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ON RESULTS OF PARAMETERIZED INEQUALITIES ASSOCIATED WITH CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In the present article, an equality is established for the case of differentiable convex functions associated with the conformable fractional integrals. With the help of this equality, several parameterized inequalities are proved by using the conformable fractional integrals. More precisely, some inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Moreover, previously achieved results and new results are given by using special cases of the obtained theorems.

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1. Introduction & preliminaries

Convex theory is an effective methodology in the case of considering the great issues that origin in several areas of the pure and applied sciences. This theory has a rich history and also has been the motivation of special mathematical research. Moreover, convexity theory has a critical place in the advancement of the idea of inequality. There are several types of convexity in the literature.

Definition 1 ([17])**.** Let I be an interval of real numbers. Then, a function $\mathcal{F}: I \to \mathbb{R}$ is said to be convex, if

$$\mathcal{F}\left(\mu x+(1-\mu)y\right) \leq \mu f\left(x\right)+(1-\mu)\mathcal{F}\left(y\right)$$

is valid $\forall x, y \in I$ and $\forall \mu \in [0,1]$.

The inequalities, introduced by C. Hermite and J. Hadamard for the case of convex functions, are significant subject in the literature. These inequalities state that if $\mathcal{F}: I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\sigma, \delta \in I$ with

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$\sigma < \delta$, then the following double inequality holds:

$$\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) \leq \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta} \mathcal{F}(x)\mathrm{d}x \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}. \tag{1.1}$$

Trapezoid-type and midpoint-type inequalities for various types of convex functions have been studied extensively by many mathematicians. To be more precise, an upper bound for which is the right-hand side of inequality (1.1) first was established by Dragomir and Agarwal in paper [6]. An upper bound for which is the left-hand side of inequality (1.1) first was proved by Kırmacı in paper [16]. Qaisar and Hussain [18] investigated several generalized midpoint-type inequalities. Moreover, Sarikaya et al. and Iqbal et al. proved some fractional trapezoid-type and midpoint-type inequalities to the case of convex functions in papers [20] and [12], respectively.

Large numbers of papers are focused on the results of Simpson-type inequalities for the case of convex functions. For example, several Simpson-type inequalities proved by the case of s-convex differentiable functions [2]. Moreover, Sarikaya et al. [19] presented the new variants of Simpson-type inequalities based on differentiable convex function. Some recent results connected with these type of inequalities, see [3] and the references cited therein.

Bullen [4] proved the inequality which is known as Bullen-type inequalities for the case of convex function. Erden and Sarikaya presented the generalized Bullen-type inequalities for the case of local fractional integrals on fractal sets in paper [8]. Moreover, Du et al. [7] used the generalized fractional integrals to get Bullen-type inequalities. In paper [5], several generalizations of integral inequalities of Bullen-type inequalities were obtained for the case of twice differentiable functions including Riemann-Liouville fractional integrals.

Kilbas et al. [15] presented fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 2 ([15])**.** Let $\mathcal{F} \in L_1[\sigma,\delta]$, $\sigma,\delta \in \mathbb{R}$ with $\sigma < \delta$. The Riemann-Liouville integrals $J^{\beta}_{\sigma^+}\mathcal{F}$ and $J^{\beta}_{\delta^-}\mathcal{F}$ of order $\beta > 0$ are defined by

$$J^{\beta}_{\sigma^+}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}(x-\mu)^{\beta-1}\mathcal{F}(\mu)\mathrm{d}\mu, \quad x > \sigma \tag{1.2}$$

and

$$J^{\beta}_{\delta^-}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}(\mu-x)^{\beta-1}\mathcal{F}(\mu)\mathrm{d}\mu, \quad x < \delta, \tag{1.3}$$

respectively. Here, Γ denotes the Gamma function defined by $\Gamma(\beta) = \int_0^{\infty} e^{-u}u^{\beta-1}\mathrm{d}u$, where $\beta > 0$.

Jarad et al. [13] established the fractional conformable integral operators and they also provided some characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined as follows:

Definition 3 ([13]). Let $\beta > 0$ and $\alpha \in (0,1]$. For $\mathcal{F} \in L_1[\sigma,\delta]$, the generalized fractional Riemann-Liouville integrals ${}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}$ and ${}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta}\mathcal{F}$ are defined by

$$ {}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}\left(\frac{(x-\sigma)^{\alpha}-(\mu-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(\mu)}{(\mu-\sigma)^{1-\alpha}}d\mu, \quad x>\sigma, \tag{1.4} $$

and

$$ {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}\left(\frac{(\delta-x)^{\alpha}-(\delta-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(\mu)}{(\delta-\mu)^{1-\alpha}}d\mu, \quad x<\delta, \tag{1.5} $$

respectively.

If we assign $\alpha = 1$, then the fractional integral in (1.4) and (1.5) equals to the Riemann-Liouville fractional integral in (1.2) and (1.3), respectively.

Several midpoint-type inequalities are proved by using the conformable fractional integrals in paper [10]. Set et al. [21] establish a new generalization of Hermite-Hadamard inequality for fractional conformable integrals. In addition to these, conformable derivative is improved in paper [14] by using the derivative's fundamental limit formulation. Some important requirements that cannot be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. In paper [1], Abdelhakim indicates that the conformable approach in [14] cannot yield good results when compared to the Caputo definition for the case of specific functions.

2. Main results

Throughout the paper, we assume that $\alpha \in (0,1]$, $\beta \in \mathbb{R}^{+}$, and $\lambda \in [0,1]$.

Lemma 1. *Let $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ be a differentiable mapping on (σ,δ) with $\beta>0$ and $\alpha\in(0,1]$. If $\mathcal{F}'\in L[\sigma,\delta]$, then the following equality for fractional integrals hold:*

$$ \begin{aligned} &(1-\lambda)\,\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right) \\ &\quad-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right] \\ &=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_{0}^{1}\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu\right. \end{aligned} $$

$$-\int_0^1 \left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)d\mu\Bigg], \tag{2.1}$$

where $\Gamma(\beta)$ *is Euler Gamma function.*

Proof. Assume that I is the integral on the right side of (2.1) and consider

$$I := I_1 - I_2.$$

Using integrating by parts and changing variables with $x = \frac{\mu}{2}\sigma + \frac{2-\mu}{2}\delta$, we get

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu \\
&= -\frac{2}{\delta-\sigma}\left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\Bigg|_0^1 \\
&\quad -\frac{2\beta}{\delta-\sigma}\int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu \\
&= \frac{2(1-\lambda)}{(\delta-\sigma)\alpha^\beta}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{2\lambda}{(\delta-\sigma)\alpha^\beta}\mathcal{F}(\delta) \\
&\quad -\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\beta\int_{\frac{\sigma+\delta}{2}}^{\delta}\left(\frac{\left(\frac{\delta-\sigma}{2}\right)^\alpha-\left(x-\frac{\sigma+\delta}{2}\right)^\alpha}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{\left(x-\frac{\sigma+\delta}{2}\right)^{1-\alpha}}dx \\
&= \frac{2(1-\lambda)}{(\delta-\sigma)\alpha^\beta}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{2\lambda}{(\delta-\sigma)\alpha^\beta}\mathcal{F}(\delta) \\
&\quad -\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\Gamma(\beta+1)\,{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta).
\end{aligned}$$

Similar to foregoing process, changing variables with $x = \frac{2-\mu}{2}\sigma + \frac{\mu}{2}\delta$ and integrating on $[0,1]$, we have

$$\begin{aligned}
I_2 &= \int_0^1 \left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)d\mu \\
&= \frac{2}{\delta-\sigma}\left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\Bigg|_0^1
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\,\mathcal{F}\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\mathrm{d}\mu\\
&=\frac{2(\lambda-1)}{(\delta-\sigma)\alpha^{\beta}}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{2\lambda}{(\delta-\sigma)\alpha^{\beta}}\mathcal{F}(\sigma)\\
&+\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\beta\int_{\sigma}^{\frac{\sigma+\delta}{2}}\left(\frac{\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\sigma+\delta}{2}-x\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{\left(\frac{\sigma+\delta}{2}-x\right)^{1-\alpha}}\mathrm{d}x\\
&=-\frac{2(1-\lambda)}{(\delta-\sigma)\alpha^{\beta}}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{2\lambda}{(\delta-\sigma)\alpha^{\beta}}\mathcal{F}(\sigma)\\
&+\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\Gamma(\beta+1)\,{}_{-}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\sigma).
\end{aligned}
$$

Finally, if we multiply I_1-I_2 by $\frac{\alpha^{\beta}(\delta-\sigma)}{4}$, then we have (2.1). This completes the proof of Lemma 1. □

Remark 1. If we choose $\lambda=0$ in (2.1), then the equality reduces to

$$
\begin{aligned}
&\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\sigma)\right]\\
&=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\mathrm{d}\mu\right.\\
&\qquad\left.-\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\mathrm{d}\mu\right].
\end{aligned}
$$

This coincides with [9, Lemma 1].

Remark 2. Let us consider $\lambda=1$ in (2.1). Then, we have

$$
\begin{aligned}
&\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\sigma)\right]\\
&=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\mathrm{d}\mu\right.\\
&\qquad\left.-\int_0^1\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\mathrm{d}\mu\right],
\end{aligned}
$$

which is investigated by E. Set et al. [22, Lemma 2.1 $\left(\text{for } x=\frac{a+b}{2}\right)$].

Corollary 1. *Consider* $\lambda = \frac{1}{3}$ *in* (2.1). *Then,* (2.1) *becomes to*

$$\frac{1}{6}\left[\mathcal{F}(\sigma)+\mathcal{F}(\delta)+4f\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]$$
$$=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left(\frac{1}{3\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu\right.$$
$$\left.+\int_0^1\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)d\mu\right].$$

Remark 3. For $\lambda = \frac{1}{2}$, equality (2.1) reduces to

$$\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]$$
$$=\frac{\alpha^{\beta}(\delta-\sigma)}{4}$$
$$\cdot\int_0^1\left(\frac{1}{2\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\left[\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)-\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right]d\mu.$$

This coincides with [11, Lemma 1].

Theorem 1. *Assume that all assumptions of Lemma 1 hold. Moreover, let* $|\mathcal{F}'|$ *be a convex function on* $[\sigma,\delta]$. *Then, we have*

$$\left|(1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right)\right.$$
$$\left.-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1(\alpha,\beta,\lambda)\left(\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right), \tag{2.2}$$

where

$$A_1(\alpha,\beta,\lambda)=\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu$$
$$=\frac{1}{\alpha^{\beta}}\left[\lambda(2C-1)+\frac{1}{\alpha}\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\lambda^{\frac{1}{\beta}}\right)\right)\right]$$

with $C = 1 - \left(1 - \lambda^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$, *the functions* $\mathfrak{B}(\cdot,\cdot)$ *and* $\mathscr{B}(\cdot,\cdot,\cdot)$ *are the Beta function and the incomplete Beta function defined as*

$$\begin{cases} \mathfrak{B}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\mathrm{d}u, \\ \mathscr{B}(x,y,r) = \int_0^r u^{x-1}(1-u)^{y-1}\mathrm{d}u \end{cases}$$

for $x,y>0$ *and* $r \in [0,1]$.

Proof. By Lemma 1, integrating by parts and convexity of $|\mathcal{F}'|$, we get

$$\begin{aligned}
&\left|(1-\lambda)\,\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right)\right. \\
&\left.-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\left|\int_0^1\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\mathrm{d}\mu\right|\right. \\
&\qquad\left.+\left|\int_0^1\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\mathrm{d}\mu\right|\right] \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|\mathrm{d}\mu\right. \\
&\qquad\left.+\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|\mathrm{d}\mu\right] \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\frac{\mu}{2}\left|\mathcal{F}'(\sigma)\right|+\frac{2-\mu}{2}\left|\mathcal{F}'(\delta)\right|\right]\mathrm{d}\mu\right. \\
&\qquad\left.+\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\frac{2-\mu}{2}\left|\mathcal{F}'(\sigma)\right|+\frac{\mu}{2}\left|\mathcal{F}'(\delta)\right|\right]\mathrm{d}\mu\right] \\
&= \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1(\alpha,\beta,\lambda)\left(\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right).
\end{aligned}$$

That is the desired result. □

Remark 4. If we consider (2.2) as $\lambda = 0$, then the inequality reduces to

$$\left| \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta} 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}J^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\delta) + {}^{\beta}_{-}J^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\sigma) \right] \right|$$
$$\leq \frac{\delta-\sigma}{4\alpha} \mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) \left(\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right| \right).$$

This coincides with [9, Theorem 2].

Remark 5. If we select $\lambda = 1$ in (2.2), then we obtain

$$\left| \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} - \frac{\alpha^{\beta} 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\delta) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\sigma) \right] \right|$$
$$\leq \frac{\delta-\sigma}{4} \left[1 - \frac{1}{\alpha} \mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) \right] \left(\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right| \right),$$

which is established by E. Set et al. [22, Theorem 2.1 $\left(\text{for } x = \frac{a+b}{2}\right)$].

Corollary 2. *Let us consider* $\lambda = \frac{1}{3}$ *in inequality* (2.2). *Then, it follows*

$$\left| \frac{1}{6} \left[\mathcal{F}(\sigma) + \mathcal{F}(\delta) + 4f\left(\frac{\sigma+\delta}{2}\right) \right] - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2} \left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta} \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\delta) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\sigma) \right] \right|$$
$$\leq \frac{(\delta-\sigma)}{4} \left[\frac{2D-1}{3} + \frac{1}{\alpha} \left(\mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) - 2\mathscr{B}\left(\beta+1, \frac{1}{\alpha}, \left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right) \right) \right] \left(\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right| \right).$$

Here, $D = 1 - \left(1 - \left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$.

Remark 6. For $\lambda = \frac{1}{2}$, inequality (2.2) reduces to

$$\left| \frac{1}{2} \left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} \right] - \frac{\alpha^{\beta} 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\delta) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\sigma) \right] \right|$$
$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} \left\{ \frac{1}{\alpha^{\beta}} \left[\frac{1}{2} - \left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} \right] \right.$$
$$\left. + \frac{1}{\alpha^{\beta+1}} \left[\mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) - 2\mathscr{B}\left(\beta+1, \frac{1}{\alpha}, \left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right) \right] \right\} \left(\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right| \right).$$

This coincides with [11, Theorem 1].

Theorem 2. *Suppose that all assumptions of Lemma 1 hold. Moreover, let* $|\mathcal{F}'|^q$ *be a convex function on* $[\sigma,\delta]$ *where* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p, q > 1$. *Then, we have*

$$\left| (1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right) \right.$$

$$
\begin{aligned}
&-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\sigma)\right]\Bigg| \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda) \\
&\quad\cdot\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\} \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[4A_2(\alpha,\beta,p,\lambda)\right]^{\frac{1}{p}}\left(\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right), \qquad (2.3)
\end{aligned}
$$

where

$$
A_2(\alpha,\beta,p,\lambda)=\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu.
$$

Proof. If we consider Lemma 1, then we get

$$
\begin{aligned}
&\Bigg|(1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right) \\
&-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\sigma)\right]\Bigg| \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|d\mu\right. \\
&\qquad\left.+\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|d\mu\right]. \qquad (2.4)
\end{aligned}
$$

Now, we consider the integrals on the right side of (2.4). Using the convexity of $|\mathcal{F}'|^q$ and the well-known Hölder inequality, we have

$$
\begin{aligned}
&\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|d\mu \\
&\leq\left(\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\leq A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left(\int_0^1\left(\frac{\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{2-\mu}{2}\left|\mathcal{F}'(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}
\end{aligned}
$$

$$= A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left[\frac{|\mathcal{F}'(\sigma)|^q + 3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}} \tag{2.5}$$

and similarly

$$\begin{aligned}
&\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| \left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma + \frac{\mu}{2}\delta\right)\right| d\mu \\
&\leq \left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|^p d\mu\right)^{\frac{1}{p}} \left(\int_0^1 \left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma + \frac{\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\leq A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left(\int_0^1 \left(\frac{2-\mu}{2}|\mathcal{F}'(\sigma)|^q + \frac{\mu}{2}|\mathcal{F}'(\delta)|^q\right) d\mu\right)^{\frac{1}{q}} \\
&= A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left[\frac{3|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.6}$$

If we consider (2.5) and (2.6) in (2.4), then we get

$$\begin{aligned}
&\left|(1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right)\right. \\
&\left. - \frac{\alpha^\beta 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_+^\beta \mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_-^\beta \mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\leq \frac{\alpha^\beta(\delta-\sigma)}{4} A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda) \\
&\quad \cdot \left\{\left[\frac{|\mathcal{F}'(\sigma)|^q + 3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}} + \left[\frac{3|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

The remaining part of Theorem 2 can be easily seen by setting $a_1 = 3|\mathcal{F}'(\sigma)|^q$, $a_2 = |\mathcal{F}'(\sigma)|^q$, $b_1 = 3|\mathcal{F}'(\delta)|^q$, $b_2 = |\mathcal{F}'(\delta)|^q$ and applying the well-known inequality

$$\sum_{k=1}^n (a_k + b_k)^s \leq \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s$$

with $0 \leq s < 1$. This finishes the proof of Theorem 2. □

Corollary 3. *If we take* $\lambda = 1$ *in* (2.3)*, then we obtain*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} - \frac{\alpha^\beta 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_+^\beta \mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_-^\beta \mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} A_2^{\frac{1}{p}}(\alpha,\beta,p,1)\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[4A_2(\alpha,\beta,p,1)\right]^{\frac{1}{p}}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],$$

where

$$A_2(\alpha,\beta,p,1)=\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu.$$

Remark 7. If we consider (2.3) as $\lambda=0$, then (2.3) reduces to

$$\left|\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} A_2^{\frac{1}{p}}(\alpha,\beta,p,0)\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left(4\frac{1}{\alpha^{\beta p+1}}\mathfrak{B}\left(\beta p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\left(\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right).$$

This coincides with [9, Theorem 4].

Corollary 4. *Note that the inequalities* (2.3) *become to*

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+\mathcal{F}(\delta)+4f\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2}\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta}\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left(\int_0^1\left|\frac{1}{3\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\cdot\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[4A_2\left(\alpha,\beta,p,\frac{1}{3}\right)\right]^{\frac{1}{p}}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right]$$

for $\lambda=\frac{1}{3}$.

Remark 8. For $\lambda=\frac{1}{2}$, the inequalities (2.3) equal to

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} A_2^{\frac{1}{p}}\left(\alpha,\beta,p,\frac{1}{2}\right)\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left(4\int_0^1\left|\frac{1}{2\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],$$

which is presented in [11, Theorem 2].

Theorem 3. *Let us consider that all assumptions of the Lemma 1 hold. If $|\mathcal{F}'|^q$ is convex on $[\sigma,\delta]$ where $q\geq 1$, then we have the following inequality*

$$\begin{aligned}
&\left|(1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right)\right.\\
&\left.-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\sigma)\right]\right|\\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left\{\left(A_3(\alpha,\beta,\lambda)\left|\mathcal{F}'(\sigma)\right|^q+A_4(\alpha,\beta,\lambda)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(A_4(\alpha,\beta,\lambda)\left|\mathcal{F}'(\sigma)\right|^q+A_3(\alpha,\beta,\lambda)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}\tag{2.7}$$

where A_1 is defined in Theorem 1 and

$$\begin{aligned}
A_3(\alpha,\beta,\lambda)&=\int_0^1\frac{\mu}{2}\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu\\
&=\frac{1}{2\alpha^{\beta}}\left\{\lambda\left(C^2-\frac{1}{2}\right)-\frac{1}{\alpha}\left[2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\lambda^{\frac{1}{\beta}}\right)-2\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\lambda^{\frac{1}{\beta}}\right)\right.\right.\\
&\qquad\left.\left.-\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)+\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right]\right\},\\
A_4(\alpha,\beta,\lambda)&=\int_0^1\frac{2-\mu}{2}\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu=A_1(\alpha,\beta,\lambda)-A_3(\alpha,\beta,\lambda)\\
&=\frac{1}{2\alpha^{\beta}}\left\{\lambda\left(4C-C^2-\frac{3}{2}\right)-\frac{1}{\alpha}\left[2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\lambda^{\frac{1}{\beta}}\right)+2\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\lambda^{\frac{1}{\beta}}\right)\right.\right.\\
&\qquad\left.\left.-\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right]\right\}.
\end{aligned}$$

Proof. If we consider the convexity of $|\mathcal{F}'|^q$ and power mean inequality, then we obtain

$$\begin{aligned}
&\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|d\mu \\
&\leq \left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|d\mu\right)^{1-\frac{1}{q}} \\
&\quad\cdot\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\leq A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left[\frac{\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{2-\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q\right]d\mu\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(A_3(\alpha,\beta,\lambda)\left|\mathcal{F}'(\sigma)\right|^q+A_4(\alpha,\beta,\lambda)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}} \qquad (2.8)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|d\mu \\
&\leq \left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|d\mu\right)^{1-\frac{1}{q}} \\
&\quad\cdot\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\leq A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left[\frac{\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{2-\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q\right]d\mu\right)^{\frac{1}{q}}
\end{aligned}$$

$$= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(A_4(\alpha,\beta,\lambda)\left|\mathcal{F}'(\sigma)\right|^q + A_3(\alpha,\beta,\lambda)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}. \tag{2.9}$$

If we consider (2.8) and (2.9) in (2.4), then the proof is completed. □

Corollary 5. *If we take* $\lambda = 1$ *in* (2.7)*, then we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} - \frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\le \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1^{1-\frac{1}{q}}(\alpha,\beta,1)\left\{\left(A_3(\alpha,\beta,1)\left|\mathcal{F}'(\sigma)\right|^q + A_4(\alpha,\beta,1)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left. + \left(A_4(\alpha,\beta,1)\left|\mathcal{F}'(\sigma)\right|^q + A_3(\alpha,\beta,1)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\begin{aligned}
A_1(\alpha,\beta,1) &= \frac{1}{\alpha^{\beta}}\left\{1 - \frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right\}, \\
A_3(\alpha,\beta,1) &= \frac{1}{2\alpha^{\beta}}\left\{\frac{1}{2} - \frac{1}{\alpha}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) - \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right]\right\}, \\
A_4(\alpha,\beta,1) &= \frac{1}{2\alpha^{\beta}}\left\{\frac{3}{2} - \frac{1}{\alpha}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) + \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right]\right\}.
\end{aligned}$$

Remark 9. If we consider (2.7) as $\lambda = 0$, then inequality (2.7) reduces to

$$\begin{aligned}
&\left|\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\le \frac{\delta-\sigma}{4\alpha}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right]^{1-\frac{1}{q}} \\
&\cdot\left\{\left[\left(\frac{\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) - \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)}{2}\right)\left|\mathcal{F}'(\sigma)\right|^q + \left(\frac{\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) + \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)}{2}\right)\left|\mathcal{F}'(\delta)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left. + \left[\left(\frac{\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) + \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)}{2}\right)\left|\mathcal{F}'(\sigma)\right|^q + \left(\frac{\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) - \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)}{2}\right)\left|\mathcal{F}'(\delta)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

This coincides with [9, Theorem 3].

Corollary 6. *For* $\lambda = \frac{1}{3}$*, inequality* (2.7) *becomes to*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+\mathcal{F}(\delta)+4f\left(\frac{\sigma+\delta}{2}\right)\right] - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2}\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta}\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\le \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1^{1-\frac{1}{q}}\left(\alpha,\beta,\frac{1}{3}\right)\left\{\left(A_3\left(\alpha,\beta,\frac{1}{3}\right)\left|\mathcal{F}'(\sigma)\right|^q + A_4\left(\alpha,\beta,\frac{1}{3}\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left. + \left(A_4\left(\alpha,\beta,\frac{1}{3}\right)\left|\mathcal{F}'(\sigma)\right|^q + A_3\left(\alpha,\beta,\frac{1}{3}\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Here, $D = 1-\left(1-\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$,

$$A_1\left(\alpha,\beta,\frac{1}{3}\right)=\frac{1}{\alpha^{\beta}}\left[\frac{2D-1}{3}+\frac{1}{\alpha}\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)\right)\right],$$

$$A_3\left(\alpha,\beta,\frac{1}{3}\right)=\frac{1}{2\alpha^{\beta}}\left\{\frac{2D^2-1}{6}-\frac{2}{\alpha}\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)+\frac{2}{\alpha}\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)+\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right\},$$

$$A_4\left(\alpha,\beta,\frac{1}{3}\right)=\frac{1}{2\alpha^{\beta}}\left\{\frac{-2D^2+8D-3}{6}-\frac{2}{\alpha}\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)-\frac{2}{\alpha}\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)+\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)+\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right\}.$$

Remark 10. For $\lambda=\frac{1}{2}$, inequality (2.7) reduces to

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1^{1-\frac{1}{q}}\left(\alpha,\beta,\frac{1}{2}\right)\left\{\left(A_3\left(\alpha,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+A_4\left(\alpha,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}+\left(A_4\left(\alpha,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+A_3\left(\alpha,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right\},$$

where

$$A_1\left(\alpha,\beta,\frac{1}{2}\right)=\frac{1}{\alpha^{\beta}}\left[\frac{1}{2}-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]+\frac{1}{\alpha^{\beta+1}}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right],$$

$$A_3\left(\alpha,\beta,\frac{1}{2}\right)=\frac{1}{2\alpha^{\beta}}\left[\left(1-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)^{2}-\frac{1}{2}\right]+\frac{1}{\alpha^{\beta+1}}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)+2\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)\right],$$

$$A_4\left(\alpha,\beta,\frac{1}{2}\right)=\frac{1}{2\alpha^{\beta}}\left[2\left(1-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)-\frac{1}{2}\left(1-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)^{2}-\frac{3}{4}\right]$$
$$+\frac{1}{\alpha^{\beta+1}}\left[-\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right.$$
$$\left.+2\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)+2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)\right].$$

This coincides with [11, Theorem 3].

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