Research Article

Nilay Sahin Bayram* and Sevda Yıldız

$\mathcal{I}$-*αβ*-statistical relative uniform convergence for double sequences and its applications

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Abstract: This article introduces a novel concept of convergence, referred to as $\mathcal{I}$-$\alpha\beta$-statistical relative uniform convergence, for double sequences of functions. This notion, which is proposed for the first time in this article, is explored in depth, leading to the establishment of a Korovkin-type approximation theorem within this framework. The illustrative example demonstrates that the proposed convergence is indeed stronger than previously known forms. Additionally, this article investigates the rate of $\mathcal{I}_2$-$\alpha\beta$-statistical relative uniform convergence, providing explicit computations to support the findings. The results contribute to the understanding of ideal statistical convergence and open up new perspectives for approximation theory.

Keywords: Korovkin-type theorem, positive linear operator, ideal relative convergence for double sequences, $\mathcal{I}_2$-$\alpha\beta$-statistical relative uniform convergence

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1 Introduction

The concept of statistical convergence was initially introduced by Steinhaus [1] and independently by Fast [2] as a generalization of classical convergence for real sequences. Over time, various generalizations and applications of statistical convergence have been explored in different spaces, notably through the significant works of Fridy [3], Fridy and Orhan [4], and Šalát [5]. In recent years, the concept of statistical convergence has been further extended in several important directions, such as:

- Lacunary statistical convergence introduced by Fridy and Orhan [4];
- λ-statistical convergence, explored by Çolak and Bektaş [6];
- Statistical relative uniform convergence introduced by Demirci and Orhan [7];
- Statistical convergence using the power series method, pioneered by Ünver and Orhan [8], with subsequent developments in the objectives of this method by various authors in 2022 and 2023 [8–11];
- Ideal convergence, a concept developed by Kostyrko et al. [12], as well as independently by Nuray and Ruckle [13].

In 2008, ideal convergence was extended to sequences of real functions and double sequences by Das et al. [14]. Subsequently, Dündar and Altay [15] investigated various ideal convergence notions for double sequences of real-valued functions. Building on these advances, Aktuğlu [16] introduced a new variant of statistical convergence, known as $\alpha\beta$-statistical convergence of order γ, which has proven to be a significant extension

*** Corresponding author: Nilay Sahin Bayram,** Department of Electrical and Electronics Engineering, Faculty of Engineering, Ankara, Turkey, e-mail: nsbayram@baskent.edu.tr
Sevda Yıldız: Department of Mathematics, Sinop University, Sinop, Turkey, e-mail: sevdaorhan@sinop.edu.tr

of both classical and statistical convergences. This concept also encompasses several other statistical-type convergences discussed in this article.

Further developments in double sequences came from Altundağ and Sözbir [17], who introduced $\alpha\beta$-statistical convergence and strong $\alpha\beta$-summability for double sequences, exploring the interplay between these concepts. Savas and Das [18] later combined statistical convergence with $\mathcal{I}$-convergence, leading to the emergence of $\mathcal{I}$-statistical convergence, which was subsequently generalized to double sequences by Belen and Yildirim [19]. In a more recent advancement, Ghosal and Mandal [20] introduced $\mathcal{I}$-$\alpha\beta$-statistical convergence for single sequences, which further generalizes the notion of $\mathcal{I}$-statistical convergence. Then, $\mathcal{I}$-$\alpha\beta$-statistical convergence for double sequences came from Yıldız and Šahin Bayram [21].

Based on the research mentioned earlier, this article introduces a novel type of convergence called $\mathcal{I}$-$\alpha\beta$-statistical relative uniform convergence for double sequences of functions, a concept presented for the first time. Then, the approximation theorem of the Korovkin-type via this new type of convergence has been proved. An example is provided to illustrate that this new convergence is indeed stronger than previous types. Finally, the rate of $\mathcal{I}_2$-$\alpha\beta$-statistical relative uniform convergence is calculated, contributing to a deeper understanding of this new convergence concept.

2 Preliminaries

For our investigation, the following is requisite

2.1 Statistical-type convergences

First, recall the main concept of convergence methods for double sequences.

Definition 1. [22] A double sequence $x = (x_{ij})$ is said to be convergent in Pringsheim's sense (P-convergent) if

$$\forall \varepsilon > 0,\ \exists J = J(\varepsilon),\ \forall i, j > J,\ |x_{ij} - L| < \varepsilon.$$

This limit is represented as $P - \lim_{i,j} x_{ij} = L$.

Definition 2. [22] A double sequence $x = (x_{ij})$ is called bounded if

$$\exists K > 0,\ \forall (i,j) \in \mathbb{N}^2, \quad |x_{ij}| \leq K, \quad \text{i.e.,}$$

if $\|x\|_{(\infty,2)} = \sup_{i,j} |x_{ij}| < \infty$.

Definition 3. [23] Consider the double sequence $x = (x_{ij})$. It is said to be statistically convergent to the limit L if for every $\varepsilon > 0$, the following holds:

$$P - \lim_{m,n\to\infty} \frac{|\{(i,j) : i \leq m, j \leq n, |x_{ij} - L| \geq \varepsilon\}|}{mn} = 0.$$

In this setting, the convergence is expressed as $st_2 - \lim_{i,j} x_{ij} = L$.

Now, let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying the following conditions:

$$\begin{aligned} &C_1: \ \alpha, \beta \ \text{are both non-decreasing,}\\ &C_2: \ \beta(n) \geq \alpha(n),\\ &C_3: \ \beta(n) - \alpha(n) \to \infty, \ \text{as } n \to \infty, \end{aligned}$$

and let Λ denote the set of pairs (α, β) satisfying C_1, C_2, and C_3 [16].

Throughout this article, let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \Lambda$.

Definition 4. [17] A double sequence $x = (x_{ij})$ is said to be $\alpha\beta$-statistically convergent to L, if for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} |\{(i,j),\ i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : |x_{ij} - L| \geq \varepsilon\}| = 0,$$

which is denoted by $st_2^{\alpha\beta} - \lim_{i,j} x_{ij} = L$, where $D_m^{\alpha_1\beta_1}$ and $D_n^{\alpha_2\beta_2}$ are closed intervals $[\alpha_1(m), \beta_1(m)]$ and $[\alpha_2(n), \beta_2(n)]$, respectively.

Definition 5. [17] A double sequence $x = (x_{ij})$ is said to be strongly $\alpha\beta$-summable to L, if for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \sum_{i \in D_m^{\alpha_1\beta_1}} \sum_{j \in D_n^{\alpha_2\beta_2}} |x_{ij} - L| = 0.$$

2.2 Ideal-type convergences

We begin by recalling some basic terminology and notation regarding ideals.

The notion of $\mathcal{I}$-convergence, introduced by Kostyrko et al. [12], generalizes statistical convergence through the use of an ideal $\mathcal{I}$. Let S be a non-empty set. A family $\mathcal{I}$ of subsets of S is called an *ideal* on S if it satisfies the following properties:

(i) $\emptyset \in \mathcal{I}$,

(ii) if $A_1, A_2 \in \mathcal{I}$, then $A_1 \cup A_2 \in \mathcal{I}$,

(iii) if $A_1 \in \mathcal{I}$ and $A_2 \subset A_1$, then $A_2 \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called *admissible* if $\{x\} \in \mathcal{I}$ for every $x \in S$. Moreover, $\mathcal{I}$ is said to be *non-trivial* if $S \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$.

For such a non-trivial ideal $\mathcal{I}$, the associated filter is defined as

$$\mathcal{F} = \{U \subset S : (\exists A_1 \in \mathcal{I}) \text{ such that } U = S \backslash A_1\}.$$

Now, consider a non-trivial ideal $\mathcal{I}_2$ on the set $\mathbb{N}^2$. This ideal is said to be *strongly admissible* if for every $i \in \mathbb{N}$, the sets $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ are in $\mathcal{I}_2$. It is evident that any strongly admissible ideal is also admissible.

Define the collection

$$\mathcal{I}_2^0 = \{A_2 \subset \mathbb{N}^2 : (\exists m(A_2) \in \mathbb{N}) \text{ such that } (i,j) \notin A_2 \text{ whenever } i,j \geq m(A_2)\}.$$

This set forms a strongly admissible, non-trivial ideal on $\mathbb{N}^2$ [14]. Furthermore, an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Throughout the rest of this article, we shall assume that $\mathcal{I}_2$ is a non-trivial, strongly admissible ideal on $\mathbb{N}^2$.

Definition 6. [19] A double sequence $x = (x_{ij})$ of real numbers is said to be $\mathcal{I}_2$-statistically convergent to a number L, if for each $\varepsilon > 0$ and $\eta > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{mn} |\{i \leq n,\ j \leq m : |x_{ij} - L| \geq \varepsilon\}| \geq \eta\right\} \in \mathcal{I}_2.$$

Yıldız and Šahin Bayram [21] introduced the following definition of $\mathcal{I}_2$-$\alpha\beta$-statistical convergence and gave well-known three important results that are stated as follows:

Definition 7. A double sequence $x = (x_{ij})$ is said to be $\mathcal{I}_2$-$\alpha\beta$-statistically convergent to L, if for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \times |\{(i,j), i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : |x_{ij} - L| \geq \varepsilon\}| \geq \eta\right\} \in \mathcal{I}_2.$$

In this case, we write $st(\mathcal{I}_2^{\alpha\beta}) - \lim_{i,j} x_{ij} = L$.

(i) If we take $\alpha_1(m) = 1, \beta_1(m) = m$ for all $m \in \mathbb{N}$ and $\alpha_2(n) = 1, \beta_2(n) = n$ for all $n \in \mathbb{N}$, then $\mathcal{I}_2$-$\alpha\beta$-statistical convergence is reduced to $\mathcal{I}_2$-statistical convergence introduced in Definition 6.
(ii) Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive numbers tending to ∞ such that

$$\lambda_{m+1} \leq \lambda_m + 1, \quad \lambda_1 = 1,$$
$$\mu_{n+1} \leq \mu_n + 1, \quad \mu_1 = 1.$$

Then, in the case of $\alpha_1(m) = m - \lambda_m + 1, \beta_1(m) = m$ for all $m \in \mathbb{N}$, and $\alpha_2(n) = n - \mu_n + 1, \beta_1(n) = n$ for all $n \in \mathbb{N}$, $\mathcal{I}_2 - \alpha\beta$-statistical convergence is reduced to $\mathcal{I}_2 - (\lambda, \mu)$-statistical convergence introduced in [19].
(iii) Recall that a double lacunary sequence $\theta_{r,s} = \{(k_r, l_s)\}$, which means there exist two increasing integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty, \quad \text{as } r \to \infty,$$
$$l_0 = 0, \quad \bar{h}_s = l_s - l_{s-1} \to \infty, \quad \text{as } s \to \infty,$$

If we take $\alpha_1(m) = k_{m-1} + 1, \beta_1(m) = k_m$ for all $m \in \mathbb{N}$ and $\alpha_2(n) = l_{n-1} + 1, \beta_1(n) = l_n$ for all $n \in \mathbb{N}$, then $\mathcal{I}_2$-$\alpha\beta$-statistical convergence of double sequence is reduced to $\mathcal{I}_2$-lacunary statistical convergence of double sequence introduced in [24].

The concept of uniform convergence of a sequence of functions relative to a scale function first given by Moore in [25]. Then, Chittenden [26] gave the definition that is equivalent to the definition given by Moore for single sequences. Demirci and Orhan [7] extended this idea to statistical convergence and gave approximation results, and then, Okçu Šahin and Dirik [27] introduced this type of convergence for double sequences as follows (see also [28]):

Definition 8. [27] The double function sequence (f_{ij}) defined on any compact subset of the real two-dimensional space converges to the limit function f relatively uniformly if there exists a function $\sigma(x, y)$ defined on any compact subset of the real two-dimensional space (which in the literature is called the scale function) such that for every $\varepsilon > 0$, there is an integer n_ε such that for $m, n > n_\varepsilon$, the inequality

$$|f_{i,j}(x, y) - f(x, y)| < \varepsilon|\sigma(x, y)|$$

holds uniformly in (x, y). The double sequence (f_{ij}) is said to converge *uniformly relatively to the scale function* σ, briefly, relatively uniformly convergent.

Ideal relative uniform convergence for double sequences of functions was introduced for the first time by Yıldız [29] in 2021. Then, in 2024, Yıldız and Šahin Bayram [21] introduced novel concept of $\mathcal{I}_2$-$\alpha\beta$-statistical convergence. In view of these new-type of convergence methods in this article, we study an interesting type of convergence called $\mathcal{I}_2 - \alpha\beta$-statistical relative uniform convergence.

3 Ideal $\alpha\beta$-statistical-type convergences

Now, we introduce our new convergence methods. Let X^2 be a compact subset of $\mathbb{R}^2$.

Definition 9. A double sequence (f_{ij}) is said to be $\mathcal{I}_2 - \alpha\beta$-statistically pointwise convergent to f on X^2, written in short as $st(\mathcal{I}_2^{\alpha\beta}) - f_{ij} \to f$ for each $(x, y) \in X^2$,

$$\left\{(m, n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \times |\{(i, j), i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : |f_{ij}(x, y) - f(x, y)| \geq \varepsilon\}| \geq \eta\right\} \in \mathcal{I}_2,$$

written in short as $st(\mathcal{I}_2^{\alpha\beta}) - f_{ij} \to f$ on X^2.

Definition 10. A double sequence (f_{ij}) is said to be $\mathcal{I}_2$-$\alpha\beta$-statistically uniformly convergent to f on X^2, written in short as $st(\mathcal{I}_2^{\alpha\beta}) - f_{ij} \rightrightarrows f$ if for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \times |\{(i,j), i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2} |f_{ij}(x,y) - f(x,y)| \geq \varepsilon\}| \geq \eta\right\} \in \mathcal{I}_2.$$

Then, it is denoted $st(\mathcal{I}_2^{\alpha\beta}) - f_{ij} \rightrightarrows f$.

Definition 11. A double sequence (f_{ij}) is said to be $\mathcal{I}_2$-$\alpha\beta$-statistically relatively uniformly convergent to f on X^2, if for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \times \left|\left\{(i,j), i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2} \left|\frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)}\right| \geq \varepsilon\right\}\right| \geq \eta\right\} \in \mathcal{I}_2.$$

In this case, we write $st(\mathcal{I}_2^{\alpha\beta}) - f_{ij} \rightrightarrows f \ (X^2; \sigma)$.

It is important to mention that choosing $\mathcal{I}_2 = \mathcal{I}_2^0$ leads to the notion of $\alpha\beta$-statistical relative uniform convergence for double sequences – a recently introduced type of convergence. Moreover, if the scale function is substituted by a nonzero constant, this framework reduces to the classical $\alpha\beta$-statistical convergence of sequences of functions [16,17].

Based on the preceding definition, we can directly state the following result.

Lemma 1. *$st_2^{\alpha\beta} - f_{ij} \rightrightarrows f$ on X^2 implies $st(\mathcal{I}_2^{\alpha\beta}) - f_{ij} \rightrightarrows f$ on X^2, which also implies $st(\mathcal{I}_2^{\alpha\beta}) - f_{ij} \rightrightarrows f(X^2; \sigma)$.*

Nonetheless, the reverse implication of Lemma 1 does not hold in general, as demonstrated by the counterexample presented in the following.

Example 1. Let $\mathcal{I}_2 = \mathcal{I}_2^{\phi}$, set of all subsets of $\mathbb{N}^2$ with double natural density zero. For each $(i,j) \in \mathbb{N}^2$, let $\alpha_1(m) = m^2$, $\beta_1(m) = (m+1)^2$, $\alpha_2(n) = n^2$, $\beta_2(n) = (n+1)^2$ and define $h_{ij} : [0,1]^2 \to \mathbb{R}$ by

$$h_{ij}(x,y) = \begin{cases} ij; & \begin{array}{c} (x,y) \notin \left[0, \frac{1}{i}\right) \times \left[0, \frac{1}{j}\right), \\ (i \in [\alpha_1(m), \beta_1(m)],\ j \in [\alpha_2(n), \beta_2(n)] \\ \wedge\, m = l^2, n = k^2, l, k \in \mathbb{N}), \end{array} \\ ij(1 - ijxy); & (x,y) \in \left[0, \frac{1}{i}\right) \times \left[0, \frac{1}{j}\right), \\ 0; & \text{otherwise.} \end{cases}$$

Here, (h_{ij}) is neither $\mathcal{I}_2 - \alpha\beta$-statistically uniformly convergent nor $\alpha\beta$-statistically uniformly convergent to the function $h = 0$ on the interval $[0,1]^2$. In addition, (h_{ij}) is not classically uniformly convergent to the function $h = 0$. But it is seen to be $st(\mathcal{I}_2^{\alpha\beta}) - h_{ij} \rightrightarrows h = 0 \ ([0,1]^2; \sigma)$, with

$$\sigma(x,y) = \begin{cases} \frac{1}{x^2y^2}, & (x,y) \in (0,1] \times (0,1], \\ 1, & x = 0 \text{ or } y = 0. \end{cases}$$

Indeed, $\exists J = J(\varepsilon)$,

$$\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \left|\left\{(i,j), i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in S^2} \left|\frac{h_{ij}(x,y)}{\sigma(x,y)}\right| \geq \varepsilon\right\}\right|$$
$$\leq \begin{cases} \frac{([\beta_1(m) - \alpha_1(m)] + 1)([\beta_2(n) - \alpha_2(n)] + 1)}{(\beta_1(m) - \alpha_1(m) + 1)(\beta_2(n) - \alpha_2(n) + 1)}; & m = l^2, n = k^2, l, k \in \mathbb{N}, \\ \frac{[J]}{(\beta_1(m) - \alpha_1(m) + 1)(\beta_2(n) - \alpha_2(n) + 1)}; & \text{otherwise.} \end{cases}$$

Example 2. Let $\mathcal{I}_2 = \mathcal{I}_2^{\phi}$ and $A \in \mathcal{I}_2^{\phi}$. Let us take $\alpha_1(m) = \alpha_2(n) = 1$, $\beta_1(m) = m$, $\beta_2(n) = n$ and for each $(i,j) \in \mathbb{N}^2$, define $h_{ij} : [0,1]^2 \to \mathbb{R}$ by

$$h_{ij}(x,y) = \begin{cases} ij, & \begin{array}{c}\beta_1(m) - \sqrt{\beta_1(m) - \alpha_1(m) + 1} + 1 \le i \le \beta_1(m), \\ \beta_2(n) - \sqrt{\beta_2(n) - \alpha_2(n) + 1} + 1 \le j \le \beta_2(n) \wedge (m,n) \notin A\end{array} \\ ij, & \alpha_1(m) \le i \le \beta_1(m), \alpha_2(n) \le j \le \beta_2(n) \wedge (m,n) \in A \\ \dfrac{ijxy}{2 + i^2j^2x^2y^2}, & \text{otherwise}, \end{cases} \tag{1}$$

and the scale function given by

$$\sigma(x,y) = \begin{cases} \dfrac{1}{xy}, & (x,y) \in (0,1] \times (0,1], \\ 1, & x = 0 \text{ or } y = 0. \end{cases} \tag{2}$$

Then, for every $\varepsilon > 0$ $(0 < \varepsilon < 1)$, since

$$\begin{aligned} &\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \left| \left\{ (i,j),\, i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in S^2} \left| \frac{h_{ij}(x,y)}{\sigma(x,y)} \right| \ge \varepsilon \right\} \right| \\ &\le \frac{\sqrt{\beta_1(m) - \alpha_1(m) + 1}\sqrt{\beta_2(n) - \alpha_2(n) + 1}}{(\beta_1(m) - \alpha_1(m) + 1)(\beta_2(n) - \alpha_2(n) + 1)} \to 0, \end{aligned}$$

as $m, n \to \infty$ and $(m,n) \notin A$. Hence,

$$\begin{aligned} &\left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \times \right. \\ &\left. \left| \left\{ (i,j),\, i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in S^2} \left| \frac{h_{ij}(x,y)}{\sigma(x,y)} \right| \ge \varepsilon \right\} \right| \ge \eta \right\} \\ &\subset A \cup \{1, 2 \ldots, J_1\}, \end{aligned}$$

for some $J_1 \in \mathbb{N}$. Clearly, the set on the right-hand side belongs to $\mathcal{I}_2^{\phi}$ and so the set on the left-hand side also belongs to $\mathcal{I}_2^{\phi}$. This shows that $st(\mathcal{I}_2^{\alpha\beta}) - h_{ij} \rightrightarrows h = 0$ $([0,1]^2; \sigma)$. However, (h_{ij}) does not converge $\mathcal{I}_2$-relatively uniformly to h.

Nevertheless, the sequence (h_{ij}) fails to exhibit $\mathcal{I}_2$-$\alpha\beta$ statistical uniform convergence, as well as $\alpha\beta$ statistical uniform convergence, toward the zero function on the domain $[0,1]^2$. Furthermore, (h_{ij}) does not converge uniformly in the classical sense to $h = 0$ over the same domain.

Definition 12. A double sequence (f_{ij}) of functions defined on X^2 is said to be $\mathcal{I}_2 - \alpha\beta$-relatively uniformly summable to f, if for every $\varepsilon > 0$,

$$\left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \sum_{i\in D_m^{\alpha_1\beta_1}} \sum_{j\in D_n^{\alpha_2\beta_2}} \sup_{(x,y)\in X^2} \left| \frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)} \right| \ge \varepsilon \right\} \in \mathcal{I}_2.$$

Based upon the aforementioned definitions, we shall give the following special cases and some inclusion relations to show the effectiveness of newly proposed methods introduced in Definitions 11 and 12.

- Let us take $\alpha_1(m) = 1, \beta_1(m) = m$ for all $m \in \mathbb{N}$ and $\alpha_2(n) = 1, \beta_2(n) = n$ for all $n \in \mathbb{N}$; then, $\mathcal{I}_2$-$\alpha\beta$-statistical relative uniform convergence given in Definition 11, is reduced to $\mathcal{I}_2$-statistical relative uniform convergence. If the case of the scale function is taken to be a constant different from zero, then we obtain ideal statistical convergence and ideal summability for function sequences (cf. [18,19,30,31]).

- In the case of $\alpha_1(m) = k_{m-1} + 1$, $\beta_1(m) = k_m$ for all $m \in \mathbb{N}$ and $\alpha_2(n) = l_{n-1} + 1$, $\beta_1(n) = l_n$ for all $n \in \mathbb{N}$, then, strong $\mathcal{I}_2 - \alpha\beta$-relative uniform summability given in Definition 12 is reduced to its lacunary version. Again, if the choice of the scale function is taken to be a constant different from zero, then we have the notions of ideal lacunary statistical convergence and ideal lacunary summability for function sequences (cf. [24,31,32]).
- Let us take $\alpha_1(m) = m - \lambda_m + 1$, $\beta_1(m) = m$ for all $m \in \mathbb{N}$ and $\alpha_2(n) = n - \mu_n + 1$, $\beta_1(n) = n$ for all $n \in \mathbb{N}$. Then, strong $\mathcal{I}_2 - \alpha\beta$-relative uniform summability is reduced to strong $\mathcal{I}_2 - (\lambda, \mu)$-relative uniform summability. If the scale function is considered to be a constant different from zero, we have the concepts of ideal (λ, μ)-statistical convergence and ideal (λ, μ)-summability for function sequences (cf. [18,19,33]).

Theorem 1. *Suppose that*

$$\sup_{(x,y)\in X^2} \left| \frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)} \right| \le K, \quad \text{for all } i,j \in \mathbb{N},\ |\sigma(x,y)| \neq 0.$$

If a double sequence (f_{ij}) of functions on X^2 is $\mathcal{I}_2 - \alpha\beta$-statistically relatively uniformly convergent to a function f, then it is $\mathcal{I}_2 - \alpha\beta$-relatively uniformly summable to the function f, but not conversely.

Proof. Let us set

$$H_{st(\mathcal{I}_2^{\alpha\beta})} \coloneqq \left\{ (i,j),\, i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2} \left| \frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)} \right| \ge \varepsilon \right\}$$

and

$$H^C_{st(\mathcal{I}_2^{\alpha\beta})} \coloneqq \left\{ (i,j),\, i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2} \left| \frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)} \right| < \varepsilon \right\},$$

with for given $\varepsilon > 0$ and enough large m and n. Also, we obtain

$$\begin{aligned}
&\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \sum_{i\in D_m^{\alpha_1\beta_1}} \sum_{j\in D_n^{\alpha_2\beta_2}} \sup_{(x,y)\in X^2} \left| \frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)} \right| \\
&= \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \sum_{\substack{i\in D_m^{\alpha_1\beta_1} \\ ((i,j)\in H_{st(\mathcal{I}_2^{\alpha\beta})}}} \sum_{j\in D_n^{\alpha_2\beta_2}} \sup_{(x,y)\in X^2} \left| \frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)} \right| \\
&\quad + \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \sum_{\substack{i\in D_m^{\alpha_1\beta_1} \\ ((i,j)\in H^C_{st(\mathcal{I}_2^{\alpha\beta})}}} \sum_{j\in D_n^{\alpha_2\beta_2}} \sup_{(x,y)\in X^2} \left| \frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)} \right| \\
&\le K \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} |H_{st(\mathcal{I}_2^{\alpha\beta})}| + \varepsilon.
\end{aligned}$$

From the hypotheses, we have

$$H^1_{st(\mathcal{I}_2^{\alpha\beta})} \coloneqq \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} |H_{st(\mathcal{I}_2^{\alpha\beta})}| \ge \frac{\varepsilon}{K} \right\} \in \mathcal{I}_2.$$

If $(m,n) \in \mathbb{N}^2 \backslash H^1_{st(\mathcal{I}_2^{\alpha\beta})}$, then

$$\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \sum_{i\in D_m^{\alpha_1\beta_1}} \sum_{j\in D_n^{\alpha_2\beta_2}} \sup_{(x,y)\in X^2} \left| \frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)} \right| \le 2\varepsilon.$$

Hence, we have

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}\sup_{(x,y)\in X^2}\left|\frac{f_{ij}(x,y)-f(x,y)}{\sigma(x,y)}\right|\geq 2\varepsilon\right\}\subset H^1_{st(\mathcal{I}_2^{\alpha\beta})},$$

and therefore, (f_{ij}) is $\mathcal{I}_2$-$\alpha\beta$-relatively uniformly summable to the function f. □

For converse, we consider the following example:

Example 3. Let $\mathcal{I}_2=\mathcal{I}_2^{\phi}$, set of all subsets of $\mathbb{N}^2$ with double natural density zero. For each $(i,j)\in\mathbb{N}^2$, let $\alpha_1(m)\leq 1\leq\beta_1(m)$, $\alpha_2(n)\leq 1\leq\beta_2(n)$ and define $h_{ij}:[0,1]^2\to\mathbb{R}$ by

$$h_{ij}(x,y)=\begin{cases} ij^2xy^2; & \begin{array}{l}1\leq i\leq[\sqrt{\beta_1(m)-\alpha_1(m)+1}],\\ 1\leq j\leq[\sqrt{\beta_2(n)-\alpha_2(n)+1}],\\ \wedge\quad m=2l,n=2k,\qquad l,k\in\mathbb{N}\end{array}\\ \dfrac{3ij^2xy^2}{2+i^2j^3x^2y^4}; & \text{otherwise.}\end{cases}$$

One may deduce that $st(\mathcal{I}_2^{\alpha\beta})$–$h_{ij}\rightrightarrows h=0$ uniformly on the domain $[0,1]^2$, where

$$\sigma(x,y)=\begin{cases}\dfrac{1}{xy^2}, & (x,y)\in(0,1]\times(0,1],\\ 1, & x=0\text{ or }y=0.\end{cases}$$

Indeed, $\exists J=J(\varepsilon)$,

$$\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}\left|\left\{(i,j),i\in D_m^{\alpha_1\beta_1}\text{ and }j\in D_n^{\alpha_2\beta_2}:\sup_{(x,y)\in X^2}\left|\frac{h_{ij}(x,y)}{\sigma(x,y)}\right|\geq\varepsilon\right\}\right|$$
$$=\begin{cases}\dfrac{[\sqrt{\beta_1(m)-\alpha_1(m)+1}][\sqrt{\beta_2(n)-\alpha_2(n)+1}]}{(\beta_1(m)-\alpha_1(m)+1)(\beta_2(n)-\alpha_2(n)+1)}; & m=2l,n=2k,\ l,k\in\mathbb{N}\\ \dfrac{[J]}{(\beta_1(m)-\alpha_1(m)+1)(\beta_2(n)-\alpha_2(n)+1)}; & \text{otherwise.}\end{cases}\to 0$$

However,

$$\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}\sup_{(x,y)\in X^2}\left|\frac{h_{ij}(x,y)}{\sigma(x,y)}\right|$$
$$=\frac{[\sqrt{\beta_1(m)-\alpha_1(m)+1}]([\sqrt{\beta_1(m)-\alpha_1(m)+1}]+1)[\sqrt{\beta_2(n)-\alpha_2(n)+1}]([\sqrt{\beta_2(n)-\alpha_2(n)+1}]+1)}{4(\beta_1(m)-\alpha_1(m)+1)(\beta_2(n)-\alpha_2(n)+1)}\nrightarrow 0,$$

which means (h_{ij}) is not $\mathcal{I}_2-\alpha\beta$-relatively uniformly summable to the function h.

4 Ideal relative Korovkin-type approximation

Let the space of all continuous real-valued functions on X^2 be named $C(X^2)$. As it is known, it is a Banach space with the norm $\|.\|$ defined by $\|f\|\coloneqq\sup_{(x,y)\in X^2}|f(x,y)|$, $f\in C(X^2)$. In this section, we explore the concept of $\mathcal{I}_2$-$\alpha\beta$-statistical relative uniform convergence to establish a Korovkin-type approximation theorem for a double sequence of positive linear operators acting on the space $C(X^2)$. Consider a linear operator $T:C(X^2)\to C(X^2)$. For any function $f\in C(X^2)$ and point $(x,y)\in X^2$, we write the operator's value as

$T(f; x, y)$ or equivalently, $T(f(s, t); x, y)$. Moreover, throughout this article, the test functions are given by

$$e_0(x, y) = 1, \quad e_1(x, y) = x, \quad e_2(x, y) = y, \quad \text{and} \quad e_3(x, y) = x^2 + y^2.$$

Prior to proceeding, we formally review the fundamental Korovkin-type approximation theorems relevant to our study.

Theorem 2. [34] *Let (T_{ij}) be a double sequence of positive linear operators acting from $C(X^2)$ into itself. Then, for all $f \in C(X^2)$,*

$$P - \lim_{ij} \|T_{ij}(f) - f\| = 0, \quad \text{on } X^2,$$

iff

$$P - \lim_{ij} \|T_{ij}(e_z) - e_z\|, \quad \text{on } X^2, \ z = 0, 1, 2, 3.$$

Theorem 3. [17] *Let (T_{ij}) be a double sequence of positive linear operators acting from $C(X^2)$ into itself. Then, for all $f \in C(X^2)$,*

$$st_2^{\alpha\beta} - \lim_{ij} \|T_{ij}(f) - f\|, \quad \text{on } X^2$$

iff

$$st_2^{\alpha\beta} - \lim_{ij} \|T_{ij}(e_z) - e_z\|, \quad \text{on } X^2, \ z = 0, 1, 2, 3.$$

Now, we give the main approximation result of this section.

Theorem 4. *Let (T_{ij}) be a double sequence of positive linear operators acting from $C(X^2)$ into itself, and σ and σ_z are the scale functions (possibly unbounded). Then, for all $f \in C(X^2)$,*

$$st(\mathcal{I}_2^{\alpha\beta}) - T_{ij}(f) \rightrightarrows f \quad (X^2; \sigma), \tag{3}$$

iff

$$st(\mathcal{I}_2^{\alpha\beta}) - T_{ij}(e_z) \rightrightarrows e_z \quad (X^2; \sigma_z), \ z = 0, 1, 2, 3. \tag{4}$$

Proof. Based on the assumptions, note that $e_z \in C(X^2)$ holds for each $z = 0, 1, 2, 3$. Hence, condition (3) ensures the validity of condition (4). The main objective now is to establish the converse implication.

As a starting point, since f is continuous on X^2, it must be bounded over this domain. Thus, there exists a constant $K_f > 0$ such that

$$|f(x, y)| \le K_f,$$

where $K_f = \|f\|$ denotes the supremum norm of f on X^2.

Moreover, the continuity of f implies that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - f(s, t)| < \varepsilon,$$

whenever $(x, y), (s, t) \in X^2$ and $|x - s| < \delta, |y - t| < \delta$.

On the other hand, for all $(x, y), (s, t) \in X^2$ satisfying $|x - s| > \delta$ and $|y - t| > \delta$, we can estimate:

$$|f(x, y) - f(s, t)| \le \frac{2K_f}{\delta^2}\{(x - s)^2 + (y - t)^2\}.$$

Consequently, for $(x, y), (s, t) \in X^2$, the following inequality holds:

$$|f(x, y) - f(s, t)| < \varepsilon + \frac{2K_f}{\delta^2}\{(x - s)^2 + (y - t)^2\}.$$

Given that T_{ij} is both linear and positive, it follows that

$$\begin{aligned}|T_{ij}(f;\,x,y)-f(x,y)| &\le T_{ij}(|f(x,y)-f(s,t)|;\,x,y)+|f(x,y)||T_{ij}(e_0;\,x,y)-e_0(x,y)|\\ &\le T_{ij}(\varepsilon+\frac{2K_f}{\delta^2}\{(x-s)^2+(y-t)^2\};\,x,y)+K_f|T_{ij}(e_0;\,x,y)-e_0(x,y)|\end{aligned}$$

holds for every $(x,y)\in X^2$ and $i,j\in\mathbb{N}$. Moreover, the following inequality directly follows from the preceding one:

$$\begin{aligned}|T_{ij}(f;\,x,y)-f(x,y)| \le \varepsilon+\left\{\varepsilon+K_f+\frac{4K_f}{\delta^2}E^2\right\}|T_{ij}(e_0;\,x,y)-e_0(x,y)|+\frac{4K_f}{\delta^2}E|T_{ij}(e_1;\,x,y)-e_1(x,y)|\\ +\frac{4K_f}{\delta^2}E|T_{ij}(e_2;\,x,y)-e_2(x,y)|+\frac{2K_f}{\delta^2}|T_{ij}(e_3;\,x,y)-e_3(x,y)|,\end{aligned}$$

where $E := \max\{|x|,|y|\}$. Now, define $\sigma(x,y)=\max\{|\sigma_z(x,y)|;\,z=0,1,2,3\}$. By multiplying both sides of the preceding inequality by $\frac{1}{|\sigma(x,y)|}$, we arrive at the following conclusion:

$$\begin{aligned}\left|\frac{T_{ij}(f;\,x,y)-f(x,y)}{\sigma(x,y)}\right| \le \frac{\varepsilon}{|\sigma(x,y)|}+K\left\{\left|\frac{T_{ij}(e_0;\,x,y)-e_0(x,y)}{\sigma_0(x,y)}\right|+\left|\frac{T_{ij}(e_1;\,x,y)-e_1(x,y)}{\sigma_1(x,y)}\right|\right.\\ \left.+\left|\frac{T_{ij}(e_2;\,x,y)-e_2(x,y)}{\sigma_2(x,y)}\right|+\left|\frac{T_{ij}(e_3;\,x,y)-e_3(x,y)}{\sigma_3(x,y)}\right|\right\},\end{aligned}$$

where $K := \max\left\{\varepsilon+K_f+\frac{4K_f}{\delta^2}E^2,\frac{4K_f}{\delta^2}E,\frac{2K_f}{\delta^2}\right\}$. Thus, taking supremum over $(x,y)\in X^2$, we have

$$\begin{aligned}&\sup_{(x,y)\in X^2}\left|\frac{T_{ij}(f;\,x,y)-f(x,y)}{\sigma(x,y)}\right|\\ &\le \sup_{(x,y)\in X^2}\frac{\varepsilon}{|\sigma(x,y)|}+K\left\{\sup_{(x,y)\in X^2}\left|\frac{T_{ij}(e_0;\,x,y)-e_0(x,y)}{\sigma_0(x,y)}\right|+\sup_{(x,y)\in X^2}\left|\frac{T_{ij}(e_1;\,x,y)-e_1(x,y)}{\sigma_1(x,y)}\right|\right. \quad (5)\\ &\left.+\sup_{(x,y)\in X^2}\left|\frac{T_{ij}(e_2;\,x,y)-e_2(x,y)}{\sigma_2(x,y)}\right|+\sup_{(x,y)\in X^2}\left|\frac{T_{ij}(e_3;\,x,y)-e_3(x,y)}{\sigma_3(x,y)}\right|\right\} \quad (6)\end{aligned}$$

Now, let $r>0$ be arbitrary. Select $\varepsilon>0$ such that $\sup_{(x,y)\in X^2}\frac{\varepsilon}{|\sigma(x,y)|}<r$. Then,

$$N := \left\{(i,j),\, i\in D_m^{\alpha_1\beta_1} \text{ and } j\in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2}\left|\frac{T_{ij}(f;\,x,y)-f(x,y)}{\sigma(x,y)}\right|\ge r\right\}$$

and

$$N_z := \left\{(i,j),\, i\in D_m^{\alpha_1\beta_1} \text{ and } j\in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2}\left|\frac{T_{ij}(e_z;\,x,y)-e_z(x,y)}{\sigma_z(x,y)}\right|\ge \frac{z-\sup_{(x,y)\in X^2}\frac{\varepsilon}{|\sigma(x,y)|}}{3K}\right\},$$

$z=0,1,2,3$. In light of (5), it is evident that $N\subset\bigcup_3^{z=0}N_z$ and

$$\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N| \le \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N_0|+\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N_1|+\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N_2|+\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N_3|.$$

Then, we can write

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N|\ge\eta\right\}\subset\bigcup_{z=0}^{3}\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N_z|\ge\frac{\eta}{4}\right\}.$$

By (4),

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N_z|\geq\frac{\eta}{4}\right\}\in\mathcal{I}_2,\quad \text{for } z=0,1,2,3.$$

Hence, by the definition of an ideal,

$$\bigcup_{z=0}^{3}\left\{(m,n)\in\mathbb{N}^2:\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N_z|\geq\frac{\eta}{4}\right\}\in\mathcal{I}_2,$$
$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|}|N|\geq\eta\right\}\in\mathcal{I}_2.$$

So, we obtain

$$st(\mathcal{I}_2^{\alpha\beta})-T_{ij}(f)\rightrightarrows f\ \ (X^2;\sigma).$$

Hence, the desired conclusion is obtained. □

Moreover, if the scale function is taken to be a nonzero constant, the subsequent result can be directly deduced from Theorem 4.

Corollary 1. *Let (T_{ij}) be a double sequence of positive linear operators acting from $C(X^2)$ into itself. Then, for all $f\in C(X^2)$,*

$$st(\mathcal{I}_2^{\alpha\beta})-T_{ij}(f)\rightrightarrows f,\quad \text{on } X^2,$$

if and only if

$$st(\mathcal{I}_2^{\alpha\beta})-T_{ij}(e_z)\rightrightarrows e,\quad \text{on } X^2,\ z=0,1,2,3.$$

5 Application of Korovkin-type approximation

Our attention is now paid to an example that shows our main theorem is a non-trivial generalization of the classical and the ideal cases of the Korovkin results.

Example 4. Consider the following Bernstein operators (see [35]) given by

$$B_{ij}(f;x,y)=\sum_{k=0}^{i}\sum_{z=0}^{j}f\left(\frac{k}{i},\frac{l}{j}\right)\binom{i}{k}\binom{j}{l}x^k(1-x)^{i-k}y^l(1-y)^{j-l},\tag{7}$$

where $(x,y)\in X^2=[0,1]^2$; $f\in C(X^2)$. Based on the aforementioned polynomials, we define the following class of positive linear operators on $C(X^2)$:

$$T_{ij}(f;x,y)=(1+h_{ij}(x,y))B_{ij}(f;x,y).\tag{8}$$

Here, the auxiliary function $h_{ij}(x,y)$ is specified in equation (1). We now examine the behavior of these operators on the test functions:

$$T_{ij}(e_0;x,y)=(1+h_{ij}(x,y))e_0(x,y),$$
$$T_{ij}(e_1;x,y)=(1+h_{ij}(x,y))e_1(x,y),$$
$$T_{ij}(e_2;x,y)=(1+h_{ij}(x,y))e_2(x,y),$$

$$T_{ij}(e_3;\,x,y) = (1 + h_{ij}(x,y))\left[e_3(x,y) + \frac{x - x^2}{i} + \frac{y - y^2}{j}\right].$$

In the view of $st(\mathcal{I}_2^{\alpha\beta}) - h_{ij} \rightrightarrows h = 0 \ \ (X^2;\,\sigma)$, $\sigma(x,y)$ is given by (2), we conclude that

$$st(\mathcal{I}_2^{\alpha\beta}) - T_{ij}(e_z) \rightrightarrows e_z \quad (X^2;\,\sigma), \ \text{ for each } \ z = 0,1,2,3.$$

Thus, by Theorem 4, we immediately see that

$$st(\mathcal{I}_2^{\alpha\beta}) - T_{ij}(f) \rightrightarrows f \quad (X^2;\,\sigma), \ \text{ for all } f \in C(X^2).$$

Unfortunately, since (h_{ij}) is neither $\alpha\beta$-statistically uniformly convergent nor $\mathcal{I}_2 - \alpha\beta$-statistically uniformly convergent to the function $h = 0$ on the interval X^2, we see that Theorem 3 and Corollary 1 do not work for our operators defined by (8). In addition, one can see that (h_{ij}) does not satisfy the classical Korovkin theorem in the Pringsheim' sense, i.e., Theorem 2 does not work. As a result, it is demonstrated that the presented formulation constitutes a broader generalization than the previously established one.

6 Rate of $\mathcal{I}_2$-$\alpha\beta$-statistical relative uniform convergence

This section aims to analyze the convergence rate corresponding to the $\mathcal{I}_2$-$\alpha\beta$-statistical relative uniform convergence.

Let us recall that the modulus of continuity of a function $f \in C(X^2)$ is defined by

$$\omega(f,\delta) = \sup_{\sqrt{(x-s)^2+(y-t)^2} \le \delta} |f(x,y) - f(s,t)|, \quad (\delta > 0), \ f \in C(X^2).$$

Definition 13. Let (a_{ij}) denote a positive, non-increasing double sequence. We define that the sequence (f_{ij}) converges to f in the sense of $\mathcal{I}_2$-$\alpha\beta$-statistical relative uniform convergence at the rate $o(a_{ij})$ if for any $\varepsilon > 0$ and $\eta > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \times \left|\left\{(i,j),\, i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2} \left|\frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)}\right| \ge \varepsilon\right\}\right| \ge \eta a_{mn}\right\} \in \mathcal{I}_2,$$

and written in short as

$$st(\mathcal{I}_2^{\alpha\beta}) - (f_{ij} - f) = o(a_{ij}) \quad (X^2;\,\sigma).$$

Definition 14. Let (a_{ij}) denote a positive, non-increasing double sequence. We define that the sequence (f_{ij}) converges to f in the sense of $\mathcal{I}_2$-$\alpha\beta$-statistical relative uniform convergence at the rate $o_i(a_{ij})$ if for any $\varepsilon > 0$ and $\eta > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \times \left|\left\{(i,j),\, i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2} \left|\frac{f_{ij}(x,y) - f(x,y)}{\sigma(x,y)}\right| \ge \varepsilon a_{ij}\right\}\right| \ge \eta\right\} \in \mathcal{I}_2,$$

and written in short as

$$st(\mathcal{I}_2^{\alpha\beta}) - (f_{ij} - f) = o_i(a_{ij}) \quad (X^2;\,\sigma).$$

Lemma 2. *Let (f_{ij}) and (h_{ij}) be double sequences of functions belonging to $C(X^2)$. Suppose (a_{ij}) and (β_{ij}) are positive non-increasing double sequences with the property that*

$$st(\mathcal{I}_2^{\alpha\beta}) - (f_{ij} - f) = o(a_{ij}) \quad (X^2;\,\sigma_0).$$

and

$$st(\mathcal{I}_2^{\alpha\beta}) - (h_{ij} - h) = o(\beta_{ij}) \quad (X^2; \sigma_1),$$

where $|\sigma_z(x,y)| > 0$ *and* $\sigma_z(x,y)$ *is unbounded,* $z = 0, 1$. *As a result, the following properties are satisfied*:

(i) $st(\mathcal{I}_2^{\alpha\beta}) - (f_{ij} + h_{ij}) - (f + h) = o(\max\{\alpha_{ij}, \beta_{ij}\})$ $(X^2; \max\{|\sigma_z(x,y)|; z = 0, 1\})$,

(ii) $st(\mathcal{I}_2^{\alpha\beta}) - (f_{ij} - f)(h_{ij} - h) = o(\alpha_{ij}\beta_{ij})$ $(X^2; \sigma_0(x,y)\sigma_1(x,y))$,

(iii) $st(\mathcal{I}_2^{\alpha\beta}) - (c(f_{ij} - f)) = o(\alpha_{ij})$ $(X^2; \sigma_0(x,y))$ *for any real number* c,

(iv) $st(\mathcal{I}_2^{\alpha\beta}) - \sqrt{|f_{ij} - f|} = o(\alpha_{ij})$ $(X^2; \sqrt{|\sigma_0(x,y)|})$.

In addition, a comparable assertion can be established when the notation "o" is appropriately replaced by "o_i."

Proof. The proof is straightforward and so is omitted. □

Lemma 3. *Let* (f_{ij}) *and* (h_{ij}) *be sequences of functions in* $C(X^2)$ *such that* $0 \le f_{ij} \le h_{ij}$. *Suppose* (α_{ij}) *is a positive non-increasing double sequence with the property that*

$$st(\mathcal{I}_2^{\alpha\beta}) - h_{ij} = o(\alpha_{ij}) \quad (X^2; \sigma).$$

Then, it follows that

$$st(\mathcal{I}_2^{\alpha\beta}) - f_{ij} = o(\alpha_{ij}) \quad (X^2; \sigma),$$

where $|\sigma(x,y)| > 0$ *and* $\sigma(x,y)$ *is unbounded. Furthermore, the statement remains valid when the notation "o" is replaced by "o_i."*

Proof. Since the inequalities $0 \le f_{ij}(x,y) \le h_{ij}(x,y)$ are satisfied for every $(x,y) \in X^2$ and for all indices $(m,n) \in \mathbb{N}^2$, it holds that for arbitrary $\varepsilon > 0$ and $\eta > 0$,

$$\begin{aligned}&\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \times \left|\left\{(i,j), i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2}\left|\frac{f_{ij}(x,y)}{\sigma(x,y)}\right| \ge \varepsilon\right\}\right| \ge \eta\alpha_{mn}\right\} \\ &\subset \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{|D_m^{\alpha_1\beta_1}||D_n^{\alpha_2\beta_2}|} \times \left|\left\{(i,j), i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : \sup_{(x,y)\in X^2}\left|\frac{h_{ij}(x,y)}{\sigma(x,y)}\right| \ge \varepsilon\right\}\right| \ge \eta\alpha_{mn}\right\}.\end{aligned}$$

Utilizing the relation $st(\mathcal{I}2^{\alpha\beta}) - hij = o(\alpha_{ij})$ on $(X^2; \sigma)$, we arrive at the intended conclusion. □

Consequently, we arrive at the following conclusion.

Theorem 5. *Let* (T_{ij}) *denote a double sequence of positive linear operators mapping* $C(X^2)$ *into itself. In addition, let* (α_{ij}) *and* (β_{ij}) *represent positive and non-increasing double sequences. Suppose that the following assumptions are satisfied*:

(a) $st(\mathcal{I}_2^{\alpha\beta}) - (T_{ij}(e_0) - e_0) = o(\alpha_{ij})$ $(X^2; \sigma_0)$,

(b) $st(\mathcal{I}_2^{\alpha\beta}) - \omega(f, \delta_{ij}) = o(\beta_{ij})$ $(X^2; \sigma_1)$, *where* $\delta_{ij}(x,y) = \sqrt{T_{ij}(\varphi_{(x,y)}; x, y)}$ *with* $\varphi_{(x,y)}(s,t) = (x-s)^2 + (y-t)^2$.

Accordingly, for every $f \in C(X^2)$, *we obtain*

$$st(\mathcal{I}_2^{\alpha\beta}) - (T_{ij}(f) - f) = o(\gamma_{mn}) \quad (X^2; \sigma),$$

where $\gamma_{mn} = \max\{\alpha_{ij}, \beta_{ij}, \alpha_{ij}\beta_{ij}\}$, *and* $\sigma(x,y) = \max\{|\sigma_0(x,y)|, |\sigma_1(x,y)|, |\sigma_0(x,y)\sigma_1(x,y)|\}$, *with* $|\sigma_z(x,y)| > 0$ *and* $\sigma_z(x,y)$ *is unbounded for* $z = 0, 1$.

Proof. Let $f \in C(X^2)$ and consider any point $(x, y) \in X^2$. Given that T_{ij} is a positive linear operator and by utilizing the characteristic property of the modulus of continuity, we have

$$\omega(f, \sqrt{\varphi_{(x,y)}(s,t)}) \leq \left(1 + \frac{\varphi_{(x,y)}(s,t)}{\delta^2}\right)\omega(f,\delta).$$

It follows that, for any δ,

$$\begin{aligned}|T_{ij}(f; x,y) - f(x,y)| &\leq T_{ij}(|f(x,y) - f(s,t)|; x,y) + |f(x,y)||T_{ij}(e_0; x,y) - e_0(x,y)|\\ &\leq T_{ij}(\omega(f, \sqrt{\varphi_{(x,y)}(s,t)}); x,y) + |f(x,y)||T_{ij}(e_0; x,y) - e_0(x,y)|\\ &\leq \omega(f,\delta)T_{ij}\left(\left(1 + \frac{\varphi_{(x,y)}(s,t)}{\delta^2}\right); x,y\right) + |f(x,y)||T_{ij}(e_0; x,y) - e_0(x,y)|\\ &\leq \omega(f,\delta)\left\{T_{ij}(e_0; x,y) + \frac{1}{\delta^2}T_{ij}(\varphi_{(x,y)}(s,t); x,y)\right\} + |f(x,y)||T_{ij}(e_0; x,y) - e_0(x,y)|.\end{aligned}$$

Setting $\delta \coloneqq \delta_{ij}(x,y) = \sqrt{T_{ij}(\varphi_{(x,y)}; x,y)}$, we may write that

$$|T_{ij}(f; x,y) - f(x,y)| \leq 2\omega(f,\delta_{ij}) + \omega(f,\delta_{ij})|T_{ij}(e_0; x,y) - e_0(x,y)| + M|T_{ij}(e_0; x,y) - e_0(x,y)|,$$

where $M = \|f\|$. Taking into account the preceding inequality, we can express the following:

$$\begin{aligned}&\sup_{(x,y)\in X^2}\left|\frac{T_{ij}(f; x,y) - f(x,y)}{\sigma(x,y)}\right|\\ &\leq 2\sup_{(x,y)\in X^2}\frac{\omega(f,\delta_{ij})}{|\sigma_1(x,y)|} + \sup_{(x,y)\in X^2}\frac{\omega(f,\delta_{ij})}{|\sigma_1(x,y)|}\sup_{(x,y)\in X^2}\left|\frac{T_{ij}(e_0; x,y) - e_0(x,y)}{\sigma_0(x,y)}\right|\\ &\quad + M\sup_{(x,y)\in X^2}\left|\frac{T_{ij}(e_0; x,y) - e_0(x,y)}{\sigma_0(x,y)}\right|.\end{aligned}$$

Thus, under assumptions (a) and (b), and by invoking Lemmas 2 and 3, the result follows. □

The proof of the following theorem is analogous to the proof of Theorem 5, and so is omitted.

Theorem 6. *Let (T_{ij}) be a double sequence of positive linear operators mapping $C(X^2)$ into itself. Furthermore, let (α_{ij}) and (β_{ij}) be positive, non-increasing double sequences. Suppose that the following assumptions are satisfied:*

(a) $st(\mathcal{I}_2^{\alpha\beta}) - (T_{ij}(e_0) - e_0) = o_i(\alpha_{ij})$ $(X^2; \sigma_0)$,

(b) $st(\mathcal{I}_2^{\alpha\beta}) - \omega(f,\delta_{ij}) = o_i(\beta_{ij})$ $(X^2; \sigma_1)$, *where* $\delta_{ij}(x,y) = \sqrt{T_{ij}(\varphi_{(x,y)}; x,y)}$ *with* $\varphi_{(x,y)}(s,t) = (x-s)^2 + (y-t)^2$.

Then, we have, for all $f \in C(X^2)$,

$$st(\mathcal{I}_2^{\alpha\beta}) - (T_{ij}(f) - f) = o_i(\gamma_{mn}) \quad (X^2; \sigma),$$

where $\gamma_{mn} = \max\{\alpha_{ij}, \beta_{ij}, \alpha_{ij}\beta_{ij}\}$ *and* $\sigma(x,y) = \max\{|\sigma_0(x,y)|, |\sigma_1(x,y)|, |\sigma_0(x,y)\sigma_1(x,y)|\}$, $|\sigma_z(x,y)| > 0$ *and $\sigma_z(x,y)$ is unbounded, $z = 0, 1$.*

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