On the Expansion Formula for a Class of Dirac Operator with Discontinuous Coefficient

Kh. R. Mamedov and Aynur ÇÖL

***Abstract*— In this paper we consider a first order differential equation system with a discontinuous coefficient and spectral parameter dependent boundary condition in the half line. The operator interpretation of the given boundary value problem is investigated in the Hilbert space $H = L_{2,\rho}\left(0,\infty;\mathbb{C}^2\right)\times\mathbb{C}$. The resolvent operator is constructed and the expansion formula with respect to eigenfunctions is obtained.** *Copyright © 2009 Yang's Scientific Research Institute, LLC. All rights reserved.*

***Index Terms*— Dirac operator, expansion formula, resolvent operator.**

I. Introduction

We consider the boundary value problem

$$BY' + \Omega(x)Y = \lambda\rho(x)Y, \quad 0 < x < \infty, \tag{1}$$

$$Y_1(0) - \lambda Y_2(0) = 0 \tag{2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

$$Y = \begin{pmatrix} Y_1(x) \\ Y_2(x) \end{pmatrix},$$

$p(x)$, $q(x)$ are real measurable functions, λ is a spectral parameter and

$$\rho(x) = \begin{cases} \alpha, & 0 \le x \le a, \\ 1, & a < x < \infty, \end{cases}$$

and $1 \ne \alpha > 0$.

Assume that the condition

$$\int_0^\infty \|\Omega(x)\|\,dx < \infty \tag{3}$$

is satisfied for Euclidean norm of the matrix function $\Omega(x)$.

The spectral analysis of the boundary-value problem (1)-(2) in the half line in the case of $\rho(x) \equiv 1$ in the equation (1) was researched in [5], [8], [7], [1], [12]. In finite interval the expansion formula for Dirac operator with a spectral parameter in the boundary conditions was examined in [11] and a discontinuous Sturm-Liouville problem with a spectral parameter in boundary condition was investigated in [3], [10]. In the half line the spectral analysis of Sturm-Liouville operator was investigated in [4]. The discontinuous inverse problem of scattering theory for the system (1) with the boundary condition not containing a spectral parameter was studied in [6], [9]. The spectral properties of Dirac operators on $(0,1)$ with potentials that belong to entrywise to $L_p(0,1)$, for some $p \in [1,\infty)$, were studied and the vast reference list about the inverse problems for Dirac operators on the finite interval were given in [2].

In this paper our aim is to obtain the expansion formula for the boundary-value problem (1)-(2) in the half line. To obtain the resolvent operator and the expansion formula we applied the method [13].

Let

$$\mu(x) = \begin{cases} a + \alpha(x-a), & 0 \le x \le a, \\ x, & x > a. \end{cases}$$

It is easily shown that the vector function

$$f^0(x,\lambda) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda\mu(x)}$$

is a solution of the equation (1) when $\Omega(x) \equiv 0$.

As known from [6], when the condition (3) is satisfied, for $\mathrm{Im}\lambda \ge 0$ the equation (1) has an solution $f(x,\lambda)$ which can be expressed uniquely as

$$f(x,\lambda) = f^0(x,\lambda) + \int_{\mu(x)}^\infty K(x,t)\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda t}dt \tag{4}$$

satisfying the condition

$$\lim_{x\to\infty} f(x,\lambda)e^{-i\lambda x} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Moreover, the elements of the matrix kernel $K(x,t)$ are summable on the positive half line and for the Euclidean norm of $K(x,t)$, the inequality

$$\int_{\mu(x)}^\infty \|K(x,t)\|\,dt \le e^{\sigma(x)} - 1 \tag{5}$$

is satisfied, where $\sigma(x) = \int\limits_x^\infty \|\Omega((t))\|\,dt$.

Manuscript received June 23, 2009; revised August 15, 2009.

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Publisher Item Identifier S 1542-5908(09)10403-7/$20.00

Let $Y(x,\lambda)$ and $Z(x,\lambda)$ be vector solutions of the equations system (1). The expression

$$\begin{aligned} W[Y(x,\lambda), Z(x,\lambda)] &= Y^T(x,\lambda) BZ(x,\lambda) \\ &= (Y_1, Y_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \\ &= Y_1 Z_2 - Y_2 Z_1 \end{aligned}$$

is called Wronskian of the vector functions $Y(x,\lambda)$ and $Z(x,\lambda)$.

Since $p(x)$ and $q(x)$ are real valued functions, the vector functions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ are fundamental solutions system of the equation (1) for real λ. The Wronskian of the vector functions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ doesn't depend on x and is equal to $2i$.

We denote by $\varphi(x,\lambda)$ the solution of the equation (1) satisfying the conditions

$$\varphi_1(0,\lambda) = \lambda, \qquad \varphi_2(0,\lambda) = 1.$$

Let us define the function

$$E(\lambda) \equiv f_1(0,\lambda) - \lambda f_2(0,\lambda).$$

It can be shown the function $E(\lambda)$ hasn't any zeros on the closed upper plane.

The paper is organized as follows: In Section II we give the theoretic formulation of operator of the boundary value problem (1), (2) in the Hilbert space $H_\rho = L_{2,\rho}\left(0,\infty;\mathbb{C}^2\right) \times \mathbb{C}$. In Section III we find the kernel for the resolvent operator and construct the resolvent operator. Finally, we obtain the expansion formula with respect to eigenfunctions in Section IV.

II. Operator Theoretic Formulation

In the Hilbert space $H_\rho = L_{2,\rho}\left(0,\infty;\mathbb{C}^2\right) \times \mathbb{C}$ an inner product is defined by

$$(F,G) := \int_0^\infty \left\{ f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)} \right\} \rho(x)\,dx + f_3 g_3,$$

for the triple component vectors

$$F := \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3 \end{pmatrix}, \qquad G := \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3 \end{pmatrix} \in H_\rho.$$

For convenience, we put

$$f := f(x) := \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

therefore

$$F = \begin{pmatrix} f(x) \\ f_3 \end{pmatrix}$$

and

$$l(y) := \frac{1}{\rho(x)}\left\{ BY' + \Omega(x) Y \right\}.$$

We define the operator L corresponding to the boundary value problem (1)-(2) as the following form

$$LF = \begin{pmatrix} -f_2'(x) + p(x) f_1(x) + q(x) f_2(x) \\ f_1'(x) + q(x) f_1(x) - p(x) f_2(x) \\ f_1(0) \end{pmatrix}, F \in D(L)$$

where the domain of definition of the operator L is

$$\begin{aligned} D(L) := \{ F \ \mid \ & F = (f_1(x), f_2(x), f_3) \in H_\rho, \\ & f_1(x), f_2(x) \text{ are absolutely continuous in} \\ & \text{every } [0,b] \subset [0,\infty), l(f) \in L_{2,\rho}\left(0,\infty;\mathbb{C}^2\right), \\ & f_3 = f_2(0) \} \end{aligned}$$

It is easy to show that the operator L with domain $D(L)$ is selfadjoint in the space H_ρ.

III. Resolvent Operator

If λ is not a spectrum point of operator L, then there exists the resolvent $R_\lambda = (L - \lambda I)^{-1}$. Now we find this expression of the operator R_λ.

Lemma III.1 *The resolvent R_λ is the integral operator with the kernel which has the following form*

$$R_\lambda(x,t) = -\frac{1}{E(\lambda)} \begin{cases} \varphi(x,\lambda)\widetilde{f}(t,\lambda), & x \le t, \\ f(x,\lambda)\widetilde{\varphi}(t,\lambda), & t \le x, \end{cases}$$

here $\widetilde{f}(t,\lambda)$ denotes the transposed vector function of $f(t,\lambda)$.

Proof: Let $F \in D(L)$ and $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ be zero in exterior of every interval. To construct the resolvent operator of L, we need to solve the boundary value problem

$$BY' + \Omega(x) Y = \lambda \rho(x) Y + \rho(x) f(x), \tag{6}$$

$$Y_1(0) = \lambda Y_2(0) + f_3. \tag{7}$$

By applying the method of variation of parameters, we want to find the solution of the problem (6),(7) which has a form

$$Y(x,\lambda) = c_1(x,\lambda)\varphi(x,\lambda) + c_2(x,\lambda) f(x,\lambda), \tag{8}$$

where $\varphi(x,\lambda)$, $f(x,\lambda)$ are solutions of homogeneous problem. Then we get the equations system

$$\begin{aligned} c_1'(x,\lambda)\widetilde{f}(x,\lambda) B\varphi(x,\lambda) &= \widetilde{f}(x,\lambda) f(x) \rho(x), \\ c_2'(x,\lambda)\widetilde{\varphi}(x,\lambda) Bf(x,\lambda) &= \widetilde{\varphi}(x,\lambda) f(x) \rho(x). \end{aligned} \tag{9}$$

Since $Y(x,\lambda) \in L_{2,\rho}\left(0,\infty;\mathbb{C}^2\right)$, then $c_1(\infty,\lambda) = 0$. Using this relation and the equation system (9) we get

$$\begin{aligned} c_1(x,\lambda) &= -\int_x^\infty \frac{\widetilde{f}(t,\lambda)}{E(\lambda)} f(t)\rho(t)\,dt, \\ c_2(x,\lambda) &= c_2(0,\lambda) - \int_0^x \frac{\widetilde{\varphi}(t,\lambda)}{E(\lambda)} f(t)\rho(t)\,dt. \end{aligned} \tag{10}$$

Substituting (10) into (8) we obtain

$$Y(x,\lambda)=\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt+c_2(0,\lambda)\,f(x,\lambda),$$

where

$$R_\lambda(x,t)=-\frac{1}{E(\lambda)}\begin{cases}\varphi(x,\lambda)\,\widetilde{f}(t,\lambda), & x\le t,\\ f(x,\lambda)\,\widetilde{\varphi}(t,\lambda), & t\le x.\end{cases} \quad (11)$$

Taking formula (7) we get $c_2(0,\lambda)=\frac{f_3}{E(\lambda)}$. Thus

$$(L-\lambda I)^{-1}f=\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt+\frac{f_3}{E(\lambda)}f(x,\lambda),\quad f\in D(L). \quad (12)$$

The lemma is proved. ∎

Lemma III.2 *Let the vector function $f(x)$ be finite at infinity and*

$$F=\begin{pmatrix} f(x)\\ f_3\end{pmatrix}\in D(L).$$

Then

$$\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt+\frac{f_3}{E(\lambda)}f(x,\lambda) \quad (13)$$

$$=\ -\frac{f(x)}{\lambda}+\frac{f(x,\lambda)\,f_1(0)}{\lambda E(\lambda)}+\frac{1}{\lambda}\int_0^\infty R_\lambda(x,t)\,g(t)\,dt,$$

where $g(x)=Bf'(x)+\Omega(x)\,f(x)$.

Proof: Using (11) and integrating by parts, we write Eq. (14) at the top of the next page. This proves the lemma. ∎

Suppose that $f(t)$ satisfies the condition of Lemma (III.2), then from Eq. (13) it follows that for $\mathrm{Im}\lambda\geqslant 0$, $|\lambda|\to\infty$,

$$\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt+\frac{f_3}{E(\lambda)}f(x,\lambda)=-\frac{f(x)}{\lambda}+O\left(\frac{1}{\lambda}\right), \quad (15)$$

since

$$\int_0^\infty R_\lambda(x,t)\,g(t)\,dt=O(1).$$

The following lemma is well known.

Lemma III.3 $\overline{R}_\lambda=R_{\overline{\lambda}}$.

IV. Expansion Formula

With the help of these lemmas we obtain the expansion formula in eigenfunctions from $D(L)$ of boundary value problem (1)-(2).

We integrate both sides of Eq. (15) with respect to λ over the circle Γ_R of radius R and center at zero. As a result we have

$$-f(x)+\frac{1}{2\pi i}\oint_{\Gamma_R}O\left(\frac{1}{\lambda}\right)d\lambda$$
$$=\frac{1}{2\pi i}\oint_{\Gamma_R}d\lambda\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt \quad (16)$$
$$+\frac{1}{2\pi i}\oint_{\Gamma_R}\frac{f_3}{E(\lambda)}f(x,\lambda)\,d\lambda$$

The function $R_\lambda(x,t)$ is analytic in the upper and lower half plane. Therefore, we have

$$\frac{1}{2\pi i}\oint_{\Gamma_R}d\lambda\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt=I_R^1+I_R^2+I_R^3, \quad (17)$$

where

$$I_R^1=\frac{1}{2\pi i}\int_{-R-i\varepsilon}^{R-i\varepsilon}d\lambda\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt, \quad (18)$$

$$I_R^2=\frac{1}{2\pi i}\int_{R+i\varepsilon}^{-R+i\varepsilon}d\lambda\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt, \quad (19)$$

$$I_R^3=\frac{1}{2\pi i}\left\{\int_{R-i\varepsilon}^{R+i\varepsilon}\left[\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt\right]d\lambda\right.$$
$$\left.+\int_{-R+i\varepsilon}^{-R-i\varepsilon}\left[\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt\right]d\lambda\right\}$$

and here ε is any positive number. From Lemma III.2 and Eq. (15) it follows that $I_R^3\to 0$ as $R\to\infty$. From Eq. (15) it follows that the limit

$$\lim_{\varepsilon\to 0}\int_0^\infty R_{\lambda\pm i\varepsilon}(x,t)\,f(t)\,\rho(t)\,dt=\int_0^\infty R_{\lambda\pm i0}(x,t)\,f(t)\,\rho(t)\,dt.$$

Therefore, by going over in Eq. (16) to the limit as $R\to\infty$ and using Eqs. (17) - (19) we find Eq. (20) in the next page from Lemma III.3, $R_{\lambda-i0}=\overline{R}_{\lambda+i0}$. Let us compute $R_{\lambda+i0}(x,t)-R_{\lambda-i0}(x,t)$. Let $\psi(x,\lambda)$ be solution of equation (1) with the initial conditions

$$\psi(0,\lambda)=\begin{pmatrix}0\\1\end{pmatrix}.$$

Then

$$f(x,\lambda)=f_2(0,\lambda)\,\varphi(x,\lambda)+E(\lambda)\,\psi(x,\lambda). \quad (21)$$

$$\begin{aligned}
&\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt \\
&= -\frac{1}{E(\lambda)}f(x,\lambda)\int_0^x \widetilde{\varphi}(t,\lambda)\,f(t)\,\rho(t)\,dt - \frac{1}{E(\lambda)}\varphi(x,\lambda)\times\int_x^\infty \widetilde{f}(t,\lambda)\,f(t)\,\rho(t)\,dt \\
&= -\frac{f(x,\lambda)}{\lambda E(\lambda)}\int_0^x \left\{-\frac{\partial}{\partial t}\widetilde{\varphi}(t,\lambda)\,B+\widetilde{\varphi}(t,\lambda)\,\Omega(t)\right\}f(t)\,dt - \frac{1}{\lambda E(\lambda)}\varphi(x,\lambda)\int_x^\infty \left\{-\frac{\partial}{\partial t}\widetilde{f}(t,\lambda)\,B+\widetilde{f}(t,\lambda)\,\Omega(t)\right\}f(t)\,dt \\
&= \frac{1}{\lambda E(\lambda)}f(x,\lambda)\,\widetilde{\varphi}(t,\lambda)\,Bf(t)\Big|_{t=0}^{x} + \frac{1}{\lambda E(\lambda)}\varphi(x,\lambda)\,\widetilde{f}(t,\lambda)\,Bf(t)\Big|_{t=x}^{\infty} - \frac{1}{\lambda}\int_0^\infty R_\lambda(x,t)\,g(t)\,dt \\
&= -\frac{f(x)}{\lambda} - \frac{f_3}{E(\lambda)}f(x,\lambda) + \frac{1}{\lambda}\int_0^\infty R_\lambda(x,t)\,g(t)\,dt + \frac{f_1(0)}{\lambda E(\lambda)}f(x,\lambda)
\end{aligned} \tag{14}$$

$$\begin{aligned}
f(x) &= -\lim_{R\to\infty}\frac{1}{2\pi i}\oint_{\Gamma_R}\left[\int_0^\infty R_\lambda(x,t)\,f(t)\,\rho(t)\,dt\right]d\lambda - \frac{1}{2\pi i}\oint_{\Gamma_R}\frac{f_3}{E(\lambda)}f(x,\lambda)\,d\lambda \\
&= -\lim_{R\to\infty}\left\{I_R^1+I_R^2+I_R^3\right\} - \frac{1}{2\pi i}\oint_{\Gamma_R}\frac{f_3}{E(\lambda)}f(x,\lambda)\,d\lambda \\
&= -\frac{1}{2\pi i}\int_{-\infty}^\infty d\lambda\int_0^\infty \left[R_{\lambda-i0}(x,t)-R_{\lambda+i0}(x,t)\right]f(t)\,\rho(t)\,dt - \frac{1}{2\pi i}\int_{-\infty}^\infty\left[\frac{f(x,\lambda+i0)}{E(\lambda+i0)}-\frac{f(x,\lambda-i0)}{E(\lambda-i0)}\right]f_3 d\lambda
\end{aligned} \tag{20}$$

Since $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are entire vector functions of λ, it follows from Eqs. (12) and (21) that

$$\begin{aligned}
&-\int_0^\infty \left[R_{\lambda+i0}(x,t)-R_{\lambda-i0}(x,t)\right]f(t)\,\rho(t)\,dt \\
=\;& -\int_0^\infty \left[R_{\lambda+i0}(x,t)-\overline{R}_{\lambda+i0}(x,t)\right]f(t)\,\rho(t)\,dt \\
=\;& \int_0^x \left[\frac{f(x,\lambda)}{E(\lambda)}-\overline{\frac{f(x,\lambda)}{E(\lambda)}}\right]\widetilde{\varphi}(t,\lambda)\,f(t)\,\rho(t)\,dt \\
&+\int_x^\infty \varphi(x,\lambda)\left[\frac{\widetilde{f}(t,\lambda)}{E(\lambda)}-\overline{\frac{\widetilde{f}(t,\lambda)}{E(\lambda)}}\right]f(t)\,\rho(t)\,dt.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\frac{f(x,\lambda)}{E(\lambda)}-\overline{\frac{f(x,\lambda)}{E(\lambda)}} \\
=\;& \left\{\frac{f_2(0,\lambda)}{E(\lambda)}-\overline{\frac{f_2(0,\lambda)}{E(\lambda)}}\right\}\varphi(x,\lambda) \\
=\;& \frac{\overline{f_1(0,\lambda)}f_2(0,\lambda)-f_1(0,\lambda)\,\overline{f_2(0,\lambda)}}{|E(\lambda)|^2}\varphi(x,\lambda) \\
=\;& -2i\frac{1}{|E(\lambda)|^2}\varphi(x,\lambda).
\end{aligned}$$

Therefore

$$\begin{aligned}
&-\frac{1}{2\pi i}\int_0^\infty \left\{R_{\lambda+i0}(x,t)-R_{\lambda-i0}(x,t)\right\}f(t)\,\rho(t)\,dt \\
&= -\frac{1}{\pi}\int_0^\infty \frac{\varphi(x,\lambda)\,\widetilde{\varphi}(t,\lambda)}{|E(\lambda)|^2}f(t)\,\rho(t)\,dt.
\end{aligned}$$

We obtained the following expansion by eigenfunctions of the operator L on the form

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_{-\infty}^\infty d\lambda\int_0^\infty \frac{\varphi(x,\lambda)\,\widetilde{\varphi}(t,\lambda)}{|E(\lambda)|^2}f(t)\,\rho(t)\,dt \\
&\quad+\frac{1}{\pi}\int_0^\infty \frac{\varphi(x,\lambda)}{|E(\lambda)|^2}f_3 d\lambda, \\
f_3 &= \frac{1}{\pi}\int_{-\infty}^\infty d\lambda\int_0^\infty \frac{\widetilde{\varphi}(t,\lambda)}{|E(\lambda)|^2}f(t)\,\rho(t)\,dt \\
&\quad+\frac{1}{\pi}\int_0^\infty \frac{f_3}{|E(\lambda)|^2}d\lambda.
\end{aligned}$$

or

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{0}^{\infty} u(x,\lambda)\, u^{*}(t,\lambda)\, f(t)\, \rho(t)\, dt \\ &+ \frac{1}{2\pi i}\int_{0}^{\infty} \frac{u(x,\lambda)}{\overline{E(\lambda)}} f_3 d\lambda, \\ f_3 &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \int_{0}^{\infty} \frac{u^{*}(t,\lambda)}{\overline{E(\lambda)}} f(t)\, \rho(t)\, dt \\ &+ \frac{1}{\pi}\int_{0}^{\infty} \frac{f_3}{|E(\lambda)|^2} d\lambda, \end{aligned}$$

where

$$\begin{aligned} u(x,\lambda) &= \overline{f(x,\lambda)} - \frac{\overline{E(\lambda)}}{E(\lambda)} f(x,\lambda), \\ u^{*}(x,\lambda) &= \widetilde{f}(x,\lambda) - \frac{\overline{E(\lambda)}}{E(\lambda)} f^{*}(x,\lambda) \end{aligned}$$

and u^{*} denotes the conjugated vector function of $\widetilde{u}$.

REFERENCES

[1] Mark J. Ablowitz. Segur, harvey solitons and the inverse scattering transform. *Society for Industrial and Applied Mathematics(SIAM) Studies in Applied Mathematics*, pages x+425, 1981.

[2] S. Albeverio, R. Hryniv, and Ya. Mykytyuk. Inverse spectral problems for dirac operators with summable potentials. *Russ. J. Math. Phys.*, 12(4):406–423, 2005.

[3] M. Demirci, Z. Akdoğan, and O. Sh. Mukhtarov. Asymptotic behavior of eigenvalues and eigenfunctions of one discontinuous boundary-value problem. *International Journal of Computational Cognition*, 2(3):101–113, 2004.

[4] Charles T. Fulton. Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions. *Proc. Roy. Soc. Edinburgh Sect. A*, 87(1-2):1–34, 1980/81.

[5] M. G. Gasymov. An inverse problem of scattering theory for a system of dirac equations of order $2n$. *Trudy Moskov. Mat. Obšč*, 19:41–112, 1968(Russian).

[6] I. M. Guseĭnov. The inverse scattering problem for a system of dirac equations with discontinuous coefficients. *Dokl. Akad. Nauk Azerbaĭdzhana*, 55(1-2):13–18, 1999(Russian).

[7] B. M. Levitan and I. S. Sargsjan. *Introduction to spectral theory, Selfadjoint ordinary differential operators*. Nauka, Moscow, 1970.

[8] B. M. Levitan and I. S. Sargsjan. *Sturm-Liouville and Dirac operators*. Mathematics and its Applications(Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1991 (translated from the Russian).

[9] Kh. R. Mamedov and Aynur Çöl. On the inverse problemof the scattering theory for a class of systems of dirac equations with discontinuous coefficient. *Eur. J. Pure Appl. Math.*, 1(3):21–32, 2008.

[10] Khanlar R. Mamedov. On a basic problem for a second order differential equation with a discontinuous coefficient and a spectral parameter in the boundary conditions. In *Geometry, integrability and quantization*, pages 218–225, Softex, Sofia, 2006.

[11] S. G. Mamedov. The inverse boundary value problem on a finite interval for dirac's system of equations. *Azerbaĭdžan. Gos. Univ. Učen. Zap. Ser. Fiz-Mat*, (5):61–67, 1975.

[12] Vladimir A. Marchenko. *Operator Theory: Advances and Applications*, chapter Sturm-Liouville operators and applications, pages xii+367. Birkhäuser Verlag, Basel, 1986(Translated from the Russian by A. Iacob).

[13] E. C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations.* Clarendon Press, Oxford, second edition, 1962.