty} A_j(x) = L$  whenever  $\lim_{k \rightarrow \infty} x_k = L$ . The well-known necessary and sufficient conditions [1] (Silverman-Toeplitz) for  $A$  to be regular are:

- R1)  $\|A\| = \sup_{j \rightarrow \infty} \sum_{k=1}^{\infty} |a_{jk}| < \infty$ ,  
 R2)  $\lim_{j \rightarrow \infty} a_{jk} = 0$  for each  $k \in \mathbb{N}$ ,  
 R3)  $\lim_{j \rightarrow \infty} \sum_{k=1}^{\infty} a_{jk} = 1$ .

Freedman and Sember [6] introduced the following extension of statistical convergence. Let  $A = \{a_{jk}\}$  be a nonnegative regular matrix. The  $A$ -density of  $K$  is defined by

$$\delta_A(K) := \lim_{j \rightarrow \infty} \sum_{k=1}^{\infty} a_{jk} \chi_K(k)$$

provided that the limit exists, where  $\chi_K$  is the characteristic function of  $K$ . Then the sequence  $x = \{x_k\}$  is said to be  $A$ -statistically convergent to the number  $L$  if for every  $\varepsilon > 0$ ,

$$\delta_A(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

or equivalently

$$\lim_{j \rightarrow \infty} \sum_{k: |x_k - L| \geq \varepsilon} a_{jk} = 0.$$

We denote this limit by  $st_A - \lim_{k \rightarrow \infty} x_k = L$  (see [6], [8], [9]). The case in which  $A = C_1$ , the Cesàro matrix of order one, reduces to the statistical convergence, and also if  $A = I$ , the identity matrix, then it coincides with the ordinary convergence.

Recently, the idea of statistical  $(C, 1)$ -summability was introduced in [11] and of statistical  $(H, 1)$ -summability in [12] by Moricz, and of statistical  $(\bar{N}, p)$ -summability by Moricz and Orhan [13]. Then these statistical summability methods were generalized by defining the statistical  $A$ -summability in [4].

Now we recall statistical  $A$ -summability for a nonnegative regular matrix  $A$ .

**Definition 1.1.** Let  $A = \{a_{jk}\}$  be a nonnegative regular matrix and  $x = \{x_k\}$  be a sequence. We say that  $x$  is statistically  $A$ -summable to  $L$  if for every  $\varepsilon > 0$ ,

$$\delta(\{j \in \mathbb{N} : |A_j(x) - L| \geq \varepsilon\}) = 0,$$

i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{j \leq n : |A_j(x) - L| \geq \varepsilon\}| = 0.$$

Thus  $x = \{x_k\}$  is statistically  $A$ -summable to  $L$  if and only if  $Ax$  is statistically convergent to  $L$ . In this case we write  $(A)_{st} - \lim_{k \rightarrow \infty} x_k = L$  or,  $st - \lim_{j \rightarrow \infty} A_j(x) = L$ .

Using the Definition 1.1, we see that if a sequence is bounded and  $A$ -statistically convergent to  $L$ , then it is  $A$ -summable to  $L$ , and hence statistically  $A$ -summable to  $L$ . However, its converse is not always true. Such examples were given in [4].

In this paper, using the concept of statistical  $A$ -summability where  $A = \{a_{jk}\}$  is a nonnegative regular matrix, we give a generalization of the classical Korovkin

approximation theorem by means of sequences of positive linear operators defined on the space of all real valued continuous and  $2\pi$  periodic functions on  $\mathbb{R}$ . We also compute the rates of statistical  $A$ -summability of sequence of positive linear operators.

2. A KOROVKIN TYPE THEOREM

We denote  $C^*(\mathbb{R})$ , the space of all real valued continuous and  $2\pi$  periodic functions on  $\mathbb{R}$ . We recall that if a function  $f$  in  $\mathbb{R}$  has period  $2\pi$ , then for all  $x \in \mathbb{R}$ ,

$$f(x) = f(x + 2\pi k)$$

holds for  $k = 0, \pm 1, \pm 2, \dots$ . This space is equipped with the supremum norm

$$\|f\|_{C^*(\mathbb{R})} = \sup_{x \in \mathbb{R}} |f(x)|, \quad (f \in C^*(\mathbb{R})).$$

Let  $L$  be a linear operator from  $C^*(\mathbb{R})$  into  $C^*(\mathbb{R})$ . Then, as usual, we say that  $L$  is a positive linear operator provided that  $f \geq 0$  implies  $L(f) \geq 0$ . Also, we denote the value of  $L(f)$  at a point  $x \in \mathbb{R}$  by  $L(f(x); x)$  or, briefly,  $L(f; x)$ .

Throughout the paper, we also use the following test functions

$$f_0(x) = 1, \quad f_1(x) = \cos x \quad f_2(x) = \sin x.$$

We also have to recall the classical Korovkin theorem [10].

**Theorem A.** *Let  $\{L_k\}$  be a sequence of positive linear operators acting from  $C^*(\mathbb{R})$  into itself. Then, for all  $f \in C^*(\mathbb{R})$ ,*

$$\lim_{k \rightarrow \infty} \|L_k(f) - f\|_{C^*(\mathbb{R})} = 0$$

*if and only if*

$$\lim_{k \rightarrow \infty} \|L_k(f_i) - f_i\|_{C^*(\mathbb{R})} = 0, \quad (i = 0, 1, 2).$$

Recently, the statistical analog of Theorem A was studied by Duman [3]. It will be read as follows.

**Theorem B.** *Let  $A = \{a_{jk}\}$  be a nonnegative regular matrix and let  $\{L_k\}$  be a sequence of positive linear operators acting from  $C^*(\mathbb{R})$  into itself. Then for all  $f \in C^*(\mathbb{R})$ ,*

$$st_A - \lim_{k \rightarrow \infty} \|L_k(f) - f\|_{C^*(\mathbb{R})} = 0$$

*if and only if*

$$st_A - \lim_{k \rightarrow \infty} \|L_k(f_i) - f_i\|_{C^*(\mathbb{R})} = 0, \quad (i = 0, 1, 2).$$

Now we study the approximation properties of sequence of positive linear operators on the space  $C^*(\mathbb{R})$  via statistical  $A$ -summability where  $A = \{a_{jk}\}$  is a nonnegative regular matrix.

**Theorem 2.1.** *Let  $A = \{a_{jk}\}$  be a nonnegative regular matrix and let  $\{L_k\}$  be a sequence of positive linear operators acting from  $C^*(\mathbb{R})$  into itself. Then, for all  $f \in C^*(\mathbb{R})$ ,*

$$(2.1) \quad st - \lim_{j \rightarrow \infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - f \right\|_{C^*(\mathbb{R})} = 0$$

if and only if

$$(2.2) \quad st - \lim_{j \rightarrow \infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_i) - (f_i) \right\|_{C^*(\mathbb{R})} = 0 \quad (i = 0, 1, 2).$$

*Proof.* Since each  $f_i$  ( $i = 0, 1, 2$ ) belongs to  $C^*(\mathbb{R})$ , the implication (2.1)  $\Rightarrow$  (2.2) is clear. Now, to prove the implication (2.2)  $\Rightarrow$  (2.1), assume that (2.2) holds. Let  $f \in C^*(\mathbb{R})$  and let  $I$  be a closed subinterval of length  $2\pi$  of  $\mathbb{R}$ . Fix  $x \in I$ . By the continuity of  $f$  at  $x$ , for given  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$|f(t) - f(x)| < \varepsilon$$

for all  $t$  satisfying  $|t - x| < \delta$ . On the other hand, by the boundedness of  $f$ , we have

$$|f(t) - f(x)| \leq 2\|f\|_{C^*(\mathbb{R})}$$

for all  $t \in \mathbb{R}$ . Now consider the subintervals  $(x - \delta, 2\pi + x - \delta]$  of length  $2\pi$ . From [3] we can see that

$$(2.3) \quad |f(t) - f(x)| < \varepsilon + \frac{2\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}} \psi(t)$$

holds for all  $t \in \mathbb{R}$ , where  $\psi(t) := \sin^2 \left( \frac{t-x}{2} \right)$ .

By using (2.3) and the positivity and monotonicity of  $L_k$  we have

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} a_{jk} L_k(f; x) - f(x) \right| \\ & \leq \sum_{k=1}^{\infty} a_{jk} L_k(|f(t) - f(x)|; x) + |f(x)| \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \\ & \leq \sum_{k=1}^{\infty} a_{jk} L_k \left( \varepsilon + \frac{2\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}} \psi(t); x \right) + |f(x)| \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \\ & \leq \varepsilon + \varepsilon \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| + \|f\|_{C^*(\mathbb{R})} \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \\ & \quad + \frac{2\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}} \sum_{k=1}^{\infty} a_{jk} L_k(\psi(t); x). \end{aligned}$$

After some simple calculations, we also get

$$\psi(t) = \frac{1}{2} (1 - \cos t \cos x - \sin t \sin x).$$

So we can get

$$(2.4) \quad \sum_{k=1}^{\infty} a_{jk} L_k(\psi(t); x) \leq \frac{1}{2} \left\{ \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \right. \\ \left. + |\cos x| \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_1; x) - f_1(x) \right| + |\sin x| \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_2; x) - f_2(x) \right| \right\}.$$

Then, using (2.4), we obtain

$$\left| \sum_{k=1}^{\infty} a_{jk} L_k(f; x) - f(x) \right| \\ \leq \varepsilon + \left( \varepsilon + \|f\|_{C^*(\mathbb{R})} + \frac{\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}} \right) \left\{ \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \right. \\ \left. + \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_1; x) - f_1(x) \right| + \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_2; x) - f_2(x) \right| \right\}.$$

Then, we obtain

$$(2.5) \quad \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - f \right\|_{C^*(\mathbb{R})} \leq \varepsilon + U \left\{ \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*(\mathbb{R})} \right. \\ \left. + \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_1) - f_1 \right\|_{C^*(\mathbb{R})} + \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_2) - f_2 \right\|_{C^*(\mathbb{R})} \right\}$$

where  $U := \varepsilon + \|f\|_{C^*(\mathbb{R})} + \frac{\|f\|_{C^*(\mathbb{R})}}{\sin^2 \frac{\delta}{2}}$ .

Now, for a given  $r > 0$ , choose  $\varepsilon > 0$  such that  $\varepsilon < r$ . By (2.5), it is easy to see that

$$\frac{1}{n} \left| \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - f \right\|_{C^*(\mathbb{R})} \geq r \right\} \right| \\ \leq \frac{1}{n} \left| \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*(\mathbb{R})} \geq \frac{r - \varepsilon}{3U} \right\} \right| \\ + \frac{1}{n} \left| \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_1) - f_1 \right\|_{C^*(\mathbb{R})} \geq \frac{r - \varepsilon}{3U} \right\} \right| \\ + \frac{1}{n} \left| \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_2) - f_2 \right\|_{C^*(\mathbb{R})} \geq \frac{r - \varepsilon}{3U} \right\} \right|.$$

Then using the hypothesis (2.2), we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - f \right\|_{C^*(\mathbb{R})} \geq r \right\} \right| = 0$$

for every  $r > 0$  and the proof is complete. □

### 3. RATE OF CONVERGENCE

In this section, using statistical  $A$ -summability we study the rate of convergence of positive linear operators defined  $C^*(\mathbb{R})$  into itself with the help of the modulus of continuity.

Demirci and Karakuş [2] introduced the rates of statistical  $A$ -summability of sequence as follows.

**Definition 3.1** ([2]). Let  $A = \{a_{jk}\}$  be a nonnegative regular matrix. A sequence  $x = \{x_k\}$  is statistical  $A$ -summable to a number  $L$  with the rate of  $\beta \in (0, 1)$  if for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{|\{j \leq n : |A_j(x) - L| \geq \varepsilon\}|}{n^{1-\beta}} = 0.$$

In this case, it is denoted by

$$x_k - L = o(n^{-\beta}) \quad ((A)_{st}).$$

Using this definition, we obtain the following auxiliary result.

**Lemma 3.2** ([2]). Let  $A = \{a_{jk}\}$  be a nonnegative regular matrix. Let  $x = \{x_k\}$  and  $y = \{y_k\}$  be bounded sequences. Assume that  $x_k - L_1 = o(n^{-\beta_1}) \quad ((A)_{st})$  and  $y_k - L_2 = o(n^{-\beta_2}) \quad ((A)_{st})$ . Let  $\beta := \min\{\beta_1, \beta_2\}$ . Then we have:

- (i)  $(x_k - L_1) \mp (y_k - L_2) = o(n^{-\beta}) \quad ((A)_{st})$
- (ii)  $\lambda(x_k - L_1) = o(n^{-\beta_1}) \quad ((A)_{st})$  for any real number  $\lambda$ .

Now we remind the concept of modulus of continuity. For  $f \in C^*(\mathbb{R})$ , the modulus of continuity of  $f$ , denoted by  $\omega(f; \delta_1)$ , is defined by

$$\omega(f; \delta_1) := \sup_{|t-x| \leq \delta_1} |f(t) - f(x)| \quad (\delta_1 > 0).$$

It is also well know that, for any  $\lambda > 0$  and for all  $f \in C^*(\mathbb{R})$

$$(3.1) \quad \omega(f; \lambda\delta_1) \leq (1 + [\lambda]) \omega(f; \delta_1)$$

where  $[\lambda]$  is defined to be the greatest integer less than or equal to  $\lambda$ .

Then we have the following result.

**Theorem 3.1.** Let  $A = \{a_{jk}\}$  be a nonnegative regular matrix and let  $\{L_k\}$  be a sequence of positive linear operators acting from  $C^*(\mathbb{R})$  into itself. Assume that the following conditions holds:

- (i)  $\|L_k(f_0) - f_0\|_{C^*(\mathbb{R})} = o(n^{-\beta_1}) \quad ((A)_{st})$  on  $\mathbb{R}$ ,

(ii)  $\omega(f; \gamma_j) = o(n^{-\beta_2}) \quad ((A)_{st})$  on  $\mathbb{R}$  where  $\gamma_j := \sqrt{\|\sum_{k=1}^{\infty} a_{jk} L_k(\varphi)\|_{C^*(\mathbb{R})}}$   
with  $\varphi(t) = \sin^2\left(\frac{t-x}{2}\right)$ .

Then we have for all  $f \in C^*(\mathbb{R})$ ,

$$\|L_k(f) - f\|_{C^*(\mathbb{R})} = o(n^{-\beta}) \quad ((A)_{st}) \quad \text{on } \mathbb{R}$$

where  $\beta := \min\{\beta_1, \beta_2\}$ .

*Proof.* Let  $f \in C^*(\mathbb{R})$  and  $x \in \mathbb{R}$  be fixed. Using (3.1) and the positivity and monotonicity of  $L_k$ , we get for any  $\delta_1 > 0$  and  $j \in \mathbb{R}$ ,

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} a_{jk} L_k(f; x) - f(x) \right| \\ \leq & \sum_{k=1}^{\infty} a_{jk} L_k(|f(t) - f(x)|; x) + |f(x)| \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \\ \leq & \sum_{k=1}^{\infty} a_{jk} L_k\left(\left(1 + \frac{(t-x)^2}{\delta_1^2}\right); x\right) \omega(f; \delta_1) + \|f\|_{C^*(\mathbb{R})} \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \\ \leq & \sum_{k=1}^{\infty} a_{jk} L_k\left(\left(1 + \frac{\pi^2}{\delta_1^2} \sin^2\left(\frac{t-x}{2}\right)\right); x\right) \omega(f; \delta_1) \\ & + \|f\|_{C^*(\mathbb{R})} \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \\ \leq & \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right| \omega(f; \delta_1) + \omega(f; \delta_1) \\ & + \frac{\pi^2}{\delta_1^2} \omega(f; \delta_1) \sum_{k=1}^{\infty} a_{jk} L_k\left(\sin^2\left(\frac{t-x}{2}\right); x\right) \\ & + \|f\|_{C^*(\mathbb{R})} \left| \sum_{k=1}^{\infty} a_{jk} L_k(f_0; x) - f_0(x) \right|. \end{aligned}$$

Hence, we get

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - f \right\|_{C^*(\mathbb{R})} & \leq \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*(\mathbb{R})} \omega(f; \gamma_j) + (1 + \pi^2) \omega(f; \gamma_j) \\ & + \|f\|_{C^*(\mathbb{R})} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*(\mathbb{R})} \end{aligned}$$

where  $\delta_1 := \gamma_j := \sqrt{\|\sum_{k=1}^{\infty} a_{jk}L_k(\varphi)\|_{C^*(\mathbb{R})}}$ . Then, we obtain

$$(3.2) \quad \left\| \sum_{k=1}^{\infty} a_{jk}L_k(f) - f \right\|_{C^*(\mathbb{R})} \leq K \left\{ \left\| \sum_{k=1}^{\infty} a_{jk}L_k(f_0) - f_0 \right\|_{C^*(\mathbb{R})} \omega(f; \gamma_j) + \omega(f; \gamma_j) + \left\| \sum_{k=1}^{\infty} a_{jk}L_k(f_0) - f_0 \right\|_{C^*(\mathbb{R})} \right\}$$

where  $K = \max \{ \|f\|_{C^*(\mathbb{R})}, 1 + \pi^2 \}$ . Hence, for given  $\varepsilon > 0$ , from (3.2) and Lemma 3.2, it follows

$$(3.3) \quad \begin{aligned} & \frac{1}{n^{1-\beta}} \left| \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk}L_k(f) - f \right\|_{C^*(\mathbb{R})} \geq \varepsilon \right\} \right| \\ & \leq \frac{1}{n^{1-\beta_1}} \left| \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk}L_k(f_0) - f_0 \right\|_{C^*(\mathbb{R})} \geq \sqrt{\frac{\varepsilon}{3K}} \right\} \right| \\ & \quad + \frac{1}{n^{1-\beta_2}} \left| \left\{ j \leq n : \omega(f; \gamma_j) \geq \sqrt{\frac{\varepsilon}{3K}} \right\} \right| \\ & \quad + \frac{1}{n^{1-\beta_2}} \left| \left\{ j \leq n : \omega(f; \gamma_j) \geq \frac{\varepsilon}{3K} \right\} \right| \\ & \quad + \frac{1}{n^{1-\beta_1}} \left| \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk}L_k(f_0) - f_0 \right\|_{C^*(\mathbb{R})} \geq \frac{\varepsilon}{3K} \right\} \right| \end{aligned}$$

where  $\beta := \min \{ \beta_1, \beta_2 \}$ . Letting  $n \rightarrow \infty$  in (3.3), from (i) and (ii), we conclude that

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1-\beta}} \left| \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk}L_k(f) - f \right\|_{C^*(\mathbb{R})} \geq \varepsilon \right\} \right| = 0,$$

which means

$$\|L_k(f) - f\|_{C^*(\mathbb{R})} = o(n^{-\beta}) \quad ((A)_{st}) \quad \text{on } \mathbb{R}.$$

The proof is completed. □

Now we give the following classical rates of convergence of a sequence of positive linear operators defined on  $C^*(\mathbb{R})$ .

**Corollary 1.** *Let  $\{L_k\}$  be a sequence of positive linear operators acting from  $C^*(\mathbb{R})$  into itself. Assume that the following conditions holds:*

(i)  $\lim_{k \rightarrow \infty} \|L_k(f_0) - f_0\|_{C^*(\mathbb{R})} = 0,$

(ii)  $\lim_{k \rightarrow \infty} \omega(f; \delta_k) = 0$  on  $\mathbb{R}$  where  $\delta_k := \sqrt{\|L_k(\varphi)\|_{C^*(\mathbb{R})}}$  with  $\varphi(t) = \sin^2(\frac{t-x}{2})$ .

Then for all  $f \in C^*(\mathbb{R})$ , we have

$$\lim_{k \rightarrow \infty} \|L_k(f) - f\|_{C^*(\mathbb{R})} = 0.$$

4. AN APPLICATION TO THEOREM 2.1 AND THEOREM 3.1

In this section, we display an example of a sequence of positive linear operators. First of all, we show that Theorem 2.1 holds, but Theorem A and Theorem B do not hold. Then, using the same sequence of positive linear operators, we show that Theorem 3.1 holds but, Corollary 1 does not hold.

Let  $A$  be Cesàro matrix, i.e.,

$$a_{jk} = \begin{cases} \frac{1}{j}, & 1 \leq k \leq j, \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$(4.1) \quad \xi_k = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ -1, & \text{if } k \text{ is even.} \end{cases}$$

Then, we observe that,  $A = \{a_{jk}\}$  is a nonnegative regular matrix and for the sequence  $\xi := \{\xi_k\}$

$$st - \lim_{j \rightarrow \infty} A_j(\xi) = 0.$$

However, the sequence  $\{\xi_k\}$  is not convergent in the usual sense and  $A$ -statistical convergent to 0. Then, consider the following Fejér operators

$$(4.2) \quad F_k(f; x) := \frac{1}{k\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin^2\left(\frac{k}{2}(t-x)\right)}{2 \sin^2\left[\frac{t-x}{2}\right]} dt$$

where  $k \in \mathbb{N}$ ,  $f \in C^*[-\pi, \pi]$ . Then, we get (see [10])

$$F_k(f_0; x) = 1, \quad F_k(f_1; x) = \frac{k-1}{k} \cos x, \quad F_k(f_2; x) = \frac{k-1}{k} \sin x.$$

Now, using (4.1) and (4.2), we introduce the following positive linear operators defined on the space  $C^*[-\pi, \pi]$

$$(4.3) \quad L_k(f; x) = (1 + \xi_k)F_k(f; x).$$

(i) Now, we consider the positive linear operators defined by (4.3) on  $C^*[-\pi, \pi]$ . Since  $st - \lim_{j \rightarrow \infty} A_j(\xi) = 0$ , we conclude that

$$st - \lim_{j \rightarrow \infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_i) - (f_i) \right\|_{C^*[-\pi, \pi]} = 0, \quad (i = 0, 1, 2).$$

Then, by Theorem 2.1, for all  $f \in C^*[-\pi, \pi]$ , we obtain

$$st - \lim_{j \rightarrow \infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f) - (f) \right\|_{C^*[-\pi, \pi]} = 0.$$

However, since  $\{\xi_k\}$  does not converge in the usual sense and  $A$ -statistical converges to 0, we conclude that Theorem A and Theorem B do not work for the operators  $L_k$  in (4.3) while our Theorem 2.1 still works.

(ii) Now, we consider the positive linear operators defined by (4.3) on  $C^*[-\pi, \pi]$ . We observe that

$$\left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*[-\pi, \pi]} = \left| \frac{1}{j} \sum_{k=1}^j (1 + \xi_k) - 1 \right| = \left| \frac{1}{j} \sum_{k=1}^j \xi_k \right|.$$

Since

$$\lim_{j \rightarrow \infty} \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*[-\pi, \pi]} = 0,$$

then we get

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1-\beta_1}} \left\{ j \leq n : \left\| \sum_{k=1}^{\infty} a_{jk} L_k(f_0) - f_0 \right\|_{C^*[-\pi, \pi]} \geq \varepsilon \right\} = 0,$$

which means that

$$(4.4) \quad \|L_k(f_0) - f_0\|_{C^*[-\pi, \pi]} = o(n^{-\beta_1}) \quad ((A)_{st}).$$

Now, we compute the quantity  $L_k(\varphi; x)$  where  $\varphi(t) = \sin^2\left(\frac{t-x}{2}\right)$ . After some calculations, we get

$$L_k(\varphi; x) = \frac{1 + \xi_k}{2k}.$$

Then, we obtain  $\gamma_j := \sqrt{\|\sum_{k=1}^{\infty} a_{jk} L_k(\varphi)\|_{C^*[-\pi, \pi]}} = \sqrt{\left| \frac{1}{j} \sum_{k=1}^j \frac{1 + \xi_k}{2k} \right|}$ . Since  $\lim_{j \rightarrow \infty} \sqrt{\left| \frac{1}{j} \sum_{k=1}^j \frac{1 + \xi_k}{2k} \right|} = 0$ , we get  $st - \lim_{j \rightarrow \infty} \sqrt{\left| \frac{1}{j} \sum_{k=1}^j \frac{1 + \xi_k}{2k} \right|} = 0$ . By the uniform continuity of  $f$  on  $[-\pi, \pi]$ , we write

$$(4.5) \quad \omega(f; \gamma_j) = o(n^{-\beta_2}) \quad ((A)_{st}).$$

From (4.4) and (4.5), the sequence of positive linear operators  $\{L_k\}$  satisfies all hypotheses of Theorem 3.1. So, for all  $f \in C^*[-\pi, \pi]$ , we have

$$\|L_k(f) - f\|_{C^*[-\pi, \pi]} = o(n^{-\beta}) \quad ((A)_{st}).$$

However, since  $\{\xi_k\}$  is not convergent, the conditions (i) and (ii) of Corollary 1 do not hold. So, the sequence  $\{L_k\}$  given by (4.3) does not converge uniformly to the function  $f \in C^*[-\pi, \pi]$ .

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