

ON MODULE CLASSES OF GENERALIZED SEMIPERFECT MODULES

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Abstract. We introduce Rad-cc-supplemented module which generalizes the general concept of co-coatomically-supplemented modules; a module W is Rad-cc-supplemented if each co-coatomic submodule of W has a Rad-supplement in W . In Section 2, we present various properties of these modules. In Section 3, we examine the characterization of modules over commutative domains. In Section 4, we explore the concept of $\oplus$ -Rad-cc-supplemented modules, which generalizes a generalized notion of $\oplus$ -co-coatomically-supplemented modules in R. Alizade, S. Güngör (2018). A module W is $\oplus$ -Rad-cc-supplemented if each co-coatomic $A \leq W$ is of a Rad-supplement which is a direct summand of W . In the concluding section of this paper, we investigate into its characteristics by introducing Rad-cc-semiperfect modules.

Keywords: co-coatomic submodules; Rad-cc-supplemented modules; totally Rad-cc-supplemented modules

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1. INTRODUCTION

All rings considered in this paper are associative with an identity element. Every module is considered a unitary left module. Let S be a ring and W be a module meeting these requirements. The notation $A \leq W$ indicates that A is a submodule of W . The notation $A \leq_{\oplus} W$ means that A is a direct summand of W . $A \leq W$ is called as *small* in W provided that $A + B \neq W$ for any proper submodule B of W , denoted by $A \ll W$, and we denote by $\text{Rad}(W)$ the sum of all small submodules of W . In contrast to this concept, a submodule A of W is called *essential* in W , symbolically denoted by $A \trianglelefteq W$, if its intersection with nonzero submodules of W is nonzero. A submodule A of W is called a *cofinite submodule* whenever $\frac{W}{A}$ is finitely generated. W is called *coatomic*, if each proper submodule of W is included

in a maximal submodule in W , see [24]. Furthermore, in [2] co-coatomic submodules are defined as a generalization of cofinite submodules as follows: $A \leq W$ is called *co-coatomic*, if the factor module $\frac{W}{A}$ is coatomic. A supplement submodule T of A in W is a minimal element of the set $\{B \leq W; W = A + B\}$ which is equivalent to $W = A + T$ and $A \cap T \ll T$. A module W is called as (*cofinitely*) *supplemented* when each (cofinite) submodule of W is supplemented ([1], [23]). $A \leq W$ is referred to as a *Rad-supplement* of B in W provided that $A + B = W$ and $A \cap B \subseteq \text{Rad}(A)$. The module W is called *amply Rad-supplemented* if, for any $A, B \leq W$ with $W = A + B$, B includes a Rad-supplement of A in W . W is called *Rad-supplemented* if each $B \leq W$ has a Rad-supplement in W [8]. Additionally, as stated in [18], a module W is considered *cofinitely Rad-supplemented* when each cofinite submodule of W has a Rad-supplement within the module W .

In [2], a broader category of supplemented modules is introduced, referred to as co-coatomically supplemented modules. Specifically, a module W is classified as *co-coatomically supplemented* if each co-coatomic submodule of W has a supplement within W . Construction on this concept in [2], $\oplus$ -co-coatomically supplemented modules are presented. A module W is designated as $\oplus$ -*co-coatomically supplemented* if each co-coatomic submodule of W has a supplement that is a direct summand of W [3]. In both studies, the properties of these modules are examined in detail.

For convenience, we denote the classes of Rad-cc-semiperfect modules, (am-
ply) Rad-cc-supplemented modules, $\oplus$ -Rad-cc-supplemented modules and totally Rad-cc-supplemented modules briefly as $\mathcal{SP}_{\text{Rad-cc}}$, $(\mathcal{AS}_{\text{Rad-cc}})$, $\mathcal{S}_{\text{Rad-cc}}$, $\mathcal{S}_{\oplus\text{-Rad-cc}}$, and $\mathcal{TS}_{\text{Rad-cc}}$ in our work.

Inspired by the definitions given above, in Section 2, we introduce the class $\mathcal{S}_{\text{Rad-cc}}$ of modules W where each co-coatomic submodule has a Rad-supplement in W , if each co-coatomic submodule of W has a Rad-supplement in W . Each cofinite submodule of a module is co-coatomic, as finitely generated modules are also coatomic. So every module which belongs to the class $\mathcal{S}_{\text{Rad(cc)}}$ is cofinitely Rad-supplemented. A module W is called as belonging to the class of $\mathcal{TS}_{\text{Rad-cc}}$ if every submodule of W is in $\mathcal{S}_{\text{Rad-cc}}$. We present the main results related with these concepts. We have shown that if the W module is semilocal with $\text{Rad}(W) \ll W$, the concept of the Rad-supplemented modules coincides with the concept we have defined. Furthermore it has been shown that each w -local module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$, any factor module on finite sum of these modules also belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. It is proven that the necessary and sufficient condition for every module over a left V -ring to belong to the class $\mathcal{S}_{\text{Rad-cc}}$ is that the module has to be semisimple. We also give an example of a module that belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ but is not Rad-supplemented. It is obtained that if the factor module is formed by taking a radical submodule of $W \in \mathcal{S}_{\text{Rad-cc}}$, then the module belongs also to the

class $\mathcal{S}_{\text{Rad-cc}}$. In the last part of this section, we build rings which belong to the class $\mathcal{S}_{\text{Rad-cc}}$ as modules over themselves. A characterization of this ring is obtained using the concepts from the literature. It is proved that each module, which belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ defined over a semilocal ring, is generalized semiperfect if and only if the ring is left perfect. Furthermore, it is obtained that the concepts of modules belonging to the class $\mathcal{S}_{\text{Rad-cc}}$ and of supplemented modules coincide for every coatomic module defined over a commutative Noetherian semilocal ring.

In Section 3, we study modules which belong to the class $\mathcal{S}_{\text{Rad-cc}}$ over commutative domains. In Section 4, modules belonging to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ are investigated and some properties of these modules defined are obtained. It is proved that each w -local module belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ and, as a result, each direct summand of the w -local module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. Also, equivalent conditions are given for an indecomposable module W to belong to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$. In addition, it is shown that the factor module formed by taking the fully invariant submodule of a module W which belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ also belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$. In addition the module concept introduced in this section coincides with semisimple module over a left V -ring. Moreover, some properties of this module concept defined over Dedekind domains are also proved. In Section 5, modules that belong to the class $\mathcal{SP}_{\text{Rad-cc}}$ are defined with the help of a generalized projective cover. Again, equivalent conditions are given for a module to be co-coatomically semiperfect. Based on previous studies, similar features in the literature have been studied for this module concept.

2. MODULE CLASS $\mathcal{S}_{\text{Rad-cc}}$

Definition 2.1. If for each co-coatomic submodule $A \leq W$, there exists $T \leq B$, such that T is a Rad-supplement of A , the module W is referred as belonging to the class $\mathcal{S}_{\text{Rad-cc}}$. Furthermore, W is referred as belonging to the class $\mathcal{AS}_{\text{Rad-cc}}$ if, for any $B \leq W$ such that $W = A + B$ for a co-coatomic submodule $A \leq W$, there exists $T \leq B$, such that T is a Rad-supplement of A .

Proposition 2.1. *Let W be a semilocal module satisfying the condition*

$$\text{Rad}(W) \ll W.$$

Then $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if W is Rad-supplemented.

Proof. ($\Leftarrow$) It is clear.

($\Rightarrow$) Let W belong to the class $\mathcal{S}_{\text{Rad-cc}}$ and $A \leq W$. Since W is semilocal, then $\frac{W}{\text{Rad}(W)}$ is semisimple and so it is coatomic. Additionally,

$$\frac{\frac{W}{\text{Rad}(W)}}{\frac{A+\text{Rad}(W)}{\text{Rad}(W)}} \cong \frac{W}{A + \text{Rad}(W)}$$

is coatomic as it is isomorphic to a factor module of a coatomic module which is coatomic. Hence, $A + \text{Rad}(W) \leq W$ is co-coatomic. Based on the hypothesis, a Rad-supplement $T \leq A + \text{Rad}(W)$ in W exists satisfying that $W = [A + \text{Rad}(W)] + T$ and $[A + \text{Rad}(W)] \cap T \leq \text{Rad}(T)$. Following that, $W = A + T$ is obtained as $\text{Rad}(W) \ll W$. Additionally, we have $A \cap T \leq [A + \text{Rad}(W)] \cap T \leq \text{Rad}(T)$. Hence, W is a Rad-supplemented module. $\square$

As mentioned in [6], a module W is called *w-local* if it possesses only one exact maximal submodule. It is noteworthy that every local module is *w-local*. Both of these definitions coincide on modules over a left max ring.

Lemma 2.1. *Any w-local module belongs to the class $\mathcal{S}_{\text{Rad(cc)}}$.*

Proof. Let W be a *w-local* module, and let $A \leq W$ be co-coatomic. Since $\frac{W}{A}$ is coatomic, there exists a maximal submodule $\frac{P}{A} \leq \frac{W}{A}$ containing every proper submodule of $\frac{W}{A}$. Consequently, $P \leq W$ is maximal such that $A \leq P$. Following this, we have that $A \leq \text{Rad}(W)$ and W itself is a Rad-supplement of A in W . This implies that $W \in \mathcal{S}_{\text{Rad-cc}}$. $\square$

Proposition 2.2. *The class $\mathcal{S}_{\text{Rad-cc}}$ is closed under taking quotients.*

Proof. Let W be a module belonging to the class $\mathcal{S}_{\text{Rad-cc}}$ and assume that $A \leq W$. For any co-coatomic submodule $\frac{N}{A} \leq \frac{W}{A}$,

$$\frac{W}{N} \cong \frac{\frac{W}{A}}{\frac{N}{A}}$$

is coatomic. Thus, $N \leq W$ is co-coatomic. Consequently, there exists a Rad-supplement submodule $T \leq W$ of N . By applying Lemma 2.3 in [18], $\frac{T+A}{A}$ is a Rad-supplement of $\frac{N}{A}$ in $\frac{W}{A}$. $\square$

Proposition 2.3. *Let W belong to the class $\mathcal{S}_{\text{Rad-cc}}$, let $A \leq W$ such that $\text{Rad}(W) \leq A$. If $\text{Rad}(\frac{W}{A}) = 0$, then each co-coatomic submodule of $\frac{W}{A}$ is a direct summand.*

P r o o f. Consider $\frac{N}{A} \leq \frac{W}{A}$ as co-coatomic. Then

$$\frac{\frac{W}{A}}{\frac{N}{A}} \cong \frac{W}{N}$$

is coatomic. Thus, $N \leq W$ is co-coatomic. Under the given assumption, consider a submodule T of W where $W = N + T$ and $N \cap T \leq \text{Rad}(T)$. From [18], Lemma 2.3, $\frac{T+A}{A}$ is a Rad-supplement of $\frac{N}{A}$ in $\frac{W}{A}$. Since $\text{Rad}(W) \leq A$, it follows that $(\frac{T+A}{A}) \cap \frac{N}{A} = 0$. Hence, $\frac{N}{A} \leq \oplus \frac{W}{A}$. $\square$

Corollary 2.1. *Let W belong to the class $\mathcal{S}_{\text{Rad-cc}}$. It can be stated that each co-coatomic submodule of $\frac{W}{\text{Rad}(W)}$ is a direct summand of $\frac{W}{\text{Rad}(W)}$.*

Now, we present the following useful lemma, which allows us to prove that the finite sum of modules belonging to the class $\mathcal{S}_{\text{Rad-cc}}$ also belongs to the class $\mathcal{S}_{\text{Rad-cc}}$.

Lemma 2.2. *Let W belong to the class $\mathcal{S}_{\text{Rad-cc}}$ with $A \leq W$. If T is a co-coatomic submodule of W such that $A + T$ is a Rad-supplement in W , then T has a Rad-supplement in W .*

P r o o f. If Y is a Rad-supplement of $A + T$ in W , then we can write $W = (A + T) + Y$ and $(A + T) \cap Y \leq \text{Rad}(Y)$. $A \cap (T + Y)$ is a co-coatomic submodule of A , because

$$\frac{\frac{W}{T}}{\frac{T+Y}{T}} \cong \frac{A + (T + Y)}{T + Y} \cong \frac{A}{A \cap (T + Y)}$$

is coatomic. Let U be a Rad-supplement of $A \cap (T + Y)$ in A . Then

$$A = [A \cap (T + Y)] + U$$

and

$$[A \cap (T + Y)] \cap U = U \cap (T + Y) \leq \text{Rad}(U).$$

Further, we have

$$W = (A + T) + Y = [A \cap (T + Y)] + U + T + Y = T + (Y + U)$$

and

$$\begin{aligned} T \cap (Y + U) &\leq [Y \cap (T + U)] + [U \cap (T + Y)] \\ &\leq [Y \cap (T + A)] + [U \cap (T + Y)] \\ &\leq \text{Rad}(Y) + \text{Rad}(U) \\ &\leq \text{Rad}(Y + U). \end{aligned}$$

As desired, $Y + U$ is a Rad-supplement of T in W . $\square$

Theorem 2.1. *The class $\mathcal{S}_{\text{Rad-cc}}$ is closed under taking finite sums.*

Proof. Let W_1 and W_2 belong to the class $\mathcal{S}_{\text{Rad-cc}}$. It suffices to prove that $W = W_1 + W_2$ belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ when both modules W_1 and W_2 belong to the class $\mathcal{S}_{\text{Rad-cc}}$. Consider A as a co-coatomic submodule of W . Then $W = W_1 + W_2 + A$ can be written, and clearly 0 is a trivial Rad-supplement of W . Additionally, since $W_2 + A$ is co-coatomic in W and W_1 belongs to the class $\mathcal{S}_{\text{Rad-cc}}$, it follows that $W_2 + A$ has a Rad-supplement in W by Lemma 2.2. By applying Lemma 2.2 again, it is obvious that A has a Rad-supplement in W , since A is co-coatomic and W_2 belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. Finally, we conclude that $W \in \mathcal{S}_{\text{Rad-cc}}$. $\square$

Recall from [21] that a module N is called *finite W -generated* if an epimorphism from $W^{(I)}$ to N exists for some finite set I . The next corollary is obtained directly from Proposition 2.2 and Theorem 2.1.

Corollary 2.2. *The class $\mathcal{S}_{\text{Rad-cc}}$ is closed under taking W -generated modules for each W belonging to the class.*

Remember that a ring S is a *left V -ring* if each simple S -module is injective. This condition is equivalent to every S -module W having a zero radical and it is also equivalent to each proper submodule being the intersection of maximal submodules as noted in [14].

Proposition 2.4. *If S is a left V -ring, then the class $\mathcal{S}_{\text{Rad-cc}}$ consists of semisimple modules if and only if W is semisimple.*

Proof. Since left V -rings are characterized by the condition that the radical of every module is zero, then every nonzero module is coatomic.

($\Rightarrow$) Thus, $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if each submodule of W has a Rad-supplement. Since the radical of every module is zero, Rad-supplements are direct summands, meaning every submodule of W is a direct summand.

($\Leftarrow$) Let W be a semisimple module. Since each submodule of W is a direct summand, each submodule has a Rad-supplement. $\square$

Corollary 2.3. *If S is a left V -ring, then the class $\mathcal{S}_{\text{Rad-cc}}$ is closed under taking direct sums.*

Proof. Let S be a left V -ring and $\{W_i; i \in I\}$ be a class of S -modules which also belongs to the class $\mathcal{S}_{\text{Rad-cc}}$ for any index set I . It is a fact that any direct sum of semisimple modules is semisimple, as stated in [11], Corollary 8.1.5. On the other hand, semisimple modules and those modules belonging to the class $\mathcal{S}_{\text{Rad-cc}}$ coincide over left V -rings. Therefore, the claim is clear. $\square$

Corollary 2.4. *Let W be a module over a left V -ring. Then $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if W is the sum of local submodules of itself.*

Proof. This follows from [1], Proposition 3.6, as $\text{Soc}(W) = \text{Loc}(W)$ where $\text{Loc}(W)$ is the sum of local submodules of W . $\square$

Let us consider the $\mathbb{Z}$ -module $\frac{\mathbb{Z}}{8\mathbb{Z}}$ which belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. In contrast, $\mathbb{Z}\mathbb{Z}$ does not belong to the class $\mathcal{S}_{\text{Rad-cc}}$. Hence, this example emphasizes that a module W need not belong to the class $\mathcal{S}_{\text{Rad-cc}}$ while a factor module of W can still belong to the class $\mathcal{S}_{\text{Rad-cc}}$.

Now, we investigate the suitable conditions that make the implication given above be valid.

Theorem 2.2. *Let W be a module and let A be a submodule such that $A \in \mathcal{S}_{\text{Rad-cc}}$ of W where $\frac{W}{A}$ is radical. Then $W \in \mathcal{S}_{\text{Rad-cc}}$.*

Proof. Let $N \leq W$ be co-coatomic. Thus, so is $\frac{W}{N+A} \cong \frac{\frac{W}{N}}{\frac{N}{N+A}}$. By the assumption, $\frac{W}{N+A}$ is radical, as $\frac{W}{A}$ is radical. Therefore, we can express $W = N + A$. Since $A \in \mathcal{S}_{\text{Rad-cc}}$ and W has the trivial Rad-supplement $\{0\}$, N has a Rad-supplement in W by Lemma 2.2. $\square$

Corollary 2.5. *Let W be a module. Suppose that $\frac{W}{\text{Soc}(W)}$ has no nonzero maximal submodule. Then $W \in \mathcal{S}_{\text{Rad-cc}}$.*

It is evident that each Rad-supplemented module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. However, the converse is not universally true. To illustrate this, let us provide an example confirming this assertion.

Example 2.1. Consider the non-Noetherian commutative ring formed by the direct product $\prod_{i \geq 1}^{\infty} S_i$, where each S_i is a field denoted as S . Suppose that M is a subring of the given non-Noetherian commutative ring, consisting of all sequences $(s_i)_{i \in \mathbb{N}}$ where there exist $s \in S$ and $j \in \mathbb{N}$ with $s_i = s$ for all $i \geq j$. Let $W =_M M$. Given that $\text{Soc}(W)$ is a maximal submodule of W , it follows that the quotient module $\frac{W}{\text{Soc}(W)}$ does not possess any nonzero maximal submodule. According to Corollary 2.5, this leads to the conclusion that $W \in \mathcal{S}_{\text{Rad-cc}}$. However, it is worth noting that W is not Rad-supplemented, as demonstrated in [17], Example 2.5.

Proposition 2.5. *Let $W \in \mathcal{S}_{\text{Rad-cc}}$. If W includes a maximal submodule, then it also contains a w -local submodule.*

Proof. Consider a maximal submodule P of W . It is evident that P is also a co-coatomic submodule of W . Based on the assumption, there exists $T \leq W$ where $W = P + T$ and $P \cap T \subseteq \text{Rad}(T)$. Hence, T is a w -local by [6], Lemma 3.3. $\square$

Now, we give a characterization of the module class $\mathcal{S}_{\text{Rad-cc}}$ based on radical submodules.

Theorem 2.3. *Let W be a module and $A \leq W$ be radical. If $\frac{W}{A} \in \mathcal{S}_{\text{Rad-cc}}$, then $W \in \mathcal{S}_{\text{Rad-cc}}$.*

Proof. Consider any co-coatomic submodule N of W . It follows that $\frac{W}{N+A} \cong \frac{\frac{W}{N+A}}{\frac{N}{N+A}} \cong \frac{\frac{W}{A}}{\frac{N+A}{A}}$ is coatomic, and so $\frac{N+A}{A} \leq \frac{W}{A}$ is co-coatomic. By the assumption, there is a Rad-supplement submodule of $\frac{N+A}{A}$, say $\frac{T}{A}$, in $\frac{W}{A}$ where $\frac{W}{A} = \frac{N+A}{A} + \frac{T}{A}$ and $\frac{(N+A) \cap T}{A} \leq \text{Rad}(\frac{T}{A}) = \frac{\text{Rad}(T)}{A}$ as $A = \text{Rad}(A) \leq \text{Rad}(T)$. Therefore, $W = N + T$ and $N \cap T \leq \text{Rad}(T)$. Hence, T is a Rad-supplement of N in W . $\square$

Recall that $P(W)$ denotes the sum of all radical submodules of W . Thus, $P(W)$ represents the largest radical submodule contained within W .

Corollary 2.6. *Let W be a module, and consider submodules N and K where $N \subseteq K \subseteq W$. Then $K \in \mathcal{S}_{\text{Rad-cc}}$ if and only if $\frac{K}{P(N)} \in \mathcal{S}_{\text{Rad-cc}}$.*

Proof. The conclusion follows from Theorem 2.3. $\square$

Corollary 2.7. *Let W be a module and $N \subseteq W$. Then $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if $\frac{W}{P(N)} \in \mathcal{S}_{\text{Rad-cc}}$. In particular, $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if $\frac{W}{P(W)} \in \mathcal{S}_{\text{Rad-cc}}$.*

Proof. This is clear by Corollary 2.6. $\square$

Let us give another characterization for a ring R to belong to the class $\mathcal{S}_{\text{Rad-cc}}$, using the fact that every finitely generated module is coatomic.

Theorem 2.4. *For a ring S , the following statements are equivalent.*

- (1) ${}_S S$ is Rad-supplemented.
- (2) ${}_S S \in \mathcal{S}_{\text{Rad-cc}}$.
- (3) ${}_S S$ is cofinitely Rad-supplemented.

Proof. The proof is clear. $\square$

For a finitely generated module W , we can extend the previous theorem in the following manner.

Theorem 2.5. *Let W be a finitely generated module. The following statements are equivalent.*

- (1) W is Rad-supplemented.
- (2) $W \in \mathcal{S}_{\text{Rad-cc}}$.
- (3) W is cofinitely Rad-supplemented.

Proof. The proof is straightforward. $\square$

Proposition 2.6. *The following statements are equivalent for a coatomic module W :*

- (1) $W \in \mathcal{S}_{\text{Rad-cc}}$.
- (2) W is Rad-supplemented.

Proof. (1) $\Rightarrow$ (2) Let W be a coatomic module. Every submodule of W is co-coatomic, as each factor module of W is coatomic. By the hypothesis, each submodule of W has a Rad-supplement in W . Consequently, W is Rad-supplemented.

(2) $\Rightarrow$ (1) This is clear. $\square$

In accordance with [22], an epimorphism $\varphi: F \rightarrow W$ is called a *generalized cover* if $\ker(\varphi) \subseteq \text{Rad}(F)$, and a generalized cover $\varphi: F \rightarrow W$ is referred to as a *projective cover* if F is projective. A *generalized semiperfect* module is a module whose factor modules have a generalized projective cover. It has been established in [22], Theorem 2.2 that each generalized semiperfect module is Rad-supplemented. Given that every Rad-supplemented module is Rad-cc-supplemented, it follows that every semiperfect module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$.

Proposition 2.7. *Let S be a semilocal ring. Every left S -module, which belongs to the class $\mathcal{S}_{\text{Rad-cc}}$, is generalized semiperfect if and only if S is left perfect.*

Proof. ($\Leftarrow$) This is evident.

($\Rightarrow$) Conversely, consider W as a left S -module with $W = \text{Rad}(W)$. Since $W \in \mathcal{S}_{\text{Rad(cc)}}$, it is generalized semiperfect by the hypothesis. So there is an onto homomorphism $\varphi: P \rightarrow W$ where P is projective. Since $\ker(\varphi) \subseteq \text{Rad}(P) \neq P$, $\frac{P}{\ker(\varphi)} \cong W$ has a maximal submodule. Since $W = \text{Rad}(W)$, this implies that $W = 0$. Thus, S is left perfect by [21], 42.9. $\square$

From [21], we retrieve the definition that an S -module W is called *reduced* if every submodule of W contains a maximal submodule, denoted as $P(W) = 0$. It is important to note that for every coatomic S -module W , $\text{Rad}(W) \ll W$. Additionally, it is worth mentioning that every coatomic S -module is reduced over commutative Noetherian rings.

Proposition 2.8. *Let W be a coatomic module over a commutative Noetherian ring. Then $W \in \mathcal{S}_{\text{Rad-cc}}$ if and only if W is supplemented.*

Proof. The implication can be deduced from Lemma 4 in reference [19]. $\square$

Proposition 2.9. *Let W be a module over a commutative Noetherian ring and A be a co-coatomic submodule of W . If $W \in \mathcal{S}_{\text{Rad(cc)}}$, then $A \in \mathcal{S}_{\text{Rad-cc}}$.*

Proof. Consider $W \in \mathcal{S}_{\text{Rad-cc}}$. We can deduce from Proposition 2.2 that $\frac{W}{A} \in \mathcal{S}_{\text{Rad-cc}}$. Since $A \leq W$ is co-coatomic, the factor module $\frac{W}{A}$ is coatomic. According to Proposition 2.8, $\frac{W}{A}$ is Rad-supplemented. Furthermore, due to the coatomic $\frac{W}{A}$, it is also reduced. Consequently, by [19], Proposition 2, we can infer that A is Rad-supplemented. Hence, $A \in \mathcal{S}_{\text{Rad-cc}}$. $\square$

Remark 2.1. In view of Theorem 2.2 and Proposition 2.9, we obtain that the class of the module class $\mathcal{S}_{\text{Rad-cc}}$ is not closed under extensions, in general. Let $0 \rightarrow K \rightarrow W \rightarrow A \rightarrow 0$ be a short exact sequence. If $K \leq W$ is co-coatomic and A is radical, then the module class $\mathcal{S}_{\text{Rad-cc}}$ is closed under extensions.

Corollary 2.8. *For a module W over a commutative Noetherian ring, the following statements are equivalent:*

- (1) $W \in \mathcal{S}_{\text{Rad-cc}}$.
- (2) Each maximal submodule of W belongs to the class $\mathcal{S}_{\text{Rad-cc}}$.
- (3) Each cofinite submodule of W belongs to the class $\mathcal{S}_{\text{Rad-cc}}$.

Proof. (1) $\Rightarrow$ (3) Consider a cofinite submodule A of W . Thus, the quotient module $\frac{W}{A}$ is finitely generated. Since $\frac{W}{A}$ is coatomic, A is a co-coatomic submodule of W . The proof follows from Proposition 2.9.

(3) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Consider W as the sum of A and B , where A and B are maximal submodules of W . By the hypothesis, we have that A and B belong to the class $\mathcal{S}_{\text{Rad-cc}}$. Thus, by Theorem 2.2, $W \in \mathcal{S}_{\text{Rad-cc}}$. If W is w -local, $W \in \mathcal{S}_{\text{Rad-cc}}$ by Lemma 2.1. $\square$

Definition 2.2. We define $\mathcal{TS}_{\text{Rad-cc}} = \{M; \text{ for all } N \leq M, N \in \mathcal{S}_{\text{Rad-cc}}\}$.

It is obvious that any semisimple module belongs to the class $\mathcal{S}_{\text{Rad-cc}}$. In the following proposition, we explore a hereditary class and cohereditary class of a module.

Proposition 2.10. *The class $\mathcal{TS}_{\text{Rad-cc}}$ is closed under taking submodules and quotients.*

Proof. Let $W \in \mathcal{TS}_{\text{Rad-cc}}$ and $K \leq A \leq W$. Then $K \in \mathcal{S}_{\text{Rad(cc)}}$ by hypothesis. Therefore, A belongs to the class $\mathcal{TS}_{\text{Rad-cc}}$. Consider any factor module $\frac{W}{A}$ of W where $\frac{K}{A} \leq \frac{W}{A}$. So $K \leq W$. Since $W \in \mathcal{TS}_{\text{Rad-cc}}$, we have $K \in \mathcal{S}_{\text{Rad-cc}}$. Consequently $\frac{W}{A} \in \mathcal{TS}_{\text{Rad-cc}}$. $\square$

3. MODULE CLASS $\mathcal{S}_{\text{Rad-cc}}$ OVER COMMUTATIVE DOMAINS

In this part, we will take all rings as commutative domains, unless specified otherwise.

Remember that $T(W)$ is represented by the set of whole elements $u \in W$ for which there is a nonzero element $s \in S$ with $su = 0$, i.e., $\text{Ann}(u) \neq 0$. Here, $T(W)$ is called the *torsion submodule* of W . If $W = T(W)$, W is called a *torsion module*, while W is referred to as *torsion-free* if $T(W) = 0$.

Proposition 3.1. *Let W be an S -module over a nonsemilocal commutative domain. If $W \in \mathcal{TS}_{\text{Rad-cc}}$, then it follows that W is a torsion module.*

Proof. Assume that $0 \neq x \in W$. We show that $\text{Ann}(x) \neq 0$. Suppose that $\text{Ann}(x) = 0$, i.e., $S \cong Sx$. Since $W \in \mathcal{TS}_{\text{Rad-cc}}$, the S -module $Sx \in \mathcal{TS}_{\text{Rad-cc}}$. Then ${}_S S \in \mathcal{S}_{\text{Rad-cc}}$. Thus, Theorem 2.4 establishes that ${}_S S$ is Rad-supplemented. Additionally, according to [19], Proposition 4, it can be deduced that S is semilocal. This contradicts our assumption. Consequently, W is a torsion module. $\square$

Corollary 3.1. *Let $W \in \mathcal{TS}_{\text{Rad-cc}}$ over a nonsemilocal Dedekind domain. Then W is torsion.*

It is obvious that there is a decomposition $W = \bigoplus_{P \in \Omega} T_P(W)$ for a torsion module W over a Dedekind domain S . Here, Ω denotes the set of all maximal (prime) ideals of S , and $T_P(W)$, a submodule of W , represents the set of all elements $x \in W$ for which there exists a positive integer n such that $p^n x = 0$.

Lemma 3.1. *Let W be a coatomic module over a nonsemilocal Dedekind domain. Then W belongs to $\mathcal{S}_{\text{Rad-cc}}$ if and only if $\frac{W}{P(W)}$ is torsion and each p -primary part of $\frac{W}{P(W)} \in \mathcal{S}_{\text{Rad-cc}}$.*

Proof. It clear by Proposition 2.26 and Theorem 3.1 in [23]. $\square$

Corollary 3.2. *Let $W \in \mathcal{S}_{\text{Rad-cc}}$ be a coatomic torsion-free over a nonlocal Dedekind domain. Then it is radical.*

Proof. Given that S is a Dedekind domain, there exists $K \leq_{\oplus} W$ where $\frac{W}{P(W)} \cong K$. According to [1], Lemma 4.4, $P(W)$ is injective. As W is a torsion-free S -module, W is radical by Lemma 3.1. $\square$

Theorem 3.1. *If the ring S is a Dedekind domain, then $\mathcal{TS}_{\text{Rad-cc}} = \{M; M \text{ is a torsion module whose all of its } p\text{-primary parts belonging to } \mathcal{S}_{\text{Rad-cc}}\}$.*

Proof. Let W be a module over a nonsemilocal Dedekind domain and $W \in \mathcal{TS}_{\text{Rad-cc}}$. ($\Rightarrow$) This is obvious by Corollary 2.7.

($\Leftarrow$) Consider W as a torsion module where each p -primary part of W belongs to the class $\mathcal{TS}_{\text{Rad(cc)}}$. Let $K \leq N \leq W$ where $K \leq N$ is co-coatomic. Since $W = \bigoplus_{P \in \Omega} T_P(W)$, $N = \bigoplus_{P \in \Omega} (N \cap T_P(W))$ and $K = \bigoplus_{P \in \Omega} (K \cap T_P(W))$. It follows from $\frac{N}{K} \cong \bigoplus_{P \in \Omega} \left(\frac{N \cap T_P(W)}{K \cap T_P(W)} \right)$ that $K \cap T_P(W)$ is co-coatomic submodule of $N \cap T_P(W)$ for every $P \in \Omega$. By the hypothesis, $K \cap T_P(W)$ has a Rad-supplement X_P in $N \cap T_P(W)$. Thus,

$$N \cap T_P(W) = [K \cap T_P(W)] + X_P$$

and

$$[K \cap T_P(W)] \cap X_P = K \cap X_P \subseteq \text{Rad}(X_P).$$

Let $X = \bigoplus_{P \in \Omega} X_P$. Then $N = K + X$. Since $K \cap X_P \subseteq \text{Rad}(X_P)$ for every $P \in \Omega$,

$$K \cap X = \left[\bigoplus_{P \in \Omega} K \cap T_P(W) \right] \cap \left(\bigoplus_{P \in \Omega} X_P \right) \subseteq \text{Rad}(X)$$

by [11], Corollary 9.1.5. So, $N \in \mathcal{S}_{\text{Rad(cc)}}$. Therefore, $W \in \mathcal{TS}_{\text{Rad-cc}}$. $\square$

In the example provided below, it is demonstrated that the module classes $\mathcal{TS}_{\text{Rad-cc}}$ and $\mathcal{S}_{\text{Rad-cc}}$ are not necessarily closed under extension in general.

Example 3.1. In Example 2.1, the modules $\text{Soc}(W)$ and $\frac{W}{\text{Soc}(W)} \in \mathcal{S}_{\text{Rad-cc}}$ but W does not.

4. MODULE CLASS $\mathcal{S}_{\oplus\text{-Rad-cc}}$

Definition 4.1. $\mathcal{S}_{\oplus\text{-Rad-cc}}$ is defined as the class $\{W; \text{ for all co-coatomic } A \leq W, A \text{ has a Rad-supplement that is a direct summand of } W\}$.

With this definition, we generalize the concept of the module class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ which is introduced in [3].

It is obvious that the module class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ is a subclass of the module class $\mathcal{S}_{\text{Rad-cc}}$.

Furthermore, we have the following relations between some specific variations of supplemented modules and our modules from the first part of the paper, owing to the definitions given above.

Proposition 4.1. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ with $\text{Rad}(W) \ll W$. Then W is $\oplus$ -co-coatomically supplemented.*

Proof. Consider any co-coatomic submodule A of W . By the assumption, there exists $B \leq_{\oplus} W$ such that $W = A + B$, $A \cap B \leq \text{Rad}(B)$. Since $\text{Rad}(B) \leq \text{Rad}(W) \ll W$, $B \leq_{\oplus} W$, we have also $A \cap B \ll B$ by [21], 19.3 (5). Hence, W is $\oplus$ -co-coatomically supplemented. $\square$

Corollary 4.1. *Assume W is a coatomic module. Then, $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ if and only if W is $\oplus$ -co-coatomically supplemented.*

Proof. Since every coatomic module has a small radical, the proof follows from Proposition 4.1. $\square$

Proposition 4.2. *Let W be a projective module. The following statements are equivalent:*

- (1) $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.
- (2) *For each co-coatomic submodule A of W , there exist $Y \leq A$ and $B \leq W$ such that $W = Y \oplus B$ and $A \cap B \leq \text{Rad}(B)$.*

Proof. (1) $\Rightarrow$ (2) If A is a co-coatomic submodule of W , then there exists a Rad-supplement B of A such that is a direct summand of W . Then $A + B = W$, $A \cap B \leq \text{Rad}(B)$ and $W = Z \oplus B$ for some submodule Z . Since $\frac{A}{A \cap B} \cong \frac{A+B}{B} = \frac{W}{B} \cong Z$, then $\frac{A}{A \cap B}$ is a projective module. Hence the exact sequence $0 \rightarrow A \cap B \rightarrow A \rightarrow \frac{A}{A \cap B} \rightarrow 0$ splits, and so $A \cap B$ is a direct summand of A . Let $A = (A \cap B) \oplus Y$. Therefore, $Y \oplus B = Y + (A \cap B) + B = A + B = W$.

(2) $\Rightarrow$ (1) If A and B are as in (2), then B is a Rad-supplement of A , and a direct summand of W . $\square$

Theorem 4.1. *The following statements are equivalent for a finitely generated module W .*

- (1) W is cofinitely $\oplus$ -Rad-supplemented.
- (2) W is $\oplus$ -Rad-supplemented.
- (3) $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) is clear from [9], Proposition 2.2. $\square$

Lemma 4.1. *If W is a radical module, then $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. Consider a co-coatomic submodule A of W . Since $\text{Rad}(W) = W$, $\text{Rad}(W)$ is a Rad-supplement of A in W which is a direct summand of W , as desired. $\square$

Proposition 4.3. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. If W has a maximal submodule, then W includes a w -local direct summand.*

Proof. This follows from [16], Proposition 2.4. $\square$

Corollary 4.2. *If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ has a small radical, then W has a local direct summand.*

Lemma 4.2. *If W is a w -local module, then $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. Assume that A is a co-coatomic submodule of W . Consequently, the factor module $\frac{W}{A}$ is also coatomic. Therefore, there exists a maximal submodule $\frac{P}{A} \leq \frac{W}{A}$ containing all the proper submodules of $\frac{W}{A}$. Since W is w -local, the radical $P = \text{Rad}(W)$ stands out as the unique maximal submodule of W that encompasses A . Consequently, W is as a Rad-supplement of A in W , constituting a direct summand. This concludes the proof. $\square$

Corollary 4.3. *Every direct summand of an w -local module belongs to the class $\mathcal{S}_{\oplus\text{-Rad(cc)}}$.*

Proof. The clarity of the statement arises from the implications of Lemma 4.1 and Lemma 4.2, which establish that every direct summand of a w -local module is to either a radical or w -local. This observation aligns with the proposition stated in [16], Proposition 2.6. $\square$

Proposition 4.4. *The following statements are equivalent for an indecomposable module W :*

- (1) $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.
- (2) *Each maximal submodule of W has a Rad-supplement, that is a direct summand of W .*
- (3) $\text{Rad}(W) = W$ or W is w -local.

Proof. (1) $\Rightarrow$ (2) Consider any maximal submodule P of W . Since $\frac{W}{P}$ is cyclic, and thus coatomic, then $P \leq W$ is co-coatomic. Therefore, by the assumption, P has a Rad-supplement which is a direct summand of W .

(2) $\Rightarrow$ (3) Assume that $\text{Rad}(W) \neq W$. Then there exists a maximal submodule $P \leq W$ having a Rad-supplement $T \leq_{\oplus} W$, such that $W = P + T$, $P \cap T \leq \text{Rad}(T)$ and $W = T \oplus T_1$ for some submodule T_1 of W . Then, $T = 0$ or $T = W$ as W is indecomposable. Suppose that $T = 0$. This case contradicts that P is a maximal submodule of W . So, $T = W$. Thus, $P \leq \text{Rad}(W)$ is obtained, and this implies that $P = \text{Rad}(W)$. Hence, W is an w -local module.

(3) $\Rightarrow$ (1) This is clear from Lemma 4.1 and Lemma 4.2. $\square$

Corollary 4.4. *Let W be an indecomposable module that is not radical. Then W is w -local if and only if $W \in \mathcal{S}_{\oplus\text{-Rad(cc)}}$.*

Lemma 4.3. *Let W be a module and $A \leq W$. If A is an w -local, then X is a Rad-supplement of a proper co-coatomic submodule T of W such that $W = A + T$.*

Proof. Consider T as a co-coatomic submodule of W such that $W = A + T$. Then $\frac{W}{T} \cong \frac{A}{A \cap T}$ is coatomic and so $\frac{A}{A \cap T}$ has a maximal submodule. As A is w -local, $\text{Rad}(A)$ is the only maximal submodule of A containing $A \cap T$. Hence, A serves as a Rad-supplement of T in W . $\square$

Proposition 4.5. *Let W be a module and $A \leq W$. If A is a co-coatomic submodule of W and $\{S_i\}_{i=1}^n$ is the family of w -local submodules of W where T is a Rad-supplement of $A + \sum_{i=1}^n S_i$ in W , then $T + \sum_{i \in I} S_i$ is a Rad-supplement of A in W where $I \subseteq \{1, 2, \dots, n\}$.*

Proof. Let us point out that each submodule of W containing A is also co-coatomic in W . Using this fact, the proposition can be proved via Lemma 4.3 and [9], Lemma 2.13 by repeated operations. $\square$

Now, we investigate the properties of direct sums of modules which belong to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.

Proposition 4.6. *Let A and B belong to $\mathcal{S}_{\oplus\text{-Rad-cc}}$. Then so does $W = A \oplus B$.*

Proof. Consider Y as a co-coatomic submodule of W . By the isomorphism,

$$\frac{\frac{W}{Y}}{\frac{Y \cap A}{Y}} \cong \frac{W}{(Y + A)} = \frac{(Y + A) + B}{Y \cap A} \cong \frac{B}{B \cap (Y + A)}$$

is coatomic. Thus, $B \cap (Y + A)$ is a co-coatomic submodule of B . From the assumption, there is a direct summand H of B where

$$B = [B \cap (Y + A)] + H$$

and

$$[B \cap (Y + A)] \cap H = (Y + A) \cap H \leq \text{Rad}(H).$$

Now following this, we have

$$\begin{aligned} W = A + B &= A + [B \cap (Y + A)] + H = [(A + B) \cap (Y + A)] + H \\ &= [W \cap (Y + A)] + H = Y + A + H. \end{aligned}$$

Since $Y \leqslant W$ is co-coatomic, $\frac{W}{Y}$ is coatomic. As being coatomic is a property preserved under factor modules, we have that $A \cap (Y + H) \leqslant A$ is co-coatomic by using the isomorphism

$$\frac{A}{A \cap (Y + H)} \cong \frac{W}{Y + H} \cong \frac{\frac{W}{Y}}{\frac{Y + H}{Y}}.$$

Thus, there exists $T \leqslant_{\oplus} A$ where

$$\begin{aligned} A &= [A \cap (Y + H)] + T, \\ [A \cap (Y + H)] \cap T &= (Y + H) \cap T \leqslant \text{Rad}(T). \end{aligned}$$

Therefore,

$$W = A + (Y + H) = [A \cap (Y + H)] + T + (Y + H) = Y + (H \oplus T).$$

Furthermore

$$\begin{aligned} Y \cap (H + T) &\leqslant [H \cap (Y + T)] + [T \cap (Y + H)] \\ &\leqslant \text{Rad}(H) \oplus \text{Rad}(T) = \text{Rad}(H \oplus T). \end{aligned}$$

Therefore, $H \oplus T$ is a Rad-supplement of Y in W which is a direct summand of W . Consequently, $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. $\square$

Corollary 4.5. *A finite direct sum of modules which belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ remains in the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Corollary 4.6. *A finite direct sum of w -local modules also belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. The clarity of this statement is derived from the implications presented in Lemma 4.2 and Corollary 4.5. $\square$

Consider direct summands A and B of W such that $W = A + B$. If $A \cap B \leqslant_{\oplus} W$, then W is referred to as a module with the property (D_3) [15].

Proposition 4.7. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ with the condition (D_3) . In that case, each co-coatomic direct summand of W belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. Consider $A \leq_{\oplus} W$ as co-coatomic. Thus, $W = A \oplus X_1$ for some $X_1 \leq W$ where $\frac{W}{A} \cong X_1$ is coatomic. For any co-coatomic submodule N of A , we have $\frac{W}{N} = \frac{A \oplus X_1}{N} \cong \frac{A}{N} \oplus \frac{X_1}{X_1 \cap N}$. Since X_1 is coatomic, then $\frac{X_1}{X_1 \cap N}$ is coatomic. Therefore, $\frac{W}{N}$ is coatomic as a direct sum of two coatomic modules. Based on the hypothesis, there is a direct summand D of W such that $W = N + D$ and $N \cap D \leq \text{Rad}(D)$. It follows that $A \cap D$ is a direct summand of W since it has (D_3) . From the modularity we get $A = N + (A \cap D)$. Moreover, we have

$$N \cap (A \cap D) = N \cap D \leq \text{Rad}(D) \cap (A \cap D) = \text{Rad}(A \cap D)$$

from [8], 3.7 (3). Hence, $A \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. $\square$

A submodule A of W is called *closed* in W if it has no proper essential extension in W . According to [8], an S -module W is referred to as a *(UC)-module* if every submodule has a unique closure in W . W is called an *extending module* if each closed submodule of W is a direct summand of W .

Corollary 4.7. *Let W be a UC-extending module. If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$, then so is every co-coatomic direct summand of W .*

Now, we explore adequate conditions under which a factor module of modules, that belong to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$, also belongs to this class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.

Example 4.1. Let S be a local commutative ring whose ideals are not totally ordered with respect to inclusion. For $n \geq 2$, the finitely presented indecomposable module $W \cong \frac{S^{(n)}}{K}$ cannot be generated by fewer than n elements [20], Theorem 2. Here, $S^{(n)}$ belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ by [9], Example 4.1 but W does not belong to $\mathcal{S}_{\oplus\text{-Rad-cc}}$.

From the provided example, we observe that a submodule of a module which belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ may not belong to this class itself.

According to the information presented in reference [21], a submodule A of an S -module W is called *fully invariant* if $g(A) \leq A$ for all g in $\text{End}_S(W)$. Notably, $\text{Rad}(W)$ qualifies as a fully invariant submodule of W .

Now, we show that the implication given above is valid for the factor modules of a module constructed from fully invariant submodules.

Proposition 4.8. *Let W be a module and $A \leq W$ be a fully invariant submodule. If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$, then so does $\frac{W}{A}$.*

Proof. Let $\frac{N}{A} \leq \frac{W}{A}$ be co-coatomic. Thus, $\frac{W}{N} \cong \frac{\frac{W}{A}}{\frac{N}{A}}$ is coatomic, meaning that N is co-coatomic in W . By the hypothesis, there is a Rad -supplement K of N

in W which is a direct summand, that is $W = N + K$, $N \cap K \leq \text{Rad}(K)$ and $W = K \oplus K_1$. By [7], Lemma 3.4, $\frac{K+A}{A}$ is a Rad-supplement of $\frac{N}{A}$ in $\frac{W}{A}$. Moreover, as A is fully invariant, we have $A = (K \cap A) \oplus (K_1 \cap A)$ for some $K_1 \leq W$. Using this fact, we obtain

$$\begin{aligned} \frac{(K+A)}{X} \cap \frac{(K_1+A)}{X} &= \frac{K \oplus (K_1 \cap X)}{A} \cap \frac{K_1 \oplus (K \cap A)}{A} \\ &= \frac{(K \cap A) \oplus (((K_1 \cap A) + K) \cap K_1)}{A} \\ &= \frac{(K \cap A) \oplus (K_1 \cap A)}{A} = \frac{(K \oplus K_1) \cap A}{A} = 0. \end{aligned}$$

Therefore, $\frac{W}{A} = \frac{(K+A)}{A} \oplus \frac{(K_1+A)}{A}$ is obtained. Hence, $\frac{W}{A} \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. $\square$

Corollary 4.8. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. Then, $\frac{W}{\text{Rad}(W)}$ and $\frac{W}{\text{Soc}(M)}$ belong to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proposition 4.9. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ and $A \leq W$ be a fully invariant submodule. If $A \leq_{\oplus} W$ is co-coatomic, then $A \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. Let $Y \leq A$ be co-coatomic. Then $\frac{A}{Y}$ is coatomic. As A is co-coatomic direct summand of W , there is a submodule X_1 of W such that $W = A \oplus X_1$. It is clear that $\frac{W}{A} \cong X_1$ is also coatomic. Further, $\frac{W}{Y} = \frac{A \oplus X_1}{Y} = \frac{A}{Y} \oplus \frac{X_1+Y}{Y} \cong \frac{A}{Y} \oplus \frac{X_1}{X_1 \cap Y}$. As X_1 is co-coatomic, $\frac{X_1}{X_1 \cap Y}$ is coatomic. Since $\frac{A}{Y}$ and $\frac{X_1}{X_1 \cap Y}$ are coatomic, $\frac{W}{Y}$ is coatomic. So, Y is a co-coatomic submodule of W . From the assumption, there is a direct summand D of W such that $W = Y + D$, $Y \cap D \leq \text{Rad}(D)$ and $W = D \oplus D_1$. Since A is fully invariant submodule of W , we have $A = (A \cap D) \oplus (A \cap D_1)$. From modularity, we obtain $A = Y + (X \cap D)$ and $Y \cap (A \cap D) = Y \cap D \leq \text{Rad}(D) \leq \text{Rad}(W)$. As $A \cap D$ is a direct summand of W , we have $Y \cap (A \cap D) \leq \text{Rad}(A \cap D)$. Thus $A \cap D$ is a Rad-supplement of Y in A which is a direct summand. Hence, $A \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. $\square$

Proposition 4.10. *For a left V -ring, the class of $\mathcal{S}_{\oplus\text{-Rad-cc}}$ consists of semisimple modules.*

Proof. This is a consequence of Proposition 2.4. $\square$

Theorem 4.2. *S is a semiperfect ring if and only if each finitely generated free S -module belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. ($\Rightarrow$) Assume that S is a semiperfect ring. Then by [15], Corollary 4.2, ${}_S S$ is $\oplus$ -supplemented. It follows that ${}_S S$ is also $\oplus$ -Rad-supplemented. Thus, $S \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. Hence, any finitely generated free S -module $F = S^{(n)}$ belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ by Corollary 4.5.

($\Leftarrow$) By the hypothesis, ${}_S S \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ as a finitely generated free S -module. As each submodule of ${}_S S$ is co-coatomic, ${}_S S$ is $\oplus$ -Rad-supplemented and therefore, it is $\oplus$ -supplemented by [9], Proposition 2.2. Therefore, ${}_S S$ is supplemented. Hence, S is a semiperfect ring by [15], Corollary 4.42. $\square$

Corollary 4.9. *A ring S is semiperfect if and only if ${}_S S \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proposition 4.11. *If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ and W is coatomic, then W is an irredundant sum of local direct summands of W .*

Proof. This is clear from [9], Proposition 2.18. $\square$

The following example shows that a module, which belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$, need not be $\oplus$ -Rad-supplemented.

Example 4.2. Let us consider the $\mathbb{Z}$ -module $\mathbb{Z}_{(p)} = \{\frac{x}{y} \in \mathbb{Q} \mid p \nmid y\}$ where p is a prime number. Since $p\mathbb{Z}_{(p)}$ is the unique maximal of $\mathbb{Z}_{(p)}$ then $\mathbb{Z}_{(p)} \in \mathcal{S}_{\oplus\text{-Rad(cc)}}$ by Lemma 4.2. However, it can be seen from [4], Example 3.1 that $\mathbb{Z}_{(p)}$ is not a Rad-supplemented module. Therefore, $\mathbb{Z}_{(p)}$ is not $\oplus$ -Rad-supplemented.

Proposition 4.12. *If $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$, then $\frac{W}{P(W)}$ is $\oplus$ -co-coatomically supplemented.*

Proof. Since $f(P(W)) \leq P(W)$ for every $f \in \text{End}_R(W)$, $P(W)$ is a fully invariant submodule of W . Then, $\frac{W}{P(W)} \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ by Proposition 4.9. Consider K as a submodule of W that is both co-coatomic and contains $P(W)$. Then, there is a direct summand $\frac{T}{P(W)}$ of $\frac{W}{P(W)}$ such that $\frac{W}{P(W)} = \frac{K}{P(W)} + \frac{T}{P(W)}$ and $\frac{K \cap T}{P(W)} \subseteq \text{Rad}(\frac{T}{P(W)})$. Since $\frac{W}{P(W)}$ is reduced, $\frac{W}{P(W)}$ is coatomic by [5], Theorem 2.6. So $\text{Rad}(\frac{W}{P(W)}) \ll \frac{W}{P(W)}$. By Proposition 4.9, we have that $\frac{W}{P(W)}$ is $\oplus$ -co-coatomically supplemented. $\square$

Theorem 4.3. *Let W be a module over a Dedekind domain. The following statements are equivalent:*

- (1) $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.
- (2) $\frac{W}{P(W)} \in \mathcal{S}_{\oplus\text{-Rad-cc}}$.
- (3) $\frac{W}{P(W)}$ is $\oplus$ -co-coatomically supplemented.

Proof. (1) $\Rightarrow$ (3) This conclusion follows from Proposition 4.12.

(3) $\Rightarrow$ (2) This implication is clear.

(2) $\Rightarrow$ (1) By our assumption, there exists a decomposition $W = P(W) \oplus N$ for some submodule N of W . Since $\frac{W}{P(W)} \cong N$ belongs to the class $\mathcal{S}_{\oplus\text{-Rad-cc}}$ and $P(W)$ is a radical module, $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ by Lemma 4.1 and Corollary 4.5. $\square$

Theorem 4.4. *Over a Dedekind ring, $\mathcal{S}_{\text{Rad-cc}}$ is a subclass of $\mathcal{S}_{\oplus\text{-Rad-cc}}$.*

Proof. Assume that $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. The fact that $\frac{W}{P(W)}$ is co-coatomically supplemented is readily apparent. Since $\frac{W}{P(W)}$ is reduced, $\frac{W}{P(W)}$ is $\oplus$ -co-coatomically supplemented. By Theorem 4.3, $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$. $\square$

5. MODULE CLASS $\mathcal{SP}_{\text{Rad-cc}}$

In [13], the author introduced generalized cofinitely semiperfect modules using generalized projective covers of Rad-supplement modules. We now refine this term via Rad_{cc} -semiperfect modules as follows.

Definition 5.1. We define $\mathcal{SP}_{\text{Rad-cc}}$ as the class $\{W; \text{ each isomorphic factor of } W \text{ has a generalized projective cover}\}$.

A ring S is called a *left max ring* when every nonzero left S -module possesses a maximal submodule. This is equivalent to the claim that each nonzero left S -module is coatomic [21]. Consequently, any module which belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$ over a left max ring is considered to be generalized semiperfect.

Recall from [8] that a ring S is called *left Bass* if each nonzero S -module possesses a maximal submodule. Left Bass rings are characterized by $\text{Rad}(W) \ll W$ for each nonzero S -module W .

Proposition 5.1. *Every module which belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$ over a left Bass ring is co-coatomically semiperfect.*

Proof. The proof is obvious since $\text{Rad}(W) \ll W$ holds for every nonzero module W over a left Bass ring. $\square$

Proposition 5.2. *Let S be a regular ring. Then every S -module which belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$ is co-coatomically semiperfect.*

Proof. Consider A as a co-coatomic submodule within a generalized co-coatomically semiperfect S -module denoted by W . In such a case, there is a generalized cover f from the projective module P to the factor module $\frac{W}{A}$. Given that S is regular and $\text{Rad}(P) = \text{Rad}(S)P = 0$, then $\ker(f) \ll P$. Consequently, f is a projective cover of $\frac{W}{A}$. This implies that $W \in \mathcal{SP}_{\text{Rad-cc}}$. $\square$

Corollary 5.1. *Let W be an S -module over a left V -ring. W is co-coatomically semiperfect if and only if $W \in \mathcal{SP}_{\text{Rad-cc}}$.*

Proof. This follows directly from Proposition 5.2 as every module over a left V -ring possesses a zero radical. $\square$

Theorem 5.1. *For a module W , the following statements are equivalent:*

- (1) $W \in \mathcal{SP}_{\text{Rad-cc}}$.
- (2) $W \in \mathcal{AS}_{\text{Rad-cc}}$ with supplements that have generalized projective covers.
- (3) $W \in \mathcal{S}_{\text{Rad-cc}}$ with Rad-supplements that have generalized projective covers.

Proof. (1) $\Rightarrow$ (2) Let A be a co-coatomic submodule of W , where W can be expressed as the sum of A and a submodule Y such that $W = A + Y$. By the hypothesis, there exists an epimorphism $f: P \rightarrow \frac{W}{A}$ for a projective module P with $\ker(f) \leq \text{Rad}(P)$. As P is projective, there is a homomorphism $g: P \rightarrow Y$. Consequently, the image of g , denoted as $\text{Im}(g)$, serves as a Rad-supplement of A in W according to [13], Lemma 4.1, and it is contained in Y . Thus, P characterizes as a generalized projective cover for $\text{Im}(g)$, given by $\ker(g) \leq \ker(f) \leq \text{Rad}(P)$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Consider a co-coatomic submodule A of the module W . Under the given hypothesis, there is a Rad-supplement Y of A , and this Rad-supplement has a generalized projective cover denoted by f , specifically, the map $f: P \rightarrow Y$ where P is the generalized projective cover. Then, the map $\pi: Y \rightarrow \frac{Y}{A \cap Y}$ is a generalized cover. Consequently, the composition $\pi \circ f: P \rightarrow \frac{Y}{A \cap Y} \cong \frac{W}{A}$ serves as a generalized projective cover. This result can be deduced from [22], Lemma 1.1 and Lemma 1.2. Hence, $W \in \mathcal{SP}_{\text{Rad-cc}}$. $\square$

Proposition 5.3. $\mathcal{SP}_{\text{Rad-cc}}$ is closed under taking quotients.

Proof. Consider $W \in \mathcal{SP}_{\text{Rad-cc}}$, and $f: W \rightarrow W_1$ as a homomorphism. Assume that $A \leq f(W)$ is co-coatomic. The homomorphism $\pi: W \rightarrow \frac{f(W)}{A}$ such that $\pi(m) = f(m) + A$ is surjective for every $m \in W$. Since $W \in \mathcal{SP}_{\text{Rad-cc}}$ and $\frac{W}{f^{-1}(A)} \cong \frac{f(W)}{A}$, $\frac{f(W)}{A}$ is a generalized projective cover. Consequently, $f(W) \in \mathcal{SP}_{\text{Rad-cc}}$. $\square$

Corollary 5.2. *Each factor module of a generalized co-coatomically semiperfect module inherits the property of belonging to the class $\mathcal{SP}_{\text{Rad-cc}}$.*

Proposition 5.4. $\mathcal{SP}_{\text{Rad-cc}}$ is closed under taking generalized projective covers.

P r o o f. Consider $N \in \mathcal{SP}_{\text{Rad-cc}}$, and let $f: W \rightarrow N$ be a generalized cover. Additionally, let $A \leq W$ be co-coatomic. The homomorphism $\bar{f}: \frac{W}{A} \rightarrow \frac{N}{f(A)}$ defined by $\bar{f}(m+A) = f(m) + f(A)$ is an epimorphism. The kernel of $\bar{f}$ is given by $\ker(\bar{f}) = \frac{\ker(f)+L}{L} \leq \frac{\text{Rad}(W)+L}{L} \leq \text{Rad}(\frac{W}{L})$, as f is a generalized cover. Thus, $\bar{f}$ is a generalized cover. On the other hand, $\frac{N}{f(A)}$ is coatomic as it is homomorphic image of a coatomic factor module $\frac{W}{A}$. By our assumption, there is a generalized cover $\pi: P \rightarrow \frac{N}{f(A)}$ where P is projective. Therefore, a homomorphism $g: P \rightarrow \frac{W}{A}$ exists such that $\bar{f} \circ g = \pi$. As π is an epimorphism and $\bar{f}$ is a generalized cover, it follows that g is an epimorphism. Furthermore, $\ker(g) \leq \ker(\bar{f} \circ g) = \ker(\pi) \leq \text{Rad}(P)$. Consequently, P is a generalized projective cover of $\frac{W}{A}$. $\square$

Corollary 5.3. *If $A \leq \text{Rad}(W)$ and $\frac{W}{A}$ belong to the class $\mathcal{SP}_{\text{Rad-cc}}$, then $W \in \mathcal{SP}_{\text{Rad-cc}}$.*

A generalized cover $f: P \rightarrow W$ is called a *generalized W -projective cover* whenever P is W -projective.

Now, we give a characterization theorem for a *generalized W -projective cover*.

Theorem 5.2. *The following statements equivalent for a module W :*

- (1) *Each coatomic factor module W has a generalized W -projective cover.*
- (2) *$W \in \mathcal{AS}_{\text{Rad-cc}}$ which have generalized W -projective covers.*
- (3) *$W \in \mathcal{S}_{\text{Rad-cc}}$ which have generalized W -projective covers.*

P r o o f. The proof is analogous to the proofs of Theorem 5.1 and Proposition 5.4. $\square$

Proposition 5.5. *Let $W \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ and W be projective. Then $W \in \mathcal{SP}_{\text{Rad-cc}}$.*

P r o o f. Let A be a co-coatomic submodule of W . Under the given hypothesis, there exists a Rad-supplement T that serves as a direct summand in W , meaning that $W = A + T$ where $A \cap T \leq \text{Rad}(T)$, and $W = T \oplus T_1$ for some submodule $T_1 \leq W$. Consider the inclusion map $i: T \rightarrow W$ and the natural homomorphism $\pi: W \rightarrow \frac{W}{A}$. Note that T is projective. The composition map $\pi \circ i: T \rightarrow \frac{W}{A}$ is an epimorphism and $\ker(\pi \circ i) = A \cap T \leq \text{Rad}(T)$. Hence, T is a generalized projective cover of $\frac{W}{A}$ and so $W \in \mathcal{SP}_{\text{Rad-cc}}$. $\square$

Proposition 5.6. *Let $P \in \mathcal{S}_{\oplus\text{-Rad-cc}}$ and $f: P \rightarrow W$ be a generalized projective cover of W . Then $P \in \mathcal{SP}_{\text{Rad-cc}}$ if and only if $W \in \mathcal{SP}_{\text{Rad-cc}}$.*

P r o o f. ($\Rightarrow$) This is clear by Corollary 5.2 and Proposition 5.5.

($\Leftarrow$) It follows from Proposition 5.4. $\square$

Lemma 5.1. *Let W be a projective module. If W is semiperfect, then each finitely W -generated module belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$. Conversely, if W is finitely generated, the converse statement holds true.*

Proof. Clear by [3], Lemma 3.10. □

Since every finitely generated module has small radical, the notions of co-coatomically semiperfect modules and modules that belong to the class $\mathcal{SP}_{\text{Rad-cc}}$ arise in the same situation for finitely generated free modules.

Corollary 5.4. *The following statements are equivalent for a ring S :*

- (1) S is semiperfect.
- (2) Each finitely generated free S -module is semiperfect.
- (3) Each finitely generated free S -module is co-coatomically semiperfect.
- (4) Each finitely generated free S -module belongs to the class $\mathcal{SP}_{\text{Rad-cc}}$.

Proof. (1) $\Leftrightarrow$ (2) This is clear from [10], Lemma 1.2 and [12], Theorem 2.1.

(1) $\Leftrightarrow$ (3) This is clear by [3], Proposition 3.11.

(1) $\Leftrightarrow$ (4) This is clear from Lemma 5.1. □

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References

- [1] *R. Alizade, G. Bilhan, P. F. Smith:* Modules whose maximal submodules have supplements. *Commun. Algebra* **29** (2001), 2389–2405. zbl MR doi
- [2] *R. Alizade, S. Güngör:* Co-coatomically supplemented modules. *Ukr. Math. J.* **69** (2017), 1007–1018. zbl MR doi
- [3] *R. Alizade, S. Güngör:* $\oplus$ -co-coatomically supplemented and co-coatomically semiperfect modules. *Hacet. J. Math. Stat.* **47** (2018), 1417–1426. zbl MR doi
- [4] *E. Büyükaşık, C. Lomp:* On a recent generalization of semiperfect rings. *Bull. Aust. Math. Soc.* **78** (2008), 317–325. zbl MR doi
- [5] *E. Büyükaşık, E. Mermut, S. Özdemir:* Rad-supplemented modules. *Rend. Semin. Mat. Univ. Padova* **124** (2010), 157–177. zbl MR doi
- [6] *E. Büyükaşık, R. Tribak:* On w -local modules and Rad-supplemented modules. *J. Korean Math. Soc.* **51** (2014), 971–985. zbl MR doi
- [7] *H. Çalısıcı, E. Türkmen:* Generalized $\oplus$ -supplemented modules. *Algebra Discrete Math.* **10** (2010), 10–18. zbl MR
- [8] *J. Clark, C. Lomp, N. Vanaja, R. Wisbauer:* Lifting Modules: Supplements and Projectivity in Module Theory. *Frontiers in Mathematics.* Birkhäuser, Boston, 2006. zbl MR doi
- [9] *Ş. Ecevit, M. T. Koşan, R. Tribak:* Rad- $\oplus$ -supplemented modules and cofinitely Rad- $\oplus$ -supplemented modules. *Algebra Colloq.* **19** (2012), 637–648. zbl MR doi
- [10] *A. Harmanı, D. Keskin, P. F. Smith:* On $\oplus$ -supplemented modules. *Acta Math. Hung.* **83** (1999), 161–169. zbl MR doi

- [11] *F. Kasch*: Modules and Rings. London Mathematical Society Monographs 17. Academic Press, New York, 1982. [zbl](#) [MR](#)
- [12] *D. Keskin, P. F. Smith, W. Xue*: Rings whose modules are $\oplus$ -supplemented. *J. Algebra* *218* (1999), 470–487. [zbl](#) [MR](#) [doi](#)
- [13] *M. T. Koşan*: Generalized cofinitely semiperfect modules. *Int. Electron. J. Algebra* *5* (2009), 58–69. [zbl](#) [MR](#) [doi](#)
- [14] *G. O. Michler, O. E. Villamayor*: On rings whose simple modules are injective. *J. Algebra* *25* (1973), 185–201. [zbl](#) [MR](#) [doi](#)
- [15] *S. H. Mohamed, B. J. Müller*: Continuous and Discrete Modules. London Mathematical Society Lecture Note Series 147. Cambridge University Press, Cambridge, 1990. [zbl](#) [MR](#) [doi](#)
- [16] *B. Nisanci, A. Pancar*: On generalization of $\oplus$ -cofinitely supplemented modules. *Ukr. Math. J.* *62* (2010), 203–209. [zbl](#) [MR](#) [doi](#)
- [17] *P. F. Smith*: Finitely generated supplemented modules are amply supplemented. *Arab. J. Sci. Eng., Sect. C, Theme Issues* *25* (2000), 69–79. [zbl](#) [MR](#)
- [18] *E. Türkmen, A. Pancar*: On cofinitely Rad-supplemented modules. *Int. J. Pure Appl. Math.* *53* (2009), 153–162. [zbl](#) [MR](#)
- [19] *E. Türkmen, A. Pancar*: Characterizations of Rad-supplemented modules. *Miskolc Math. Notes* *13* (2012), 569–580. [zbl](#) [MR](#) [doi](#)
- [20] *R. B. Warfield, Jr.*: Decomposability of finitely presented modules. *Proc. Am. Math. Soc.* *25* (1970), 167–172. [zbl](#) [MR](#) [doi](#)
- [21] *R. Wisbauer*: Foundations of Module and Ring Theory: A Handbook for Study and Research. Algebra, Logic and Applications 3. Gordon and Breach Science Publishers, Philadelphia, 1991. [zbl](#) [MR](#) [doi](#)
- [22] *W. Xue*: Characterization of semiperfect and perfect rings. *Publ. Mat., Barc.* *40* (1996), 115–125. [zbl](#) [MR](#) [doi](#)
- [23] *H. Zöschinger*: Komplementierte Moduln über Dedekindringen. *J. Algebra* *29* (1974), 42–56. (In German.) [zbl](#) [MR](#) [doi](#)
- [24] *H. Zöschinger*: Koatomare Moduln. *Math. Z.* *170* (1980), 221–232. (In German.) [zbl](#) [MR](#) [doi](#)

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